

On the Norms of r -Hankel Matrices Involving Fibonacci and Lucas Numbers

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Abstract

Let us define $A = H_r = (a_{ij})$ to be $n \times n$ r -Hankel matrix. The entries of matrix A are $F_n = F_{i+j-2}$ or $L_n = F_{i+j-2}$ where F_n and L_n denote the usual Fibonacci and Lucas numbers, respectively. Then, we obtained upper and lower bounds for the spectral norm of matrix A . We compared our bounds with exact value of matrix A 's spectral norm. These kinds of matrices have connections with signal and image processing, time series analysis and many other problems.

Keywords

Euclidean Norm, Spectral Norm, r -Hankel Matrix, Fibonacci Numbers, Lucas Numbers

1. Introduction

“Special matrices” is a widely studied subject in the research area of matrix analysis. Especially, special matrices whose entries are well known number sequences have become a very interesting research subject in recent years and many authors have obtained some good results in this area. Some researches denoted the norms of the special matrices involving famous number sequences. The authors found lower bounds, upper bounds and exact values for the spectral norms of these matrices.

Our aim in this study is to obtain some norm bounds more closure than those in literature to the exact value of matrix A 's spectral norm. Lots of articles which concern estimates for spectral norms of special matrices have been written so far.

Solak has studied the norms of circulant matrices with fibonacci and lucas numbers in [1], Türkmen and Gökbaş have made a similar study by using the r-circulant matrix with pell and pell-lucas numbers in [2], Shen and Cen have made a similar study by using the same special matrix with k-fibonacci and k-lucas numbers in [3], Akbulak and Bozkurt found lower and upper bounds for the spectral norms of toeplitz matrices with classical fibonacci and lucas numbers entries in [4], Shen gave upper and lower bounds for the spectral norms of toeplitz matrices with k-fibonacci and k-lucas numbers entries in [5], Akbulak and Bozkurt have made a similar study by using the hankel matrix with fibonacci and lucas numbers in [6], Gökbaş and Türkmen gave upper and lower bounds for the spectral norms of r-toeplitz matrices involving fibonacci and lucas numbers in [7], Bozkurt and Tam obtained determinants and inverse of circulant matrices with jacobsthal and jacobsthal-lucas numbers in [8].

The Fibonacci and Lucas sequences F_n and L_n are defined by the recurrence relations (Table 1)

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2$$

$$L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2$$

The following sum formulas the Fibonacci and Lucas numbers are well known [9]:

$$\sum_{k=1}^{n-1} F_k^2 = F_n F_{n-1}$$

$$\sum_{k=1}^{n-1} L_k^2 = L_n L_{n-1} - 2$$

$$\sum_{k=1}^n k F_k^2 = \begin{cases} n F_n F_{n+1} - F_n^2 + 1, & \text{if } n \text{ is odd} \\ n F_n F_{n+1} - F_n^2, & \text{otherwise} \end{cases}$$

$$\sum_{k=1}^n k L_k^2 = \begin{cases} n L_n L_{n+1} - L_n^2 - 1, & \text{if } n \text{ is odd} \\ n L_n L_{n+1} - F_n^2 + 4, & \text{otherwise} \end{cases}$$

The Euclidean norm of the matrix A is defined as

$$\|A\|_E = \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2}$$

The singular values of the matrix A is

$$\sigma_i = \sqrt{\lambda_i(A^*A)}$$

where λ_i is an eigenvalues of matrix A^*A and A^* is conjugate transpose of the matrix A . The square roots of the maximum eigenvalues of A^*A are called the spectral norm of A and are induced by $\|A\|_2$.

Table 1. The Fibonacci and Lucas sequence have been given.

n	0	1	2	3	4	5	6	7	8	9	10	11	...
F_n	0	1	1	2	3	5	8	13	21	34	55	89	...
Q_n	2	1	3	4	7	11	18	29	47	76	123	199	...

