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The Exponential Attractor for a Class of Kirchhoff-Type Equations with Strongly Damped Terms and Source Terms

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Abstract

This paper studies the exponential attractor for a class of the Kirchhoff-type equations with strongly damped terms and source terms. The exponential attractor is also called the inertial fractal set, which is an intermediate step between global attractors and inertial manifolds. Obtaining a set that attracts all the trajectories of the dynamical system at an exponential rate by the methods of Eden A. Under appropriate assumptions, we firstly construct an invariantly compact set. Secondly, showing the solution semigroups of the Kirchhoff-type equations is squeezing and Lipschitz continuous. Finally, the finite fractal dimension of the exponential attractor is obtained.

Keywords

Exponential Attractor, Inertial Fractal Set, Lipschitz Continuous, Squeezing Property

1. Introduction

In this paper, we concerned the equation:

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2 + \|\nabla^m v\|^2\right)\Delta u - \beta\Delta u_t + g_1(u,v) = f_1(x), \\ v_{tt} + M\left(\|\nabla u\|^2 + \|\nabla^m v\|^2\right)(-\Delta)^m v + \beta(-\Delta)^m v_t + g_2(u,v) = f_2(x), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0, \quad \left.\dfrac{\partial_i v}{\partial \mu^i}\right|_{\partial\Omega} = 0\,(i = 1,2,\cdots,m-1), \end{cases} \tag{1}$$

where Ω is a bounded domain in Rn with a smooth boundary $\partial\Omega$, $\beta>1$ is a constant and $f_i(x)(i=1,2)$ is a given out force term. Moreover, $M\left(\|\nabla u\|^2+\|\nabla^m v\|^2\right)$ is a scalar function.

Then the assumptions on M and $g_i(u,v)$ will be specified later.

For an infinitely dynamic system with dissipative properties, studying the asymptotic behavior of its dynamical system is an important issue in mathematical physics. In generally, the asymptotic behavior of the dynamic system is characterized by global attractors, uniform attractors, pull back attractors, and random attractors. The relevant research results on the autonomous system can be found in the literature [1] [2] [3] [4]. The relevant results for non-autonomous and stochastic systems can be found in the literature [5] [6] [7]. However, the attraction rates of these attractors are low and some are even difficult to estimate. In order to overcome these difficulties, people introduced the concept of exponential attractors. The exponential attractor is a positively invariant compact set with finite fractal dimensions and attracts the solution orbit at an exponential rate. It is a tangible concept between the global attractor and the inertial manifold. It can be understood as the intersection of an absorption set and an inertial manifold. In addition, the exponential attractor has a uniform orbital exponential attraction rate, making it more stable to disturbances. Therefore, it is extremely important to study the exponential attractor of an infinite-dimensional dynamical dissipative system.

For the exponential attractor, Eden *et al.* [8] proposed the concept of inertial sets which were compact sets of finite fractal dimension and attracted all the solutions with an exponential rate of convergence as early as in 1990. They showed the long time dynamics of the dissipative evolution equations are characterized by an inertial set. Then, in 1995, A Eden *et al.* [9] first proposed the concept of an exponential attractor (also called inertial sets). In the paper, they presented a new construction of exponential attractors based on the control of Lyapunov exponents over a compact, invariant set. In the same time, they also discussed various applications to Navier-Stokes system. There are more similar references (see [10] [11]).

By the 21st century, the research on the exponential attractors of the dynamical system has been further developed. Firstly, in 2003, Shang Yadong and Guo Boling [12] considered the asymptotic behavior of solutions for the following nonclassical diffusion equation:

$$u_t-v\Delta u_t-\sum\left[\delta\left(u_{x_i}\right)\right]_{x_i}+g(u)=f(x,t). \tag{2}$$

Under appropriate assumptions, they showed the squeezing property and the existence of the exponential attractor for this equation. Meanwhile, they also made the estimates on its fractal dimension.

Secondly, in 2010, Meihua Yang and Chunyou Sun [13] studied the following strongly damped wave equation on a bounded domain $\Omega\subset R^3$ with smooth boundary $\partial\Omega$:

$$\begin{cases} \partial_{tt}u - \Delta\partial_t u - \Delta u + f(u) = g(x), \\ (u(0), \partial_t u(0)) = (u_0, v_0), \\ u|_{\partial\Omega} = 0. \end{cases} \tag{3}$$

They obtained the global attractor and exponential attractor with finite fractal dimension under appropriate conditions. Thereafter, Yang Zhijian and Li Xiao [14] studied the existence of the finite dimension global attractors and exponential attractors for the dynamical system associated with the Kirchhoff type equation with a strong dissipation.

Finally, in 2016, Ruijin Lou, Penghui Lv and Guoguang Lin [15] considered a class of generalized nonlinear Kirchhoff-Sine-Gordon equation as following:

$$u_{tt} - \beta\Delta u_t + \alpha u_t - \phi\left(\|\nabla u\|^2\right)\Delta u + g(\sin u) = f(x). \tag{4}$$

They obtained the exponential attractors and inertial manifolds for above equation. In addition, Yunlong Gao *et al.* also made their own contribution to the research of the exponential attractor (see [16] [17] [18] [19]).

