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Hopf Bifurcation Control in a Lorenz Type System

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Abstract

In this paper Hopf bifurcation control is implemented in order to change the bifurcation from supercritical to subcritical in a differential equations system of Lorenz type. To achieve this purpose: first, a region of parameters is identified where the system has a supercritical Hopf bifurcation; second, a class of non-linear feedback control laws is proposed; finally, it is shown that there are control laws which the disturbed system undergoes subcritical Hopf bifurcation.

Keywords

Lorenz Type Systems, Bifurcation Control, Hopf Bifurcation Control, Subcritical and Supercritical Hopf Bifurcation

1. Introduction

In [1] the *Bifurcation Control* is described as the task of designing a control law that modifies the bifurcation characteristics in order to achieve a desirable dynamic behaviour. To specify the above, set a given nonlinear system,

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \alpha), \tag{1}$$

with state vectors $\boldsymbol{x} \in \mathbb{R}^n$ and parameters vector $\alpha \in \mathbb{R}^m$, which presents a bifurcation. A parametric class of functions $u(\cdot, \alpha)$ is proposed, called control laws, with which (1) is perturbed, resulting the nonlinear system,

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \alpha) + u(\boldsymbol{x}, \alpha). \tag{2}$$

The bifurcation control problem consists in to find, within the parametric class, a control law that achieves a desirable behaviour or an established objective in the perturbed system.

Objectives of the bifurcation control are diverse, some of them are: to move the critical value or the bifurcation points [2] [3], introducing a new bifurcation at a preferable parameter value [4], to change the amplitude or frequency of some limit cycles emerging from bifurcation [1] [5], to stabilize the periodic orbit that emerges from the Hopf bifurcation [3] [6], to modify the shape or type of a bifurcations chain [7] [8], to change and/or create a bifurcation [4] [9].

A kind of bifurcation is the Hopf bifurcation, this corresponds to the following situation: when the parameter crosses a critical value, the Jacobian of the system evaluated in the equilibrium has a pair of conjugate complex eigenvalues, which moves from the left to the right half-plane, while the other eigenvalues remain stable; at the crossing moment the real parts of the two eigenvalues become zero, the stability of the equilibrium changes from stable to unstable and a limit cycle arises. When the limit cycle is stable the Hopf bifurcation is said to be *supercritical* and when the limit cycle is unstable it is said that the Hopf bifurcation is *subcritical* [10].

When the system (1) presents Hopf bifurcation in a equilibrium point $(\boldsymbol{x}^*,\alpha)$, the *Hopf Bifurcation Control* consists in to determine a control law $u(\cdot,\alpha^0)$ of a parametric class $u(\cdot,\alpha)$, in such a way that the equilibrium point $(\boldsymbol{x}^*,\alpha)$ of (1) moves to an equilibrium point $(\boldsymbol{x}^0,\alpha^0)$ of the perturbed system (2), the Hopf bifurcation is preserved and one of the following situations occurs:

1) Stability of the limit cycle that emerges from the Hopf bifurcation changes from unstable to stable or vice versa.

2) Change the orientation of the limit cycle.

In this paper is consider the Lorenz type system

$$\begin{aligned} \dot{x} &= a(y-x), \\ \dot{y} &= dy - xz, \\ \dot{z} &= -bz + gxy, \end{aligned} \tag{3}$$

with state vector $(x,y,z)^{\mathrm{T}} \in \mathbb{R}^3$, parameters $a>0$, $g>0$ and $b,d \in \mathbb{R}$. The system (3) is a particular case of the system studied by Xianyi Li y Qianjun Ou in [11], where is proved, under standar hypotheses, the system has a Hopf bifurcation. Recently, in [12] is proved the Hopf bifurcation in (3) is non-degenerated and supercritical.

The objective is apply Hopf Bifurcation Control to the system (3) in order to change the Hopf bifurcation from supercritical to subcritical, by using non-lineal control laws in state feedback. The used method for Hopf bifurcation analisys include the Hopf Theorem, the Central Manifold Theorem, Normal Forms, and a formule for the first Lyapunov coefficient given in [13].

On one side, the system (3) is a particular case of the system studied in [11], another side it is an extension of the Lu system [14], namely, when the parameter g takes the value 1, the system (3) is reduced to the Lu system [14]. The Lorenz type systems class is studied or taken as reference in fields of physics, engineering, and mathematics; in topics such as: chaos [15] [16], Hopf bifurcation [11] [17] [18], synchronization [14] and control [19] [20] [21] [22] [23].

On the other hand, in [3] [24] [25] [26], Hopf bifurcation control is done with the purpose of to change the Hopf bifurcation from supercritical to subcritical, for systems that are not Lorenz type.

Remainder of this paper is ordered as follows. The Section 2 describes some preliminary results concerning the existence of the Hopf bifurcation in the system (3). In Section 3 it is proved that the Hopf bifurcation is non-degenerate and supercritical. In Sections 4 and 5 the Hopf bifurcation control of the system (3) is made. Finally, in Section 6 the conclusions of this paper are presented.

2. Dynamics System

In this section, in order to make a self-contained writing, the results of [11] are specialize to the system (3), which correspond to the Hopf bifurcation.

System that Li and Ou considers is as follows,

$$\begin{aligned} \dot{x} &= a(y-x), \\ \dot{y} &= dy - xz, \\ \dot{z} &= -bz + fx^2 + gxy, \end{aligned} \tag{4}$$

where $a > 0$, $f \geq 0$, $g \geq 0$, $f + g > 0$, $b, d \in \mathbb{R}$, and states vector $(x, y, z)^{\mathrm{T}} \in \mathbb{R}^3$. It is observed that when $f = 0$ the system (3) is obtained.