The following inequality holds,

$$\frac{1}{\sqrt{n}} \|A\|_E \leq \|A\|_2 \leq \|A\|_E$$

Define the maximum row length norm r_1 and the maximum column length norm c_1 of any matrix A by

$$r_1(A) = \max_i \sqrt{\sum_j |a_{ij}|^2} \quad \text{and} \quad c_1(A) = \max_j \sqrt{\sum_i |a_{ij}|^2}$$

respectively. Let A, B and C be $m \times n$ matrices. If $A = B \circ C$ then

$$\|A\|_2 \leq r_1(B) c_1(C)$$

for the matrices $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ the Hadamard Product of these matrices is defined as $A \circ B = (a_{ij} b_{ij})_{m \times n}$ [10].

Let us give some lemmas which we will use in our result.

Lemma 1.1: Be a Hankel matrix whose entries determined by $a_{ij} = F_{i+j-2}$ where F_n stands for n th Fibonacci numbers [6].

$$\|A\|_2 \leq \sqrt{\frac{F_{4n} - 3F_{2n}^2 - 2F_{2n} + 6F_n^2 - 2 + 2[-1]^n}{2n}} \leq \|A\|_2$$

$$\|A\|_2 \leq \sqrt{(F_{2n-2}F_{2n-3} - F_{n-1}F_{n-2} + F_{2n-2}^2)(F_{2n-2}F_{2n-3} - F_{n-1}F_{n-2} + 1)}$$

Lemma 1.2: Be a Hankel matrix whose entries determined by $a_{ij} = F_{i+j-2}$ where F_n stands for n th Fibonacci numbers [11].

$$\|A\|_2 = \begin{cases} \frac{F_{2n-1} - 1 + \sqrt{F_{2n-1}^2 - 2F_{n-1} + 4F_n^2 - 3}}{2}, & \text{if } n \text{ is odd} \\ \frac{F_{2n-1} - 1 + \sqrt{F_{2n-1}^2 - 2F_{n-1} + 4F_n^2 + 1}}{2}, & \text{otherwise} \end{cases}$$

Lemma 1.3: Be a Hankel matrix whose entries determined by $a_{ij} = L_{i+j-2}$ where L_n stands for n th Lucas numbers [6].

$$\|A\|_2 \leq \sqrt{\frac{L_{2n-1}^2 - 2L_{n-1}^2 + 1 + 5[1 - (-1)^n]}{n}} \leq \|A\|_2$$

$$\|A\|_2 \leq \sqrt{(L_{2n-1}L_{2n-2} - L_{n-1}L_{n-2} - L_{2n-2}^2 + 1)(L_{2n-1}L_{2n-2} - L_{n-1}L_{n-2})}$$

Lemma 1.4: Be a Hankel matrix whose entries determined by $a_{ij} = L_{i+j-2}$ where L_n stands for n th Lucas numbers [11].

$$\|A\|_2 = \begin{cases} \frac{L_{2n-1} + 1 + \sqrt{5\left(\frac{L_{2n-2} + L_{2n} - 5}{5}\right)^2 + 4}}{2}, & \text{if } n \text{ is odd} \\ \frac{L_{2n-1} + 1 + \sqrt{5\left(\frac{L_{2n-2} + L_{2n} - 5}{5}\right)^2}}{2}, & \text{otherwise} \end{cases}$$

2. Result

In this section, we define an $n \times n$ r -Hankel matrix $A = H_r = (a_{ij})$ whose entries are $a_{ij} = F_{i+j-2}$ or $a_{ij} = L_{i+j-2}$ where F_n and L_n denote the usual Fibonacci and Lucas numbers, respectively. We obtain some inequalities related to $A = H_r = (a_{ij})$.

Definition 2.5: A matrix $H_r = (h_{ij}) \in M_{n,n}(\mathbb{C})$ is called a r -Hankel matrix if it is of the form

$$H_{ij} = \begin{cases} rh_{i+j-2}, & i + j \leq n \\ h_{i+j-2}, & i + j > n \end{cases}$$

Obviously, the r -Hankel matrix H is determined by parameter r and its first row elements h_0, h_1, \dots, h_{n-1} , thus we denote $H = H_r(h_0, h_1, \dots, h_{n-1})$. For $r = 1$, the matrix H is called a Hankel matrix.