Although the study of exponential attractors has continued to develop, the study of the exponential attractors of the system of equations is not universal. As a result, this has spurred our desire to explore the exponential attractor for a class of the Kirchhoff-type equations with strongly damped terms and source terms. In this paper, our main difficulty is the handling of $M\left(\|\nabla u\|^2 + \|\nabla^m v\|^2\right)$ and nonlinear terms $g_i(u,v)$. But after many attempts, we finally solved this problem.

The paper is arranged as follows. In Section 2, we introduced some notations and basic concepts. In Section 3, we proved the existence of the exponential attractor and estimated the fractal dimension.

2. Preliminaries

For convenience, we need to introduce the following notations:

$H = L^2(\Omega)$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{L^q} = \|\cdot\|_{L^q(\Omega)}$,

$V_0 = H_0^1(\Omega) \times H_0^m(\Omega) \times L^2(\Omega) \times L^2(\Omega)$,

$V_1 = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times \left(H^{2m}(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega) \times H_0^m(\Omega)$,

$C_i\,(i = 1, 2, \cdots)$ are denoted as different positive constants.

Next, we give some assumptions in the proof of our results.

(H$_1$) $0 \le m_0 \le M(s) \le m_1, M(s) \in C^1(\Omega)$,

(H$_2$) $g_i(u,v) \in C^1(\Omega), (i = 1, 2)$.

(H$_3$) $\beta > m_1\lambda_{N+1}^{\frac{m}{2}} + C\lambda_1^{-\frac{m}{2}}, m \ge 1$.

Then, we denote the inner product and norm in V_0 as follows:

$\forall U_i = (u_i, v_i, p_i, q_i) \in V_0, (i = 1, 2)$, we have

$$(U_1, U_2)_{V_0} = (\nabla u_1, \nabla u_2) + (\nabla^m v_1, \nabla^m v_2) + (p_1, p_2) + (q_1, q_2), \tag{5}$$

$$\|U_1\|_{V_0}^2 = \|\nabla u_1\|^2 + \|\nabla^m v_1\|^2 + \|p_1\|^2 + \|q_1\|^2. \tag{6}$$

Setting $\forall U=(u,v,p,q)^{\mathrm{T}}\in V_0, p=u_t+\varepsilon u, q=v_t+\varepsilon v$, then equation (1) can be converted into the following first-order evolution equation

$$U_t+H(U)=F(U), \tag{7}$$

where

$$H(U)=\begin{pmatrix}\varepsilon u-p\\ \varepsilon v-q\\ -\varepsilon p+\beta(-\Delta)p+\varepsilon^2u+(1-\beta\varepsilon)(-\Delta)u\\ -\varepsilon q+\beta(-\Delta)^m q+\varepsilon^2v+(1-\beta\varepsilon)(-\Delta)^m v\end{pmatrix}, \tag{8}$$

$$F(U)=\begin{pmatrix}0\\ 0\\ \left[1-M\left(\|\nabla u\|^2+\|\nabla^m v\|^2\right)\right](-\Delta)u-g_1(u,v)+f_1(x)\\ \left[1-M\left(\|\nabla u\|^2+\|\nabla^m v\|^2\right)\right](-\Delta)^m v-g_2(u,v)+f_2(x)\end{pmatrix}. \tag{9}$$

In order to accomplish the proof, we need to construct a map. Let V_0, V_1 are two Hilbert spaces with $V_1\to V_0$ is dense and continuous injection, and $V_1\to V_0$ is compact. Let $S(t)$ is a solution semigroup generated by Equation (7).

Definition 2.1 ([12]) A compact set $M\subset V$ is called an exponential attractor of $(D(A),V)$ type for $(S(t),B)$ if $A\subseteq M\subseteq B$ and

1) $S(t)M\subseteq M, \forall t\ge 0$,

2) M has finite fractal dimension, $d_F(M)<+\infty$,

3) There exist positive constants C_0, C_1 such that

$$dist_V(S(t)B,M)\le C_0\mathrm{e}^{-C_1 t},\quad \forall t>0, \tag{10}$$

where

$$dist_V(A,B)=\sup_{x\in A}\inf_{y\in B}\|x-y\|_V,$$

B is a positively invariant set for $S(t)$ in V.

Definition 2.2 ([12]) If for every $\delta\in\left(0,\frac{1}{8}\right)$, there exists a time $t^*>0$, an integer $N_0\ge 1$, and an orthogonal projection P_{N_0} of rank equal to N_0 such that for every U and V in B, either

$$\|S(t_*)U-S(t_*)V\|_V\le\delta\|U-V\|_V, \tag{11}$$

or

$$\left\|Q_{N_0}\left(S(t_*)U-S(t_*)V\right)\right\|_V\le\left\|P_{N_0}\left(S(t_*)U-S(t_*)V\right)\right\|_V, \tag{12}$$

then we call $S(t)$ is squeezing in B, where $Q_{N_0}=I-P_{N_0}$.