Li and Ou [11] found that when $(a,b,d,f,g) = (10,3,6,1,0)$ and $(x,y,z) = \left(\frac{23}{20}, \frac{33}{10}, \frac{7}{2}\right)$, the system that results presents the Lyapunov exponents: $\lambda_{LE_1} = 0.4265$, $\lambda_{LE_2} = 0.000001$, $\lambda_{LE_3} = -7.4264$, it is observed that two of these exponents are positive, which characterizes the emergence of a chaotic orbit. The **Figure 1** shows the chaotic attractor of the system (4) when the initial conditions and parameters are as indicated at the beginning of the paragraph.

Equilibrium points of the system (4) are classified according to the next cases:

- Case $bd < 0$: There is only one equilibrium point $P_0 = (0,0,0)$.
- Case $bd = 0$: If $b = 0$, all point of form $(0,0,z)$ is an equilibrium and for $d = 0$ the equilibrium is P_0.
- Case $bd > 0$: There are three equilibrium points $P_0 = (0,0,0)$,

$$P_1 = \left(\sqrt{\frac{bd}{f+g}}, \sqrt{\frac{bd}{f+g}}, d\right) \text{ and } P_2 = \left(-\sqrt{\frac{bd}{f+g}}, -\sqrt{\frac{bd}{f+g}}, d\right).$$

From now, the analysis focuses on the system (3) that results from taking $f = 0$ in (4), and is restricted to the case $bd > 0$.

Li and Ou in [11] proved that in P_0 the system (3) has a no-bifurcation dynamics and that in the equilibrium points P_1 y P_2 the system presents Hopf bifurcation, if $b > 0$ the bifurcation critical value is:

$$d_{-} = \frac{a+b}{3}.$$

Next theorem guarantees the existence of the Hopf bifurcation of the system (3) in the equilibrium points P_1 and P_2.

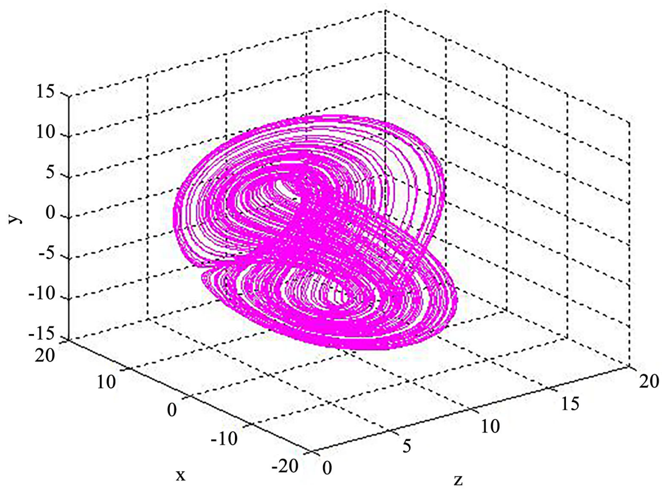


Figure 1. Chaotic attractor of Li-Ou.

Theorem 1 *When* $a > 0$, $b > 0$, $d > 0$ *and* $g > 0$, *the following statements are true to the system* (3):

1) *For* $0 < d < \frac{a+b}{3}$, *the equilibriums* P_1 *and* P_2 *are asymptotically stable.*

2) *When* $\frac{a+b}{3} < d$, *the equilibriums* P_1 *and* P_2 *are unstables.*

3) *When* $d = \frac{a+b}{3}$, *in each equilibrium* P_1 *and* P_2 *a periodic orbit emerges with period* $T = \frac{2\pi}{\sqrt{ab}}$.

Hence the system (3) *presents Hopf bifurcation at the equilibrium points* P_1 *and* P_2.

3. Hopf Bifurcation Analysis

In this section it will be showed that the Hopf bifurcation of the system (3) is non-degenerated and supercritical. To achieve this it is used the central manifold theory, it can be consulted in [10] Chapter 5.

Theorem 2 can be found in [12] and the proof presented here is in essence the same except that the Lyapunov coefficient is obtained through a different way.

Theorem 2 *If the parameters satisfy* $b > 0$, $a = b$, $d = \frac{a+b}{3}$, *and* $g > 0$, *then the system* (3) *exhibits non-degenerated and supercritical Hopf Bifurcation in the equilibrium points* P_1 *and* P_2.

Proof. Since the conditions in the Theorem 1 are met, the system (3) exhibits Hopf bifurcation in the equilibrium points P_1 and P_2. It will be demonstrated that Hopf bifurcation is supercritical, at verifying that the first Lyapunov coefficient is negative in the equilibrium point P_1. Since the symmetry of the system, the conclusion is the same for the equilibrium point P_2.

The Jacobian matrix A of the system (3) in the point P_1 is given by,

$$A=\begin{pmatrix} -a & a & 0 \\ -\frac{2a}{3} & \frac{2a}{3} & -h \\ gh & gh & -a \end{pmatrix}, \text{ with } h=\sqrt{\frac{2a^2}{3g}}, \tag{5}$$

and corresponding characteristic polynomial

$$P(\lambda)=\lambda^3+\frac{4a}{3}\lambda^2+a^2\lambda+\frac{4a^3}{3}.$$

Let P be with a couple of pure imaginary roots and a not-zero real root denoted by $\lambda_1=\omega i$, $\lambda_2=-\omega i$ and λ_3, respectively, where $\omega>0$. Due to the relationship between the roots and the P coefficients, it follows that

$$\lambda_1+\lambda_2+\lambda_3=-\frac{4a}{3}, \tag{6}$$

$$\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3=a^2, \tag{7}$$

$$\lambda_1\lambda_2\lambda_3=-\frac{4a^3}{3}. \tag{8}$$

By replacing $\lambda_1+\lambda_2=0$ into (6) is obtained the root $\lambda_3=-\frac{4}{3}a$, which is negative since a is positive. Then, by replacing in (8) $\lambda_1=\omega i$, $\lambda_2=-\omega i$ and $\lambda_3=-4a/3$, is obtained $\omega=a$.