Theorem 2.6: Let $A = H_r = (a_{ij})$ be a Hankel matrix satisfying $a_{ij} = F_{i+j-2}$, where $r \in (\mathbb{C})$.

$$\begin{aligned} 1) \quad & |r| \geq 1, \sqrt{\frac{F_{2n-1}F_{2n-2} + F_{2n-2}^2 - 3F_{n-1}F_{n-2} - 2F_{n-2}^2 + F_nF_{n-1} + [1 - (-1)^n]}{n}} \leq A_2 \\ & \|A\|_2 \leq \sqrt{(|r|^2(n-1) + 1)(F_{2n-1}F_{2n-2} - F_{n-1}F_{n-2})} \\ 2) \quad & |r| < 1, |r| \sqrt{\frac{F_{2n-1}F_{2n-2} + F_{2n-2}^2 - 3F_{n-1}F_{n-2} - 2F_{n-2}^2 + F_nF_{n-1} + [1 - (-1)^n]}{n}} \leq \|A\|_2 \\ & \|A\|_2 \leq \sqrt{n(F_{2n-1}F_{2n-2} - F_{n-1}F_{n-2})} \end{aligned}$$

where $\|\cdot\|_2$ is the spectral norm and F_n denotes the n th Fibonacci number.

Proof. The matrix A is of the form

$$A = \begin{bmatrix} rF_0 & rF_1 & \dots & F_{n-1} \\ rF_1 & & \dots & F_n \\ \vdots & \vdots & \ddots & \vdots \\ F_{n-1} & F_n & \dots & F_{2n-2} \end{bmatrix}$$

Then,

$$\|A\|_F^2 = \sum_{k=0}^{n-2} |r|^2 (k+1) F_k^2 + \sum_{k=n-1}^{2n-2} (2n-1-k) F_k^2.$$

Hence, when $|r| \geq 1$ we obtain

$$\begin{aligned} \|A\|_F^2 &\geq \sum_{k=0}^{n-2} (k+1) F_k^2 + \sum_{k=n-1}^{2n-2} (2n-1-k) F_k^2 \\ &\geq (2n-1) \sum_{k=0}^{n-2} F_k^2 - \sum_{k=1}^{2n-2} k F_k^2 - (2n-1) \sum_{k=0}^{n-2} F_k^2 + 2 \sum_{k=1}^{n-2} k F_k^2 + \sum_{k=0}^{n-2} F_k^2 \\ &\geq F_{2n-1}F_{2n-2} + F_{2n-2}^2 - 3F_{n-1}F_{n-2} - 2F_{n-2}^2 + F_nF_{n-1} + [1 - (-1)^n] \\ \|A\|_2 &\geq \sqrt{\frac{F_{2n-1}F_{2n-2} + F_{2n-2}^2 - 3F_{n-1}F_{n-2} - 2F_{n-2}^2 + F_nF_{n-1} + [1 - (-1)^n]}{n}}. \end{aligned}$$

On the other hand, let the matrices B and C be $n \times n$ matrices satisfying

$$A = B \circ C.$$

$$B = \begin{bmatrix} r & r & \cdots & 1 \\ r & & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} F_0 & F_1 & \cdots & F_{n-1} \\ F_1 & & \cdots & F_n \\ \vdots & \vdots & \ddots & \vdots \\ F_{n-1} & F_n & \cdots & F_{2n-2} \end{bmatrix}$$

Since

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{\sum_{j=0}^{n-1} |b_{nj}|^2} = \sqrt{|r|^2(n-1) + 1}$$

and

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} |c_{in}|^2} = \sqrt{\sum_{i=n-1}^{2n-2} F_i^2} = \sqrt{F_{2n-1}F_{2n-2} - F_{n-1}F_{n-2}},$$