Theorem 2.1 [20] Assume that

1) $S(t)$ possesses a (V_1,V_0)-compact attractor A,

2) $S(t)$ exists a positive invariant compact set $B\subset V_0$,

3) $S(t)$ is a Lipschitz continuous map with a Lipschitz continuous function

$l(t)$ on *B*, such that $\|S(t)u - S(t)v\|_V \le l(t)\|u-v\|_V$, and satisfied the discrete squeezing property on *B*.

Then $S(t)$ has a (V_1,V_0)-compact exponential attractor M and

$$M = \bigcup_{0\le t\le t_*} S(t)M_*, \tag{13}$$

where

$$M_* = A \cup \left(\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} S(t_*)^j \left(E^{(k)}\right)\right). \tag{14}$$

Moreover, the fractal dimension of M satisfies $d_F(M) \le 1 + cN_0$, where $N_0, E^{(k)}$ are defined as in [20]

Proposition 2.1 [12] There exists $t_0(B_0)$ such that

$$B = \overline{\bigcup_{0\le t\le t_0} S(t)B_0}$$

is the positive invariant set of $S(t)$ in V_0, and B attracts all bounded subsets of V_1, where B_0 is a closed bounded adsorbing set for $S(t)$ in V_1.

Proposition 2.2 Let B_0, B_1 respectively are closed bounded adsorbing set of Equation (7) in V_0, V_1, then $S(t)$ possesses a (V_1,V_0)-compact attractor *A*.

3. The Exponential Attractor

In [21], under of the appropriate hypothesizes, the initial boundary value problem Equation (1) exists unique smooth. This solution possesses the following properties:

$$\|U\|_{V_0}^2 = \|\nabla u\|^2 + \|\nabla^m v\|^2 + \|p\|^2 + \|q\|^2 \le C(R_0), \tag{15}$$

$$\|U\|_{V_1}^2 = \|\Delta u\|^2 + \|\Delta^m v\|^2 + \|\nabla p\|^2 + \|\nabla^m q\|^2 \le C(R_1). \tag{16}$$

We denote the solution in Theorem 2.1 by $S(t)(U_0) = U$, the $S(t)$ is a continuous semigroup in V_0, There exist the balls:

$$B_1 = \left\{U \in V_0 : \|U\|_{V_0}^2 \le C(R_0)\right\}, \tag{17}$$

$$B_2 = \left\{U \in V_1 : \|U\|_{V_1}^2 \le C(R_1)\right\}, \tag{18}$$

respectively is a absorbing set of $S(t)$ in V_0 and V_1.

Lemma 3.1 For $\forall U = (u,v,p,q)^{\mathrm{T}} \in V_0$, when

$$0 < \varepsilon < \min\left\{1, \frac{\lambda_1(3-2\beta)}{2}, \frac{\lambda_1^m(3-2\beta)}{2}, \frac{-5-2\lambda_1\beta + \sqrt{(5+2\lambda_1\beta)^2 + 16\lambda_1\beta}}{4}, \frac{-5-2\lambda_1^m\beta + \sqrt{(5+2\lambda_1^m\beta)^2 + 16\lambda_1^m\beta}}{4}\right\},$$

we can obtain

$$\left(H(U), U\right)_{V_0} \ge k_1\|U\|_{V_0}^2 + k_2\left(\|\nabla p\|^2 + \|\nabla^m q\|^2\right). \tag{19}$$

Proof. By (5), (8) we get

$$
\begin{aligned}
&\left(H(U),U\right)_{V_0}\\
&=\left(\varepsilon\nabla u-\nabla p,\nabla u\right)+\left(-\varepsilon p+\beta(-\Delta)p+\varepsilon^2 u+(1-\beta\varepsilon)(-\Delta)u,p\right)\\
&\quad+\left(\varepsilon\nabla^m v-\nabla^m q,\nabla^m v\right)+\left(-\varepsilon q+\beta(-\Delta)^m q+\varepsilon^2 v+(1-\beta\varepsilon)(-\Delta)^m v,q\right)\\
&=\varepsilon\|\nabla u\|^2-\varepsilon\|p\|^2+\beta\|\nabla p\|^2+\varepsilon^2(u,p)-\beta\varepsilon(\nabla u,\nabla p)\\
&\quad+\varepsilon\left\|\nabla^m v\right\|^2-\varepsilon\|q\|^2+\beta\left\|\nabla^m q\right\|^2+\varepsilon^2(v,q)-\beta\varepsilon\left(\nabla^m v,\nabla^m q\right).
\end{aligned}
\tag{20}
$$