Resolving the system $Aq=aiq$, it has that the complex eigenvector is,

$$q=\begin{pmatrix} ah \\ ah+ahi \\ a^2-\frac{1}{3}a^2 i \end{pmatrix}.$$

By resolving the system $A^{\mathrm{T}}p=-aip$, it has the complex eigenvector,

$$p=\begin{pmatrix} \left(\frac{5}{3}a^2-gh^2\right)+\frac{1}{3}a^2 i \\ -a^2+a^2 i \\ ah \end{pmatrix}.$$

In order to eigenvectors p and q satisfy the condition $\langle p,q\rangle=1$, the eigenvector p is rewritten in the form:

$$p=\frac{9}{50ha^3}\begin{pmatrix} \frac{5}{9}a^2+\frac{5}{3}a^2 i \\ -\frac{7}{3}a^2-\frac{1}{3}a^2 i \\ ah+\frac{4}{3}ahi \end{pmatrix}.$$

The equilibrium point $P_1=\left(h,h,\frac{2a}{3}\right)$ is moved to the origin via the transformation $x = h + y_1$, $y = h + y_2$, $z = \frac{2a}{3} + y_3$, thus the system (3) beecomes

$$\dot{Y} = \begin{pmatrix} -a & a & 0 \\ -\frac{2a}{3} & \frac{2a}{3} & -h \\ gh & gh & -a \end{pmatrix} Y + y_1 B Y, \tag{9}$$

where $Y = (y_1, y_2, y_3)^{\mathrm{T}}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & g & 0 \end{pmatrix}$.

Observe that the system (9) has the form $\dot{Y} = AY + F(Y)$, where A is the Jacobian matrix of the system evaluated in the equilibrium point P_1 and the non-linear part $F(Y) = O\left(\|Y\|^2\right)$, is a smooth function with Taylor expansion in a neighbourhood of $Y = 0$ with expression:

$$F(Y) = \frac{1}{2} B(Y,Y) + \frac{1}{6} C(Y,Y,Y) + O\left(\|Y\|^4\right),$$

where $B(Y,Y)$ and $C(Y,Y,Y)$, are multi-linear vector functions of Y.

To the system (9), the multi-linear functions B and C are:

$$B(q,q) = 2ah \begin{pmatrix} 0 \\ -\omega^2 + \frac{1}{3} a\omega i \\ h(ag + g\omega i) \end{pmatrix}, (q,\bar{q}) = 2ah \begin{pmatrix} 0 \\ -\omega^2 \\ ahg \end{pmatrix},$$

$$B(\bar{q},\bar{q}) = 2ah \begin{pmatrix} 0 \\ -\omega^2 - \frac{1}{3} a\omega i \\ h(ag - g\omega i) \end{pmatrix}, (q,q,\bar{q}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

finally, obtaining

$$F(Y) = \begin{pmatrix} 0 \\ -y_1 y_3 \\ g y_1 y_2 \end{pmatrix}.$$

First Lyapunov coefficient ℓ_1 is calculated using the formula taken from [10],

$$\begin{aligned} \ell_1(0) = \frac{1}{2\omega} Re\Big[& \langle p, C(q,q,\bar{q}) \rangle \\ & -2 \left\langle p, B\left(q, A^{-1} B(q,\bar{q})\right) \right\rangle \\ & + \left\langle p, B\left(\bar{q}, (2i\omega I_3 - A)^{-1} B(q,q)\right) \right\rangle \Big], \end{aligned} \tag{10}$$

where $\langle \cdot,\cdot \rangle$ denotes the usual inner product in the field of complex numbers and $Re(\cdot)$ denotes the real part function of a complex number. Some elements of the formula are written below. The inverse matrix of A defined in (5) is

$$A^{-1}=\frac{1}{h}\begin{pmatrix}\frac{-3gh^2+2a^2}{6ahg} & \frac{-a}{2hg} & \frac{1}{2g}\\ \frac{2a^2+3h^2g}{6ahg} & \frac{-a}{2hg} & \frac{1}{2g}\\ \frac{2}{3} & -1 & 0\end{pmatrix}.$$

The matrix $2i\omega I_3-A$ and its inverse are,

$$2i\omega I_3-A=\begin{pmatrix}2i\omega+a & -a & 0\\ \frac{2}{3}a & 2i\omega-\frac{2}{3}a & h\\ -hg & -gh & 2i\omega+a\end{pmatrix},$$

$$\left(2i\omega I_3-A\right)^{-1}=\frac{1}{r}\begin{pmatrix}\frac{-9\omega^2-2a^2+2a\omega i}{6} & \frac{a^2+2a\omega i}{2} & \frac{-ah}{2}\\ \frac{-2a^2-3h^2g-6\omega i}{6} & \frac{\left(a^2-4\omega^2\right)+4a\omega i}{2} & \frac{-h\left(a+2\omega i\right)}{2}\\ -\frac{2}{3}agh+gh\omega i & h\left(ag+g\omega i\right) & -2\omega^2+\frac{1}{3}a\omega i\end{pmatrix},$$

where $r=-2a^3-3\omega^3 i$.

By replacing this elements in (10), result the first Lyapunov coefficient

$$\ell_1(0)=-\frac{142}{65}a^2,$$

which is negative, thus the periodic orbit that arises in the equilibrium point P_1 of the system (3) is stable. Hence in the equilibrium points P_1 and P_2 of the system (3) the supercritical and non-degenerated Hopf bifurcation is presented.