we have

$$\|A\|_2 \leq \sqrt{(|r|^2(n-1) + 1)(F_{2n-1}F_{2n-2} - F_{n-1}F_{n-2})}.$$

when $|r| < 1$ we also obtain

$$\begin{aligned} A_F^2 &\geq |r|^2 \left[\sum_{k=0}^{n-2} (k+1)F_k^2 + \sum_{k=n-1}^{2n-2} (2n-1-k)F_k^2 \right] \\ &\geq |r|^2 \left[(2n-1) \sum_{k=0}^{n-2} F_k^2 - \sum_{k=1}^{2n-2} kF_k^2 - (2n-1) \sum_{k=0}^{n-2} F_k^2 + 2 \sum_{k=1}^{n-2} kF_k^2 + \sum_{k=0}^{n-2} F_k^2 \right] \\ &\geq |r|^2 \left[F_{2n-1}F_{2n-2} + F_{2n-2}^2 - 3F_{n-1}F_{n-2} - 2F_{n-2}^2 + F_nF_{n-1} + [1 - (-1)^n] \right] \\ \|A\|_2 &\geq |r| \sqrt{\frac{F_{2n-1}F_{2n-2} + F_{2n-2}^2 - 3F_{n-1}F_{n-2} - 2F_{n-2}^2 + F_nF_{n-1} + [1 - (-1)^n]}{n}}. \end{aligned}$$

On the other hand, let the matrices B and C be $n \times n$ matrices satisfying $A = B \circ C$.

$$B = \begin{bmatrix} r & r & \cdots & 1 \\ r & & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} F_0 & F_1 & \cdots & F_{n-1} \\ F_1 & & \cdots & F_n \\ \vdots & \vdots & \ddots & \vdots \\ F_{n-1} & F_n & \cdots & F_{2n-2} \end{bmatrix}$$

Since

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{\sum_{j=0}^{n-1} |b_{nj}|^2} = \sqrt{n}$$

and

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} |c_{in}|^2} = \sqrt{\sum_{i=n-1}^{2n-2} F_i^2} = \sqrt{F_{2n-1}F_{2n-2} - F_{n-1}F_{n-2}},$$

we have

$$\|A\|_2 \leq \sqrt{n(F_{2n-1}F_{2n-2} - F_{n-1}F_{n-2})}.$$

Thus, the proof is concluded.

Theorem 2.7: Let $A = H_r = (a_{ij})$ be a Hankel matrix satisfying $a_{ij} = L_{i+j-2}$, where $r \in (\mathbb{C})$.

$$\begin{aligned}
 1) \quad |r| \geq 1, \quad \|A\|_2 &\geq \begin{cases} \sqrt{\frac{L_{2n-1}L_{2n-2} - 2L_{n-1}L_{n-2} + L_{2n-2}^2 - 2L_{n-2}^2 + 1}{n}}, & \text{if } n \text{ is odd} \\ \sqrt{\frac{L_{2n-1}L_{2n-2} - 2L_{n-1}L_{n-2} + L_{2n-2}^2 - 2L_{n-2}^2 + 6}{n}}, & \text{otherwise} \end{cases} \\
 \|A\|_2 &\leq \sqrt{(|r|^2(n-1) + 1)(L_{2n-1}L_{2n-2} - L_{n-1}L_{n-2})} \\
 2) \quad |r| < 1, \quad \|A\|_2 &\geq \begin{cases} |r| \sqrt{\frac{L_{2n-1}L_{2n-2} - 2L_{n-1}L_{n-2} + L_{2n-2}^2 - 2L_{n-2}^2 + 1}{n}}, & \text{if } n \text{ is odd} \\ |r| \sqrt{\frac{L_{2n-1}L_{2n-2} - 2L_{n-1}L_{n-2} + L_{2n-2}^2 - 2L_{n-2}^2 + 6}{n}}, & \text{otherwise} \end{cases} \\
 \|A\|_2 &\leq \sqrt{n(L_{2n-1}L_{2n-2} - L_{n-1}L_{n-2})}
 \end{aligned}$$

where L_n is the spectral norm and L_n denotes the n th Lucas number.