By employing holder's inequality, Young's inequality and Poincare inequality, we process the terms in (20), we have

$$\varepsilon^2(u,p)\ge-\frac{\varepsilon^2}{2}\|u\|^2-\frac{\varepsilon^2}{2}\|p\|^2\ge-\frac{\varepsilon^2}{2\lambda_1}\|\nabla u\|^2-\frac{\varepsilon^2}{2}\|p\|^2. \tag{21}$$

$$\varepsilon^2(v,q)\ge-\frac{\varepsilon^2}{2}\|v\|^2-\frac{\varepsilon^2}{2}\|q\|^2\ge-\frac{\varepsilon^2}{2\lambda_1^m}\left\|\nabla^m v\right\|^2-\frac{\varepsilon^2}{2}\|q\|^2. \tag{22}$$

$$-\beta\varepsilon(\nabla u,\nabla p)\ge-\frac{\beta\varepsilon}{2}\|\nabla u\|^2-\frac{\beta\varepsilon}{2}\|\nabla p\|^2. \tag{23}$$

$$-\beta\varepsilon\left(\nabla^m v,\nabla^m q\right)\ge-\frac{\beta\varepsilon}{2}\left\|\nabla^m v\right\|^2-\frac{\beta\varepsilon}{2}\left\|\nabla^m q\right\|^2. \tag{24}$$

By the value of ε, and substituting (21)-(24), we have

$$
\begin{aligned}
\left(H(U),U\right)_{V_0}&\ge\left(\varepsilon-\frac{\beta\varepsilon}{2}-\frac{\varepsilon^2}{2\lambda_1}\right)\|\nabla u\|^2+\left(\frac{\beta}{2}-\frac{\beta\varepsilon}{2}\right)\|\nabla p\|^2+\left(-\frac{\varepsilon^2}{2}-\varepsilon\right)\|p\|^2\\
&\quad+\frac{\beta}{2}\|\nabla p\|^2+\left(\varepsilon-\frac{\beta\varepsilon}{2}-\frac{\varepsilon^2}{2\lambda_1^m}\right)\left\|\nabla^m v\right\|^2+\left(\frac{\beta}{2}-\frac{\beta\varepsilon}{2}\right)\left\|\nabla^m q\right\|^2\\
&\quad+\left(-\frac{\varepsilon^2}{2}-\varepsilon\right)\|q\|^2+\frac{\beta}{2}\left\|\nabla^m q\right\|^2\\
&\ge\frac{\varepsilon}{4}\left(\|\nabla u\|^2+\left\|\nabla^m v\right\|^2+\|p\|^2+\|q\|^2\right)+\frac{\beta}{2}\left(\|\nabla p\|^2+\left\|\nabla^m q\right\|^2\right)\\
&=\frac{\varepsilon}{4}\|U\|_{V_0}^2+\frac{\beta}{2}\left(\|\nabla p\|^2+\left\|\nabla^m q\right\|^2\right)=k_1\|U\|_{V_0}^2+k_2\left(\|\nabla p\|^2+\left\|\nabla^m q\right\|^2\right)
\end{aligned}
\tag{25}
$$

where $k_1=\frac{\varepsilon}{4},k_2=\frac{\beta}{2}$.

The proof is completed.

Let $S(t)U_0=U(t)=\left(u(t),v(t),p(t),q(t)\right)^{\mathrm{T}}$ where $p(t)=u_t(t)+\varepsilon u(t)$, $q(t)=v_t(t)+\varepsilon v(t)$, $S(t)V_0=V(t)=\left(\overline{u(t)},\overline{v(t)},\overline{p(t)},\overline{q(t)}\right)^{\mathrm{T}}$,

where $\overline{p(t)}=\overline{u_t(t)}+\varepsilon\overline{u(t)}$, $\overline{q(t)}=\overline{v_t(t)}+\varepsilon\overline{v(t)}$.

Next set $\phi(t)=S(t)U_0-S(t)V_0=U(t)-V(t)=\left(w_1(t),w_2(t),z_1(t),z_2(t)\right)^{\mathrm{T}}$, where $z_1(t)=w_{1t}(t)+\varepsilon w_1(t)$, $z_2(t)=w_{2t}(t)+\varepsilon w_2(t)$, then $\phi(t)$ satisfies:

$$\phi_t(t)+HU-HV+F(U)-F(V)=0, \tag{26}$$

$$\phi(0)=U_0-V_0. \tag{27}$$

In order to certify Equation (1) exists a exponential attractor, we first show the semigroup $S(t)$ of system (1) is Lipschitz continuous on *B*.

Lemma 3.2 For $\forall U_0, V_0 \in B$, where U_0, V_0 is the initial values of problem(1), and $t \ge 0$, we have

$$\left\|S(t)U_0 - S(t)V_0\right\|_{V_0}^2 \le \mathrm{e}^{kt}\left\|U_0 - V_0\right\|_{V_0}^2. \tag{28}$$

Proof. Taking the inner product of the Equation (26) with $\phi(t)$ in V_0, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\phi(t)\right\|_{V_0}^2 + \left(HU - HV, \phi(t)\right)_{V_0} - \left((-\Delta)w_1(t), z_1(t)\right) - \left((-\Delta)^m w_2(t), z_2(t)\right) \\
&+\left(M\left(\|\nabla u\|^2 + \|\nabla^m v\|^2\right)(-\Delta)u - M\left(\|\nabla \bar{u}\|^2 + \|\nabla^m \bar{v}\|^2\right)(-\Delta)\bar{u}, z_1(t)\right) \\
&+\left(M\left(\|\nabla u\|^2 + \|\nabla^m v\|^2\right)(-\Delta)^m v - M\left(\|\nabla \bar{u}\|^2 + \|\nabla^m \bar{v}\|^2\right)(-\Delta)^m \bar{v}, z_2(t)\right) \\
&+\left(g_1(\bar{u},\bar{v}) - g_1(u,v), z_1(t)\right) + \left(g_2(\bar{u},\bar{v}) - g_2(u,v), z_2(t)\right) = 0.
\end{aligned} \tag{29}$$

Next, we deal with the following items one by one.