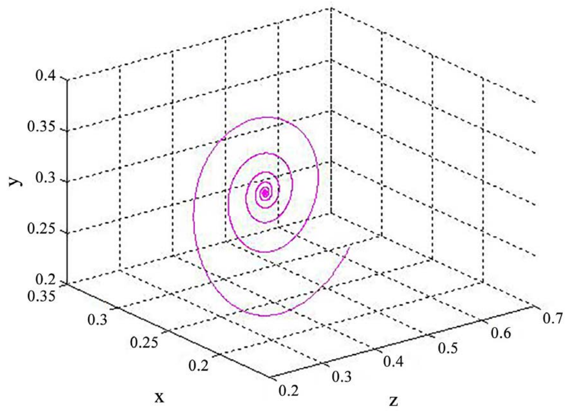
Example 1 *For each case of the Theorem* 1 *a particular system and the graph of the orbit with initial condition* $\left(x_0,y_0,z_0\right)=\left(\frac{5}{4},\frac{5}{4},\frac{89}{100}\right)$ *are presented.*

System with parameters $\left(a,b,d,g\right)=\left(\frac{9}{10},\frac{9}{10},\frac{9}{20},2\right)$ *satisfies the conditions* $a>0$, $b>0$, $g>0$ *and* $d<\frac{a+b}{3}$. *Then by the part* 1 *of the Theorem* 1, *the equilibrium points* P_1 *and* P_2 *are asymptotically stables. The* **Figure 2(a)** *illustrates the behaviour of the orbit around* P_1.

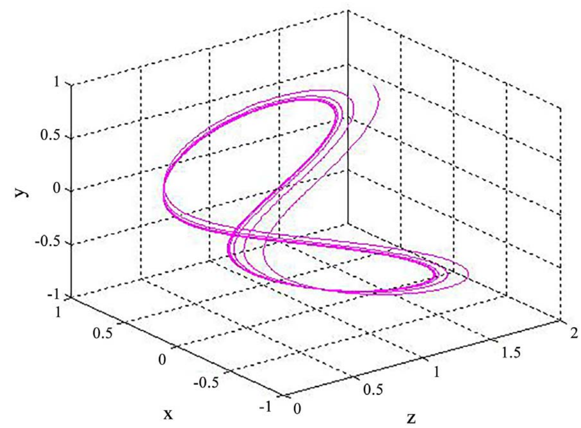
When the parameters are $\left(a,b,d,g\right)=\left(\frac{9}{10},\frac{9}{10},\frac{7}{10},2\right)$, *the conditions* $a>0$, $b>0$, $g>0$ *and* $d>\frac{a+b}{3}$ *are satisfied, and the part* 2 *of the Theorem* 1 *guaranties that the equilibrium points* P_1 *and* P_2 *are unstable, as is illustrated in the* **Figure 2(b)**.

To the parameters $\left(a,b,d,g\right)=\left(\frac{9}{10},\frac{9}{10},\frac{3}{5},2\right)$, *the conditions* $a>0$, $b>0$,

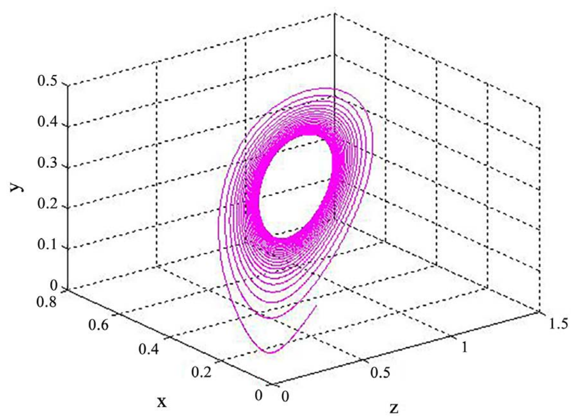
$g > 0$ *and* $d = \frac{a+b}{3}$ *are verified, and the part* 3 *of the Theorem* 1 *guaranties the existence of a periodic orbit. As consequence of the Theorem* 2 *this orbit is stable. The* **Figure 2(c)** *shows a orbit that converge at limit cycle around* P_1.



(a)



(b)



(c)

Figure 2. Dynamics of the system (3) when $(a,b,g) = \left(\frac{9}{10}, \frac{9}{10}, 2\right)$.

(a) $d = \frac{9}{20}$; (b) $d = \frac{7}{10}$; (c) $d = \frac{3}{5}$.

4. Hopf Bifurcation Control

In this section the main results about Hopf bifurcation control in the system (3) are presented, the proofs are postponed to the next section.

In order to resolve the Hopf bifurcation control problem in the system (3), the control laws class in closed loop are used,

$$u(X,d,k)=\left(0,-kx(z-d),0\right)^{\mathrm{T}}, \tag{11}$$

where $X=(x,y,z)^{\mathrm{T}}$ is the states vector, the parameter k is a positive real number called control gain, and d is a parameter of the system (3). The control laws class (11) was used by Pei in [23] to the Hopf bifurcation control in the Lü system.

Perturbed system with the control laws class (11) is the next,

$$\begin{aligned}
\dot{x}&=a(y-x),\\
\dot{y}&=dy-xz-kx(z-d),\\
\dot{z}&=-bz+gxy.
\end{aligned} \tag{12}$$

with parameters $a>0$, $g>0$, $k>0$ and $b,d\in\mathbb{R}$.

Problem consists in finding a control law of the class (11) such that the perturbed system preserves the equilibrium points of the system (3), it presents Hopf bifurcation and this is subcritical. When a control law achieve the planted objectives, it is said that it changes the Hopf bifurcation of the system (3) from supercritical to subcritical.

Observe that the perturb system preserves the equilibrium points P_1, P_2 and P_3 of the system (3), since $u(P_1,d,k)=u(P_2,d,k)=u(P_0,d,k)=0$.

Next theorem states that the perturbed system (12) presents Hopf bifurcation in the equilibrium points P_1 and P_2.

Theorem 3 *To the system* (12) *with parameters* $a>0$, $a=b$, $d>0$, $g>0$ *and* $k>0$, *the critical value of the bifurcation is* $d_H=\frac{-3+\sqrt{8k+9}}{2k}a$ *and the next sentences are true*:

1) *When* $0<d<d_H$, *the equilibrium points* P_1 *and* P_2 *are stable.*
2) *For the case* $d_H<d$, *the equilibrium points* P_1 *and* P_2 *are unstable.*
3) *When* $d=d_H$, *in each equilibrium point* P_1 *and* P_2 *arises a periodic orbit with period* $T=\frac{2\pi}{\omega}$ *where* $\omega=\sqrt{\frac{2a^2d_H(1+k)}{2a-d_H}}$.