Proof. The matrix A is of the form

$$A = \begin{bmatrix} rF_0 & rF_1 & \cdots & F_{n-1} \\ rF_1 & & \cdots & F_n \\ \vdots & \vdots & \ddots & \vdots \\ F_{n-1} & F_n & \cdots & F_{2n-2} \end{bmatrix}$$

Then,

$$\|A\|_F^2 = \sum_{k=0}^{n-2} |r|^2 (k+1)L_k^2 + \sum_{k=n-1}^{2n-2} (2n-1-k)L_k^2.$$

Hence, when $|r| \geq 1$ we obtain

$$\begin{aligned}
 A_F^2 &\geq \sum_{k=0}^{n-2} (k+1)L_k^2 + \sum_{k=n-1}^{2n-2} (2n-1-k)L_k^2 \\
 &\geq (2n-1) \sum_{k=0}^{n-2} L_k^2 - \sum_{k=n-1}^{2n-2} kL_k^2 - (2n-1) \sum_{k=0}^{n-2} L_k^2 + 2 \sum_{k=1}^{n-2} kL_k^2 + \sum_{k=0}^{n-2} L_k^2 \\
 &\geq \begin{cases} L_{2n-1}L_{2n-2} - 2L_{n-1}L_{n-2} + L_{2n-2}^2 - 2L_{n-2}^2 + 1, & \text{if } n \text{ is odd} \\ L_{2n-1}L_{2n-2} - 2L_{n-1}L_{n-2} + L_{2n-2}^2 - 2L_{n-2}^2 + 6, & \text{otherwise} \end{cases} \\
 \|A\|_2 &\geq \begin{cases} \sqrt{\frac{L_{2n-1}L_{2n-2} - 2L_{n-1}L_{n-2} + L_{2n-2}^2 - 2L_{n-2}^2 + 1}{n}}, & \text{if } n \text{ is odd} \\ \sqrt{\frac{L_{2n-1}L_{2n-2} - 2L_{n-1}L_{n-2} + L_{2n-2}^2 - 2L_{n-2}^2 + 6}{n}}, & \text{otherwise} \end{cases}
 \end{aligned}$$

On the other hand, let the matrices B and C be $n \times n$ matrices satisfying $A = B \circ C$.

$$B = \begin{bmatrix} r & r & \cdots & 1 \\ r & & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} L_0 & L_1 & \cdots & L_{n-1} \\ L_1 & & \cdots & L_n \\ \vdots & \vdots & \ddots & \vdots \\ L_{n-1} & L_n & \cdots & L_{2n-2} \end{bmatrix}$$

Since

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{\sum_{j=0}^{n-1} |b_{nj}|^2} = \sqrt{|r|^2(n-1) + 1}$$

and

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} |c_{in}|^2} = \sqrt{\sum_{i=n-1}^{2n-2} L_i^2} = \sqrt{L_{2n-1}L_{2n-2} - L_{n-1}L_{n-2}},$$

we have

$$\|A\|_2 \leq \sqrt{(|r|^2(n-1)+1)(L_{2n-1}L_{2n-2} - L_{n-1}L_{n-2})}.$$

when $|r| < 1$ we also obtain

$$\begin{aligned} \|A\|_F^2 &\geq |r|^2 \left[\sum_{k=0}^{n-2} (k+1)L_k^2 + \sum_{k=n-1}^{2n-2} (2n-1-k)L_k^2 \right] \\ &\geq |r|^2 \left[(2n-1) \sum_{k=0}^{n-2} L_k^2 - \sum_{k=1}^{2n-2} kL_k^2 - (2n-1) \sum_{k=0}^{n-2} L_k^2 + 2 \sum_{k=1}^{n-2} kL_k^2 + \sum_{k=0}^{n-2} L_k^2 \right] \\ &\geq |r|^2 \begin{cases} L_{2n-1}L_{2n-2} - 2L_{n-1}L_{n-2} + L_{2n-2}^2 - 2L_{n-2}^2 + 1, & \text{if } n \text{ is odd} \\ L_{2n-1}L_{2n-2} - 2L_{n-1}L_{n-2} + L_{2n-2}^2 - 2L_{n-2}^2 + 6, & \text{otherwise} \end{cases} \\ \|A\|_2 &\geq \begin{cases} |r| \sqrt{\frac{L_{2n-1}L_{2n-2} - 2L_{n-1}L_{n-2} + L_{2n-2}^2 - 2L_{n-2}^2 + 1}{n}}, & \text{if } n \text{ is odd} \\ |r| \sqrt{\frac{L_{2n-1}L_{2n-2} - 2L_{n-1}L_{n-2} + L_{2n-2}^2 - 2L_{n-2}^2 + 6}{n}}, & \text{otherwise} \end{cases} \end{aligned}$$