Similar to Lemma 3.1, we easily obtain

$$\begin{aligned}
\left(HU - HV, \phi(t)\right)_{V_0} &= \left(H(\phi(t)), \phi(t)\right)_{V_0} \\
&\ge k_1\left\|\phi(t)\right\|_{V_0}^2 + k_2\left(\left\|\nabla z_1(t)\right\|^2 + \left\|\nabla^m z_2(t)\right\|^2\right)
\end{aligned} \tag{30}$$

For convenience, let's call $s = \|\nabla u\|^2 + \|\nabla^m v\|^2, \bar{s} = \|\nabla \bar{u}\|^2 + \|\nabla^m \bar{v}\|^2$, then by (H_1) and using the mean value theorem, young's inequality, we have

$$\begin{aligned}
&\left(M(\bar{s})(-\Delta)\bar{u} - M(s)(-\Delta)u, z_1(t)\right) \\
&= \left(M(\bar{s})(-\Delta)\bar{u} - M(\bar{s})(-\Delta)u + M(\bar{s})(-\Delta)u - M(s)(-\Delta)u, z_1(t)\right) \\
&\le \left|\left(M(\bar{s})(-\Delta)w_1(t), z_1(t)\right)\right| + \left|\left(M'(\xi)(\bar{s}-s)(-\Delta)u, z_1(t)\right)\right| \\
&\le \frac{m_1\lambda_1^{-\frac{1}{2}}}{2}\left\|\nabla w_1(t)\right\|^2 + \frac{m_1\lambda_1^{\frac{1}{2}}}{2}\left\|\nabla z_1(t)\right\|^2 \\
&\quad + C_2\left\|M'(\xi)\right\|_\infty\left(\left\|\nabla w_1(t)\right\| + \left\|\nabla^m w_2(t)\right\|\right)\left\|(-\Delta)u\right\|\left\|z_1(t)\right\| \\
&\le \frac{m_1\lambda_1^{-\frac{1}{2}}}{2}\left\|\nabla w_1(t)\right\|^2 + \frac{m_1\lambda_1^{\frac{1}{2}}}{2}\left\|\nabla z_1(t)\right\|^2 + C_3\lambda_1^{-\frac{1}{2}}\left(\left\|\nabla w_1(t)\right\| + \left\|\nabla^m w_2(t)\right\|\right)\left\|\nabla z_1(t)\right\| \\
&\le \frac{(m_1 + C_3)\lambda_1^{-\frac{1}{2}}}{2}\left\|\nabla w_1(t)\right\|^2 + \frac{C_3\lambda_1^{-\frac{1}{2}}}{2}\left\|\nabla^m w_2(t)\right\|^2 + \left(\frac{m_1\lambda_1^{\frac{1}{2}}}{2} + C_3\lambda_1^{-\frac{1}{2}}\right)\left\|\nabla z_1(t)\right\|^2.
\end{aligned} \tag{31}$$

Similar to the above process

$$\begin{aligned}
&\left(M(\bar{s})(-\Delta)^m\bar{v} - M(s)(-\Delta)^m v, z_2(t)\right) \\
&\le \frac{(m_1 + C_4)\lambda_1^{-\frac{m}{2}}}{2}\left\|\nabla^m w_2(t)\right\|^2 + \frac{C_4\lambda_1^{-\frac{m}{2}}}{2}\left\|\nabla w_1(t)\right\|^2 + \left(\frac{m_1\lambda_1^{\frac{m}{2}}}{2} + C_4\lambda_1^{-\frac{m}{2}}\right)\left\|\nabla^m z_2(t)\right\|^2.
\end{aligned} \tag{32}$$