Thus, the system presents Hopf bifurcation in the equilibrium points P_1 *and* P_2.

Next theorem states that there are control laws for which the Hopf bifurcation of the perturbed system is subcritical.

Theorem 4 *Let the perturbed system* (12) *be, with parameters* $a>0$, $a=b$, $g>0$ *and* $d=\frac{-3+\sqrt{8k+9}}{2k}a$, *then there is at least a value of k such that the system presents subcritical Hopf bifurcation in the equilibrium points* P_1 *y* P_2.

Next corollary summarizes the obtained results.

Corollary 1 *The control law in feedback of states*

$$\boldsymbol{u} = \left(0, -9x\left(z-d\right), 0\right)^{\mathrm{T}},$$

changes the Hopf bifurcation of the system (3) *from supercritical to subcritical.*

Example 2 *In the* **Figure 3**, *a numerical simulation is presented to illustrate the Theorem* 4 *results. Taking the parameter values* $\left(a,b,d,f,g,k\right) = \left(\frac{3}{5}, \frac{3}{5}, \frac{19}{100}, 0, 3, 10\right)$ *the conditions* $a > 0$, $a = b$, $g > 0$ *and* $d = d_H$ *are met, and the Theorem* 4 *ensures the presence of an unstable periodic orbit around each equilibrium* P_1 *and* P_2. *On the graph, the orbit with initial condition* $\left(x_0, y_0, z_0\right) = \left(\frac{433}{1250}, \frac{433}{1250}, \frac{2}{5}\right)$ *converges to equilibrium point and moves off of cycle, while the orbit with initial condition* $\left(\frac{1501}{2000}, \frac{249}{1250}, \frac{1041}{10000}\right)$, *moves around of each equilibrium point, like a Lorenz attractor.*

5. Proof of Theorems 3 and 4

To proof the Theorems 3 and 4 the following lemmas are used.

Lemma 1 *Let* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *be defined as* $T\left(x,y,z\right) = \left(-x,-y,z\right)$. *The system* (12) *is invariant under T.*

The proof follows from the fact that vectorial field *f* associated to the system (12) satisfies $f\left(T\left(X\right)\right) = T\left(f\left(X\right)\right)$, for all $X \in \mathbb{R}^3$.

A consequence of the previous lemma is that the dynamic system, determined by the perturbed system, is symmetric with respect to the *z*-axis.

Lemma 2 *Polynomial with real coefficients* $P\left(\lambda\right) = \lambda^3 + l_1\lambda^2 + l_2\lambda + l_3$ *has a couple of conjugate imaginary pure roots, if and only if,* $l_3 = l_1 l_2$ *with* $l_2 > 0$.

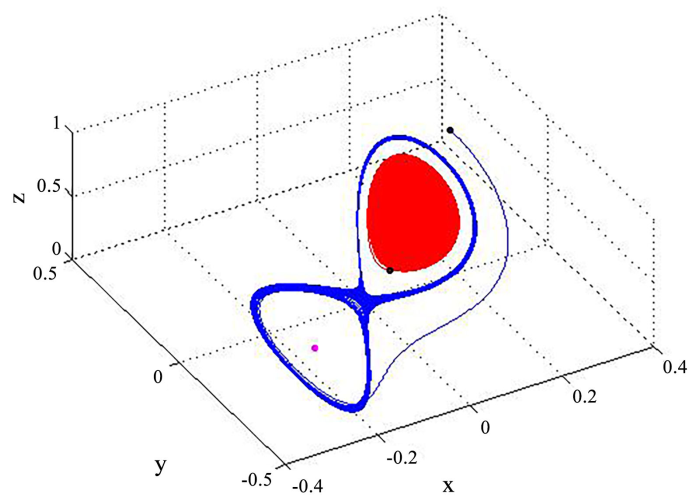


Figure 3. Unstable limit cycle for $\left(a,b,d,f,g,k\right) = \left(\frac{3}{5}, \frac{3}{5}, \frac{19}{100}, 0, 3, 10\right)$.

Lemma 3 *Let* $P(\lambda)=\lambda^3+l_1(\alpha)\lambda^2+l_2(\alpha)\lambda+l_3(\alpha)$ *be a polynomial which coefficients are differentiable real functions with respect to the parameter* α *in the neighbourhood* $V(\alpha_0;\varepsilon)$. *Suppose that for* $\alpha=\alpha_0$, P *has a couple of conjugate roots* $\lambda_1(\alpha_0)=\mu(\alpha_0)+i\beta(\alpha_0)$, $\lambda_2(\alpha_0)=\mu(\alpha_0)-i\beta(\alpha_0)$ *with* $\mu(\alpha_0)=0$ *and* $\beta(\alpha_0)=\omega>0$, *and a negative real root* $\lambda_3(\alpha_0)$. *Then, for all* α *in a neighbourhood of* α_0, *the polynomial* P *has two conjugate roots* $\lambda_1(\alpha)=\mu(\alpha)+i\beta(\alpha)$, $\lambda_2(\alpha)=\mu(\alpha)-i\beta(\alpha)$ *with* $\beta(\alpha)>0$ *and a negative real root* $\lambda_3(\alpha)$, *and the derivative of the real part of the* λ_1 *in* α_0 *is:*

$$\left.\frac{D(Re(\lambda_1))}{D\alpha}\right|_{\alpha=\alpha_0}=\left.\frac{l_3'-l_2'l_1-l_2l_1'}{2l_1^2+2l_2}\right|_{\alpha=\alpha_0}. \tag{13}$$

Proof of Theorem 3. Under the established conditions in the parameters the analysis of stability of the system in the equilibrium point P_1 is done. As the system is symmetric, the conclusion is the same to the equilibrium point P_2.