On the other hand, let the matrices B and C be $n \times n$ matrices satisfying $A = B \circ C$.

$$B = \begin{bmatrix} r & r & \cdots & 1 \\ r & & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} L_0 & L_1 & \cdots & L_{n-1} \\ L_1 & & \cdots & L_n \\ \vdots & \vdots & \ddots & \vdots \\ L_{n-1} & L_n & \cdots & L_{2n-2} \end{bmatrix}$$

Since

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{\sum_{j=0}^{n-1} |b_{nj}|^2} = \sqrt{n}$$

and

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} |c_{in}|^2} = \sqrt{\sum_{i=n-1}^{2n-2} L_i^2} = \sqrt{L_{2n-1}L_{2n-2} - L_{n-1}L_{n-2}},$$

we have $\|A\|_2 \leq \sqrt{n(L_{2n-1}L_{2n-2} - L_{n-1}L_{n-2})}$.

Thus, the proof is concluded.

3. Numerical Examples

Example 3.8: Let $A = H_r(F_0, F_1, \dots, F_{n-1})$ be a r -Hankel matrix where $r \in (\mathbb{C})$. It can easily be seen that the values obtained in theorem 2.6. are more closure than those obtained in lemma 1.1. to exact values in from **Table 2**.

Example 3.9: Let $A = H_r(L_0, L_1, \dots, L_{n-1})$ be a r -Hankel matrix where $r \in (\mathbb{C})$. It can easily be seen that the values obtained in theorem 2.7. are more closure than those obtained in lemma 1.3. to exact values in from **Table 3**.

Table 2. Numerical results of $a_{ij} = F_{i+j-2}, r = 1$.

n	Lemma 1.1.	Theorem 2.6.	Lemma 1.2. (exact value)
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Continued

2	$\sqrt{4}$	$\sqrt{4}$	$\frac{1+\sqrt{5}}{2}$
3	$\sqrt{84}$	$\sqrt{42}$	$\frac{4+\sqrt{28}}{2}$
4	$\sqrt{3978}$	$\sqrt{408}$	$\frac{12+\sqrt{180}}{2}$
5	$\sqrt{189744}$	$\sqrt{3540}$	$\frac{33+\sqrt{1185}}{2}$
6	$\sqrt{9057280}$	$\sqrt{29280}$	$\frac{88+\sqrt{7674}}{2}$
\vdots	\vdots	\vdots	\vdots

Table 3. Numerical results of $a_j = L_{i+j-2}, r = 1$.

n	Lemma 1.3.	Theorem 2.7.	Lemma 1.4. (exact value)
2	$\sqrt{20}$	$\sqrt{20}$	$\frac{5+\sqrt{5}}{2}$
3	$\sqrt{1184}$	$\sqrt{222}$	$\frac{12+\sqrt{84}}{2}$
4	$\sqrt{95370}$	$\sqrt{2040}$	$\frac{30+\sqrt{720}}{2}$
5	$\sqrt{4708068}$	$\sqrt{17720}$	$\frac{77+\sqrt{5449}}{2}$
6	$\sqrt{226236800}$	$\sqrt{146400}$	$\frac{200+\sqrt{38720}}{2}$
\vdots	\vdots	\vdots	\vdots

4. Conclusion

In this paper, we firstly define the r -Hankel matrix with entries Fibonacci and Lucas numbers. Then we introduce the Euclidean norm equality and the spectral norm inequalities of these matrices. Furthermore, we compared our finding with the exact value of matrix A 's spectral norm. In the future we shall further develop some kinds of r -Hankel matrix such as right r -Hankel, left r -Hankel and geometric r -Hankel. Also we will observe upper and lower bounds for the spectral norm of these matrices involving famous number sequences.

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