For the last two terms, we apply the mean value theorem, Young's inequality and Poincare inequality, by (H_2), we have

$$\begin{aligned}
&\sum_{i=1}^{2}\left(g_i(u,v)-g_i(\bar{u},\bar{v}),z_i(t)\right)\\
&\le\sum_{i=1}^{2}\left(\|g_{iu}(\varsigma,v)\|_\infty\|w_1(t)\|\|z_i(t)\|+\|g_{iv}(\bar{u},\eta)\|_\infty\|w_2(t)\|\|z_i(t)\|\right)\\
&\le\sum_{i=1}^{2}\left(C_5\lambda_1^{-\frac{1}{2}}\|\nabla w_1(t)\|\|z_i(t)\|+C_6\lambda_1^{-\frac{m}{2}}\|\nabla^m w_2(t)\|\|z_i(t)\|\right) \qquad (33)\\
&\le\sum_{i=1}^{2}\left(C_5\lambda_1^{-\frac{1}{2}}\left(\|\nabla w_1(t)\|^2+\|z_i(t)\|^2\right)+C_6\lambda_1^{-\frac{m}{2}}\left(\|\nabla^m w_2(t)\|^2+\|z_i(t)\|^2\right)\right)\\
&\le C_7\|\phi(t)\|_{V_0}^2,
\end{aligned}$$

where

$$C_7=\max\left\{C_5\lambda_1^{-\frac{1}{2}},C_6\lambda_1^{-\frac{m}{2}},\frac{C_5\lambda_1^{-\frac{1}{2}}+C_6\lambda_1^{-\frac{m}{2}}}{2}\right\}$$

Integrating (30)-(33) into (29), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi(t)\|_{V_0}^2+k_1\|\phi(t)\|_{V_0}^2+\left(k_2-\frac{m_1\lambda_1^{\frac{1}{2}}}{2}-C_3\lambda_1^{-\frac{1}{2}}\right)\|\nabla z_1(t)\|^2$$
$$+\left(k_2-\frac{m_1\lambda_1^{\frac{m}{2}}}{2}-C_4\lambda_1^{-\frac{m}{2}}\right)\|\nabla^m z_2(t)\|^2\le(C_7+C_8)\|\phi(t)\|_{V_0}^2$$

where

$$c_8=\max\left\{\frac{(m_1+C_3+1)\lambda_1^{-\frac{1}{2}}}{2}+\frac{C_4\lambda_1^{-\frac{m}{2}}}{2},\frac{(m_1+C_4+1)\lambda_1^{-\frac{m}{2}}}{2}+\frac{C_3\lambda_1^{-\frac{1}{2}}}{2}\right\}.$$

By (H_1), (H_3) we using Gronwall inequality, we have

$$\|\phi(t)\|_{V_0}^2\le \mathrm{e}^{2(C_7+C_8)t}\|\phi(0)\|_{V_0}^2=\mathrm{e}^{kt}\|\phi(0)\|_{V_0}^2, \qquad (34)$$

where $k=2(C_7+C_8)$, so we have

$$\|S(t)U_0-S(t)V_0\|_{V_0}^2\le \mathrm{e}^{kt}\|U_0-V_0\|_{V_0}^2. \qquad (35)$$

The proved is ended.

Now, we introduce the operator

$$A=-\Delta:D(A)\to H;D(A)=\left\{u,v\in H\,\middle|\,Au,A^m v\in H\right\}.$$

Obviously, A is an unbounded self-adjoin positive operator and A^{-1} is compact. So, there is an orthonormal basis $\{\omega_i\}_{i=1}^{\infty}$ of H consisting of eigenvectors ω_j of A such that $A\omega_j=\lambda_j\omega_j, 0<\lambda_1\le\lambda_2\le\cdots\le\lambda_j\to+\infty$. $\forall N$ denote by $P=P_n:H\to span\{\omega_1,\cdots,\omega_N\}$ the projector, $Q=Q_N=I-P_N$.

As follows, we will need

$$\left\|A^{\frac{1}{2}}u\right\|\geq\lambda_{N+1}^{\frac{1}{2}}\|u\|,\ u\in Q_N H,\quad \left\|A^{\frac{1}{2}}u\right\|=\|\nabla u\|,u\in D\left(A^{\frac{1}{2}}\right),$$

$$\|AQ_N u\|=\|Q_N Au\|\leq\|Au\|,\ u\in D(A),$$

$$\left\|A^{\frac{m}{2}}v\right\|\geq\lambda_{N+1}^{\frac{m}{2}}\|v\|,\ \mathrm{v}\in Q_N H,\quad \left\|A^{\frac{m}{2}}u\right\|=\|\nabla^m v\|,\ v\in D\left(A^{\frac{m}{2}}\right)$$

$$\|A^m Q_N v\|=\|Q_N A^m v\|\leq\|A^m v\|,\ v\in D(A),$$

Lemma 3.3 For $\forall U_0,V_0\in B$, where U_0,V_0 is the initial values of problem (1). Let

$$Q_{n_0}(t)=Q_{n_0}(U(t)-V(t))=Q_{n_0}\phi(t)=\phi_{n_0}(t)$$
$$=\left(w_{n_01}(t),w_{n_02}(t),z_{n_01}(t),z_{n_02}(t)\right)^{\mathrm{T}},$$

then we have

$$\|\phi_{n_0}(t)\|_{V_0}^2\leq\left(\mathrm{e}^{-2k_1t}+\frac{(C_{11}+C_{12})\lambda_{N+1}^{-\frac{1}{2}}}{2k_1}k\mathrm{e}^{kt}\right)\|\phi(0)\|_{V_0}^2. \tag{36}$$