The Jacobian matrix A of the system (12) in the point P_1 takes the form

$$A=\begin{pmatrix}-a & a & 0\\ -d & d & -(1+k)h\\ gh & gh & -a\end{pmatrix},\ \text{with } h=\sqrt{\frac{ad}{g}}, \tag{14}$$

and characteristic polynomial

$$p(\lambda)=\lambda^3+(2a-d)\lambda^2+(a^2+akd)\lambda+2(1+k)a^2d. \tag{15}$$

With help of the Lemma 2, the necessary and sufficient conditions so that $p(\lambda)$ to have two imaginary pure roots are

1) $(2a-d)(a^2+akd)=2(1+k)a^2d$
2) $a^2+akd>0$

From 1)

$$d_1=\frac{-3+\sqrt{8k+9}}{2k}a \text{ and } d_2=\frac{-3-\sqrt{8k+9}}{2k}a.$$

From 2) and by the hypothesis of the theorem ($a>0$ and $d>0$),

$$d_H=\frac{-3+\sqrt{8k+9}}{2k}a. \tag{16}$$

Case 1. Let $0<d<d_H$ be, that is,

$$2dk+3a<a\sqrt{9+8k}.$$

As the terms of inequality are positive,

$$kd^2+3ad-2a^2<0.$$

Thus $(2a-d)(a^2+akd)>2(1+k)a^2d$. As $k>0$ and $d>0$, it follows that $2(1+k)a^2d>0$, and $(2a-d)(a^2+akd)>0$. Hence, $2a-d>0$.

By Routh-Hurwitz criterion, [27] page 460, a necessary and sufficient condition so that eigenvalues λ_1, λ_2 and λ_3 of the matrix A to have negative real part is that the expressions $2a-d$, $2(1+k)a^2d$ and $(2a-d)(a^2+adk)-2(1+k)a^2d$ are positives, which is proved. Hence, the point P_1 of the perturbed system is stable.

Case 2. Let $d_H < d$ be, that is,

$$a\sqrt{9+8k} < 2kd + 3a.$$

Since the terms of inequality are positive,

$$0 < kd^2 + 3ad - 2a^2.$$

Then $(2a-d)(a^2+akd) - 2(1+k)a^2d < 0$. From the previous inequality and from $2(1+k)a^2d > 0$, by the Routh-Hurwitz criterion there is at least a $j \in \{1,2,3\}$ such that $Re(\lambda_j) \geq 0$. The case $\lambda_j = 0$, implies that $2(1+k)a^2d = 0$, which is not possible. In the case that λ_j is pure imaginary, by the Lemma 2, $d = d_H$, which is not possible, since that $d_H < d$. Hence, when $d_H < d$ there is at least a $j \in \{1,2,3\}$ such that $Re(\lambda_j) > 0$. Therefore the equilibrium point P_1 of the system (12) is unstable.

Case 3. Let λ_1, λ_2 and λ_3 be the roots of $p(\lambda)$. When $d = \dfrac{-3+\sqrt{8k+9}}{2k}a$, it follows that two of these are pure imaginary, of the form $\lambda_1 = \omega i$, $\lambda_2 = -\omega i$ with $\omega > 0$. To find the value of ω, it is observed that the coefficients of $p(\lambda)$ satisfy,

$$\lambda_1 + \lambda_2 + \lambda_3 = -(2a - d_H), \tag{17}$$

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = a^2 + akd_H, \tag{18}$$

$$\lambda_1\lambda_2\lambda_3 = -2(1+k)a^2d_H. \tag{19}$$

By replacing $\lambda_1 + \lambda_2 = 0$ in (17), the real root $\lambda_3 = -(2a - d_H)$ is obtained. By replacing λ_1, λ_2 and λ_3 in (19), $\omega = \sqrt{\dfrac{2a^2d_H(1+k)}{2a-d_H}} > 0$ is obtained. Hence, as $d = d_H$, the polynomial (15) has two conjugate eigenvalues $\lambda_1 = \omega i$, $\lambda_2 = -\omega i$ with $\omega = \sqrt{\dfrac{2a^2d_H(1+k)}{2a-d_H}} > 0$ and $\lambda_3 = -(2a-d_H) < 0$. Since to the Lemma 3 the transversality value is,

$$\left.\frac{DRe(\lambda_1)}{Dd}\right|_{d=d_H} = \frac{2(1+k)a^2 + \omega^2 + 2k(2a-d_H)}{2\left[(2a-d_H)^2 + \omega^2\right]}.$$

Due to $2a - d_H > 0$, $2(1+k)a^2 > 0$ and $\omega > 0$, it follows that the transversality value is positive. And by the Hopf Theorem, in the equilibrium point P_1 of the system (12) arises a periodic orbit with period $T = \dfrac{2\pi}{Im(\lambda_1)}$, where $Im(\lambda_1) = \sqrt{\dfrac{2a^2d_H(1+k)}{2a-d_H}}$.

Proof of Theorem 4. Let the parameters be as in the Theorem 4. According to Theorem 3, the system (12) exhibit Hopf bifurcation in the equilibriums P_1 and P_2. Using the central manifold theory and the same steps that in the proof of Theorem 2, below is shown that the periodic orbit which arises in P_1 is unstable.

When solving the system $(A-\omega iI)q=0$, with A the Jacobian matrix of the system (12) evaluated in P_1, the complex eigenvector

$$q=\begin{pmatrix} ah(1+k) \\ ah(1+k)+\omega(1+k)hi \\ \omega^2+(d_H-a)\omega i \end{pmatrix},$$

is obtained, with $h=\sqrt{\frac{ad_H}{g}}$. When solving the system $A^{\mathrm{T}}p=-i\omega p$, the eigenvector,

$$p=\frac{1}{r}\begin{pmatrix} \omega\left[d_H(2a-d_H)-a(a-d_H)\right]+\left[(a-d_H)\omega^2+a^2(2a-d_H)\right]i \\ -a\omega(3a-d_H)+\left[a\omega^2-a^2(2a-d_H)\right]i \\ ah(1+k)\omega+ah(2a-d_H)(1+k)i \end{pmatrix},$$

is obtained, where $r=2ah(1+k)\left[\omega^3+(2a-d_H)^2\omega\right]$. Also $\langle p,q\rangle=1$ is satisfied.