Proof. Applying Q_{n_0} to (26), we have

$$\phi_{n_0t}(t)+Q_{n_0}(HU-HV)+Q_{n_0}(F(U)-F(V))=0. \tag{37}$$

Taking the inner product of (37) with $\phi_{n_0t}(t)$ in V_0, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi_{n_0}(t)\|_{V_0}^2+k_1\|\phi_{n_0}(t)\|_{V_0}^2+k_2\left(\|\nabla z_{n_01}(t)\|^2+\|\nabla^m z_{n_02}(t)\|^2\right)\\&-\left((-\Delta)w_{n_01}(t),z_{n_01}(t)\right)-\left((-\Delta)^m w_{n_02}(t),z_{n_02}(t)\right)\\&+\left(Q_{n_0}\left(M\left(\|\nabla u\|^2+\|\nabla^m v\|^2\right)(-\Delta)u-M\left(\|\nabla\bar{u}\|^2+\|\nabla^m\bar{v}\|^2\right)(-\Delta)\bar{u}\right),z_{n_01}(t)\right)\\&+\left(Q_{n_0}\left(M\left(\|\nabla u\|^2+\|\nabla^m v\|^2\right)(-\Delta)^m v-M\left(\|\nabla\bar{u}\|^2+\|\nabla^m\bar{v}\|^2\right)(-\Delta)^m\bar{v}\right),z_{n_02}(t)\right)\\&+\left(Q_{n_0}\left(g_1(u,v)-g_1(\bar{u},\bar{v})\right),z_{n_01}(t)\right)+\left(Q_{n_0}\left(g_2(u,v)-g_2(\bar{u},\bar{v})\right),z_{n_02}(t)\right)=0.\end{aligned} \tag{38}$$

Next, we deal with the following items one by one.

$$\begin{aligned}&\left(Q_{n_0}\left(M(\bar{s})(-\Delta)\bar{u}-M(s)(-\Delta)u\right),z_{n_01}(t)\right)\\&=\left(M\left(\overline{s_{n_0}}\right)(-\Delta)\overline{u_{n_0}}-M(s_{n_0})(-\Delta)u_{n_0},z_{n_01}(t)\right)\\&\leq\frac{(m_1+C_9)\lambda_{N+1}^{-\frac{1}{2}}}{2}\|\nabla w_{n_01}(t)\|^2+\frac{C_9\lambda_{N+1}^{-\frac{1}{2}}}{2}\|\nabla^m w_{n_02}(t)\|^2\\&\quad+\left(\frac{m_1\lambda_{N+1}^{\frac{1}{2}}}{2}+C_9\lambda_{N+1}^{-\frac{1}{2}}\right)\|\nabla z_{n_01}(t)\|^2.\end{aligned} \tag{39}$$

Similar to the above process

$$\left(Q_{n_0}\left(M(\bar{s})(-\Delta)^m\bar{v}-M(s)(-\Delta)^m v\right),z_{n_02}(t)\right)$$

$$= \left(M\left(\overline{s_{n_0}}\right)(-\Delta)^m \overline{v_{n_0}} - M\left(s_{n_0}\right)(-\Delta)^m v_{n_0}, z_{n_0 2}(t) \right)$$

$$\le \frac{(m_1 + C_{10})\lambda_{N+1}^{-\frac{m}{2}}}{2} \left\| \nabla^m w_{n_0 2}(t) \right\|^2 + \frac{C_{10}\lambda_{N+1}^{-\frac{m}{2}}}{2} \left\| \nabla^m w_{n_0 1}(t) \right\|^2 \tag{40}$$

$$+ \left(\frac{m_1 \lambda_{N+1}^{\frac{m}{2}}}{2} + C_{10}\lambda_{N+1}^{-\frac{m}{2}} \right) \left\| \nabla z_{n_0 1}(t) \right\|^2 .$$

For the last two terms, we apply the mean value theorem, Young's inequality and Poincare inequality, by (H_2), we have

$$\sum_{i=1}^{2} \left(Q_{n_0}\left(g_i(u,v) - g_i(\overline{u},\overline{v}) \right), z_{n_0 i}(t) \right)$$
$$= \sum_{i=1}^{2} \left(g_i\left(u_{n_0}, v_{n_0}\right) - g_i\left(\overline{u_{n_0}}, \overline{v_{n_0}}\right), z_{n_0 i}(t) \right)$$
$$\le \sum_{i=1}^{2} \left(C_5 \lambda_{N+1}^{-\frac{1}{2}} \left\| \nabla w_{n_0 1}(t) \right\| \left\| z_{n_0 i}(t) \right\| + C_6 \lambda_{N+1}^{-\frac{m}{2}} \left\| \nabla^m w_{n_0 2}(t) \right\| \left\| z_{n_0 i}(t) \right\| \right) \tag{41}$$
$$\le \sum_{i=1}^{2} \left(C_5 \lambda_{N+1}^{-\frac{1}{2}} \left(\left\| \nabla w_{n_0 1}(t) \right\|^2 + \left\| z_{n_0 i}(t) \right\|^2 \right) + C_6 \lambda_{N+1}^{-\frac{m}{2}} \left(\left\| \nabla^m w_{n_0 2}(t) \right\|^2 + \left\| z_{n_0 i}(t) \right\|^2 \right) \right)$$
$$\le C_{11} \lambda_{N+1}^{-\frac{1}{2}} \left\| \phi_{n_0}(t) \right\|_{V_0}^2 ,$$

where

$$C_{11} = \max\left\{ C_5, C_6, \frac{C_5 + C_6}{2} \right\}$$

Integrating (39)-(41) into (38), by (H_3) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \left\| \phi_{n_0}(t) \right\|_{V_0}^2 + k_1 \left\| \phi(t) \right\|_{V_0}^2 \le (C_{11} + C_{12}) \lambda_{N+1}^{-\frac{1}{2}} \left\| \phi(t) \right\|_{V_0}^2$$
$$\le (C_{11} + C_{12}) \lambda_{N+1}^{-\frac{1}{2}} \mathrm{e}^{kt} \left\| U_0 - V_0 \right\|_{V_0}^2 = (C_{11} + C_{12}) \lambda_{N+1}^{-\frac{1}{2}} \mathrm{e}^{kt} \left\| \phi(0) \right\|_{V_0}^2 , \tag{42}$$

where

$$C_{12} = \frac{m_1 + C_3 + C_4 + 1}{2}$$

Using Gronwall inequality, we have

$$\left\| \phi_{n_0}(t) \right\|_{V_0}^2 \le \left(\mathrm{e}^{-2k_1 t} + \frac{(C_{11} + C_{12}) \lambda_{N+1}^{-\frac{1}{2}}}{2k_1} k\mathrm{e}^{kt} \right) \left\| \phi(0) \right\|_{V_0}^2 , \tag{43}$$

The proved is ended.

Lemma 3.4 (squeezing property) For $\forall U_0, V_0 \in B$, if

$$\left\| P_{n_0}\left(S(t_*)U_0 - S(t_*)V_0 \right) \right\|_{V_0}^2 \le \left\| \left(I - P_{n_0} \right) \left(S(t_*)U_0 - S(t_*)V_0 \right) \right\|_{V_0}^2 , \tag{44}$$

then we have

$$\left\|S(t_*)U_0 - S(t_*)V_0\right\|_{V_0} \le \frac{1}{8}\left\|U_0 - V_0\right\|_{V_0}. \tag{45}$$

Proof. If $\left\|P_{n_0}\left(S(t_*)U_0 - S(t_*)V_0\right)\right\|_{V_0}^2 \le \left\|\left(I-P_{n_0}\right)\left(S(t_*)U_0 - S(t_*)V_0\right)\right\|_{V_0}^2$, then

$$\begin{aligned}
&\left\|S(t_*)U_0 - S(t_*)V_0\right\|_{V_0}^2 \\
&\le \left\|P_{n_0}\left(S(t_*)U_0 - S(t_*)V_0\right)\right\|_{V_0}^2 + \left\|\left(I-P_{n_0}\right)\left(S(t_*)U_0 - S(t_*)V_0\right)\right\|_{V_0}^2 \\
&\le 2\left\|\left(I-P_{n_0}\right)\left(S(t_*)U_0 - S(t_*)V_0\right)\right\|_{V_0}^2 \\
&\le 2\left(\mathrm{e}^{-2k_1t_*} + \frac{(C_{11}+C_{12})\lambda_{N+1}^{-\frac{1}{2}}}{2k_1}k\mathrm{e}^{kt_*}\right)\left\|U_0 - V_0\right\|_{V_0}^2.
\end{aligned} \tag{46}$$

Let t_* be large enough

$$\mathrm{e}^{-2k_1t_*} \le \frac{1}{256}. \tag{47}$$

Also let n_0 be large enough

$$\frac{(C_{11}+C_{12})\lambda_{N+1}^{-\frac{1}{2}}}{2k_1}k\mathrm{e}^{kt_*} \le \frac{1}{256}. \tag{48}$$

Subsituting (46), (47) into (45), we have

$$\left\|S(t_*)U_0 - S(t_*)V_0\right\|_{V_0} \le \frac{1}{8}\left\|U_0 - V_0\right\|_{V_0}. \tag{49}$$

The prove to complete.

Theorem 3.1 Under the above assumptions, $U_0 \in V_k, k = 1,2, f \in H$. Then the initial boundary value problem (1) the solution semigroup has a (V_1, V_0)-compact exponential attractor M on B,

$$M = \bigcup_{0\le t\le t_*} S(t)\left(A \cup \left(\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} S(t_*)^j\left(E^{(k)}\right)\right)\right),$$

and the fractal dimension is satisfied $d_F(M) \le 1 + cN_0$.

Proof. According to Theorem 2.1, Lemma 3.2, Lemma 3.3, Theorem 3.1 is easily proven.

4. Conclusion

In this paper, we studied the exponential attractor for a class of the Kirchhoff-type equations with strongly damped terms and source terms, and obtained the finite fractal dimension of the exponential attractor. Next, we will study the existence of random attractors for this dynamic system.

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