Equilibrium point P_1 is moved to the origin via the transformation $X=P_1+X$, with $X=(x,y,z)^{\mathrm{T}}$, resulting the system (12) in $X=AX+F(X)$, where A is given by (14), the non-linear part $F(X)=0\left(\|X\|^2\right)$ is a smooth function with Taylor expansion in a neighbourhood of $X=0$,

$$F(X)=\frac{1}{2}B(X,X)+\frac{1}{6}C(X,X,X)+0\left(\|X\|^4\right),$$

where $B(X,X)$ y $C(X,X,X)$ are vectorial multi-linear functions with $X\in\mathbb{R}^3$.

To the system (12) in the origin, the vectorial multi-linear functions B and C are:

$$B(q,q)=2ah(1+k)^2\begin{pmatrix} 0 \\ -\omega^2-(d_H-a)\omega i \\ h(ag+g\omega i) \end{pmatrix},$$

$$B(q,\overline{q})=2ah(1+k)^2\begin{pmatrix} 0 \\ -\omega^2 \\ ahg \end{pmatrix},$$

$$B(\overline{q},\overline{q})=2ah(1+k)^2\begin{pmatrix} 0 \\ -\omega^2+(d_H-a)\omega i \\ h(ag-g\omega i) \end{pmatrix},C(q,q,\overline{q})=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

resulting,

$$F(X)=\begin{pmatrix} 0 \\ -xz-kxz \\ gxy \end{pmatrix}.$$

First Lyapunov coefficient $\ell_1(0)$ is founded using the formula (10), below the elements of this formula are described.

Inverse matrix A^{-1} of the matrix A definite in (14) is,

$$A^{-1}=\frac{1}{h}\begin{pmatrix}\frac{-gh^2(1+k)+ad_H}{2ahg(1+k)} & \frac{-a}{2hg(1+k)} & \frac{1}{2g}\\ \frac{ad_H+h^2g(1+k)}{2ahg(1+k)} & \frac{-a}{2hg(1+k)} & \frac{1}{2g}\\ \frac{d_H}{a(1+k)} & \frac{-1}{1+k} & 0\end{pmatrix}.$$

Matrix $2i\omega I_3-A$ and her inverse are:

$$2i\omega I_3-A=\begin{pmatrix}2i\omega+a & -a & 0\\ d_H & 2i\omega-d_H & h(1+k)\\ -hg & -gh & 2i\omega+a\end{pmatrix},$$

$$(2i\omega I_3-A)^{-1}=\frac{1}{s}\begin{pmatrix}\frac{-3\omega^2-ad_H+2(a-d_H)\omega i}{2} & \frac{a^2+2a\omega i}{2} & \frac{-ah(1+k)}{2}\\ \frac{-\left[ad_H+h^2g(1+k)+2\omega i\right]}{2} & \frac{a^2-4\omega^2+4a\omega i}{2} & \frac{-h(1+k)(a+2\omega i)}{2}\\ h\left[-d_Hg+g\omega i\right] & h\left[ag+g\omega i\right] & \omega\left[-2\omega+(a-d_H)i\right]\end{pmatrix},$$

where $s=-3\left[a^2d_H(1+k)+\omega^3 i\right]$.

Finally, these are replaced in (10), resulting the Lyapunov coefficient

$$\begin{aligned}\ell_1(0)=2\sqrt{2}a^2\big(&-196830-1277208k-3519855k^2-4921830k^3\\&-2091738k^4+5196138k^5+12026872k^6+13293502k^7\\&+9317612k^8+4357822k^9+1338423k^{10}+245120k^{11}\\&+18252k^{12}-808k^{13}+65610t+396576kt+1003509k^2t\\&+1230894k^3t+233486k^4t-1749622k^5t-3243200k^6t\\&-3154458k^7t-1952508k^8t-792330k^9t-202421k^{10}t\\&-27452k^{11}t-796k^{12}t+72k^{13}t\big)\Big/\Big[\big(-2025-6930k-11144k^2\\&-10514k^3-5423k^4-1164k^5-24k^6+675t+2010kt+2888k^2t\\&+2390k^3t+941k^4t+96k^5t\big)\big(-27-63k-56k^2-18k^3\\&+9t+17kt+12k^2t+6k^3t\big)(k-3+t+kt)\sqrt{\frac{(1+k)(-3+t)}{k-3+t+kt}}\Big],\end{aligned} \tag{20}$$

with $t=\sqrt{8k+9}$.

When $k=9$, the expression (20) takes the form,

$$\ell_1(0)=\frac{14175}{58}\sqrt{5}a^2>0.$$

Due to the first Lyapunov coefficient is positive, the periodic orbit is unstable. Hence, in the equilibrium P_1 of the system (12) arises sub-critical non-degenerated

Hopf bifurcation. By Lemma 1, the conclusion is the same to the equilibrium P_2.

6. Conclusions

Via the Theorem 2, the existence of the supercritical and non-degenerate Hopf Bifurcation to the system (3) in the equilibrium points P_1 and P_2 was proved.

It was proved that at perturbing the system (3) in the system (12), the control law $u=\left(0,-9x\left(z-d\right),0\right)^{\mathrm{T}}$ changes the Hopf bifurcation of the system (3) from supercritical to subcritical, in the parameters region $a>0$, $a=b$, $d=\frac{-3+\sqrt{8k+9}}{2k}a$ and $g>0$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Chen, G., Moiola, J. and Wang, H. (2000) Bifurcation Control: Theories, Methods and Applications. *International Journal of Bifurcation and Chaos*, **10**, 511-548. https://doi.org/10.1142/S0218127400000360

[2] Nguyen, L. and Hong, K. (2012) Hopf Bifurcation Control via a Dynamic State-feedback Control. *Physics Letters A*, **376**, 442-446. https://doi.org/10.1016/j.physleta.2011.11.057

[3] Wang, H. and Abed, E.H. (1995) Bifurcation Control of a Chaotic Systems. *Automatica* (*Journal of IFAC*), **31**, 1213-1226. https://doi.org/10.1016/0005-1098(94)00146-A

[4] Chen, G., Yap, K.C. and Lu, J. (1998) Feedback Control of Hopf Bifurcations. In: *Proceedings of the* 1998 *IEEE International Symposium on Circuits and Systems*, Monterey, 31 May-3 June 1998, 639-642.

[5] Chen, G., Moiola, J.L. and Wang, H.O. (1999) Controlling Bifurcations. *PINSA*, **65**, 585-612.

[6] Verduzco, F. and Alvarez, J. (2005) Hopf Bifurcation Control: A New Approach. *Systems* & *Control Letters*, **55**, 437-451. https://doi.org/10.1016/j.sysconle.2005.09.007

[7] Krener, A., Kang, W. and Chang, D. (2004) Control Bifurcations. *IEEE Transactions on Automatic Control*, **49**, 1231-1246. https://doi.org/10.1109/TAC.2004.832199

[8] Xiao, M. and Kang, W. (2000) Feedback Control of Hopf Bifurcations via Integral Averaging Method. *Proceedings of the* 39*th IEEE Conference on Decision and Control*, **3**, 2066-2071.

[9] Chen, D., Wang, H. and Chen, G. (2001) Anti-Control of Hopf Bifurcations. *IEEE Transactions on Circuits and Systems I*: *Fundamental Theory and Applications*, **48**, 661-672. https://doi.org/10.1109/81.928149

[10] Kuznetsov, Y.A. (2004) Elements of Applied Bifurcation Theory. Springer, New York. https://doi.org/10.1007/978-1-4757-3978-7

[11] Li, X. and Ou, Q. (2011) Dynamical Properties and Simulation of a New Lorenzlike Chaotic System. *Nonlinear Dynamical*, **65**, 255-270. https://doi.org/10.1007/s11071-010-9887-z

[12] Calderón-Saavedra, P.E., Muñoz-Aguirre, E. and Alvarez-Mena, J. (2018) Tratamiento Analítico De La Bifurcación De Hopf En Una Extensión Del Sistema De Lü. *Revista de Matem ´atica: Teor ´ıa y Aplicaciones*, **25**, 29-40. https://doi.org/10.15517/rmta.v1i25.32230

[13] Guckenheimer, J. and Holmes, P. (1990) Nonlinear Oscillations Dynamical Systems and Bifurcations of Vectors Fields. Spriger-Verlag, Berlin.

[14] Chen, G. and Lü, J. (2003) Dynamics of the Lorenz Family: Analysis, Control and Synchronization. Chinese Science Press, Beijing.

[15] Lorenz, E.N. (1963) Deterministic Nonperiodic Flow. *Journal of the Atmospheric Sciences*, **20**, 130-141. https://doi.org/10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2

[16] Lorenz, E.N. (1993) The Essence of Chaos. The Jessie and John Danz Lectures. University of Washington Press, Seattle. https://doi.org/10.4324/9780203214589

[17] Algaba, A., Domınguez-Moreno, M.C., Merino, M. and Rodrıguez-Luis, A.J. (2015) Study of the Hopf Bifurcation in the Lorenz, Chen and Lu Systems. *Nonlinear Dynamics*, **79**, 885-902. https://doi.org/10.1007/s11071-014-1709-2

[18] Li, H. and Wang, M. (2013) Hopf Bifurcation Analysis in a Lorenz-Type Systems. *Nonlinear Dynamics*, **71**, 235-240. https://doi.org/10.1007/s11071-012-0655-0

[19] Cruz-Villar, C.A. (2007) Optimal Stabilization of Unstable Periodic Orbits Embedded Chaotic Systems. *Revista Mexicana de Fısica*, **53**, 415-420.

[20] Pyragas, K. (2006) Delayed Feedback Control of Chaos. *Philosophical Transactions of the Royal Society A*, **364**, 2309-2334.

[21] Saito, T., Itoh, S. and Yagi, M. (2004) Control of Chaos by Linear and Nonlinear Feedback Methods. *Journal of Plasma and Fusion Research Series*, **6**, 263-266.

[22] Yu, P. and Chen, G. (2004) Hopf Bifurcation Control Using Nonlinear Feedback with Polynomial Functions. *International Journal of Bifurcation and Chaos*, **14**, 1683-1704. https://doi.org/10.1142/S0218127404010291

[23] Yu, P. and Lu, J. (2011) Bifurcation Control for a Class of Lorenz-Like Systems. *International Journal of Bifurcation and Chaos*, **21**, 2647-2664. https://doi.org/10.1142/S0218127411030003

[24] Abed, E.H. and Fu, J. (1986) Local Feedback Stabilization and Bifurcation Control. I. Hopf Bifurcation. *Systems & Control Letters*, **7**, 11-17. https://doi.org/10.1016/0167-6911(86)90095-2

[25] Fu, J. and Abed, E.H. (1993) Families of Lyapunov Functions for Nonlinear Systems in Critical Cases. *IEEE Transactions on Automatic Control*, **38**, 3-16. https://doi.org/10.1109/9.186308

[26] Vahdati, P.M., Kazemi, A., Amini, M.H. and Vanfretti, L. (2017) Hopf Bifurcation Control of Power System Nonlinear Dynamics via a Dynamic State Feedback Controller—Part I: Theory and Modeling. *IEEE Transactions on Power Systems*, **32**, 3217-3228.

[27] Chen, C.T. (1994) System and Signal Analysis. Oxford University Press, New York, Oxford.