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Quantum Measurements Generating Structures of Numerical Events

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Abstract

Let S be a set of states of a physical system and $p(s)$ the probability of an occurrence of an event when the system is in state $s \in S$. The function p from S to $[0,1]$ is called a numerical event, multidimensional probability or, more precisely, S-probability. If a set of numerical events is ordered by the order of real functions one obtains a partial ordered set P in which the sum and difference of S-probabilities are related to their order within P. According to the structure that arises, this further opens up the opportunity to decide whether one deals with a quantum mechanical situation or a classical one. In this paper we focus on the situation that P is generated by a given set of measurements, *i.e.* S-probabilities, without assuming that these S-probabilities can be complemented by further measurements or are embeddable into Boolean algebras, assumptions that were made in most of the preceding papers. In particular, we study the generation by S-probabilities that can only assume the values 0 and 1, thus dealing with so called concrete logics. We characterize these logics under several suppositions that might occur with measurements and generalize our findings to arbitrary S-probabilities, this way providing a possibility to distinguish between potential classical and quantum situations and the fact that an obtained structure might not be sufficient for an appropriate decision. Moreover, we provide some explanatory examples from physics.

Keywords

Ordered Structures of Numerical Events, Generating Processes, Quantum Logics, Concrete Logics, Examples from Physics

1. Introduction

Let S be a set of states of a physical system and $p(s)$ the probability of the occurrence of an event when the system is in state $s \in S$. (For example, think of the probability that the numerical value of an observable is inside a given set of values.) Next, consider $p(s)$ for all $s \in S$. This way one obtains a function p from S to $[0,1]$. In many cases the set S will be finite so that one obtains an n-tuple of probabilities which can be thought of as an event arising when measuring an observable. In their papers of 1991 and 1993 ([1] and [2]) E. G. Beltrametti and M. J. Maczyński introduced the notion of numerical events or S-probabilities for the functions $p(s)$ for arbitrary S. This notion was taken up later and will be used in this paper.

Given S, a set P of numerical events can be ordered by the order $\le$ of functions, similar to the ordering of events of a classical event space (σ-algebra) by the set-theoretic inclusion $\subseteq$. We will require some of the characteristic features of classical event spaces to belong to the partially ordered set $(P,\le)$.

Definition 1. (cf. [1]) We denote the constant functions $p(s)=0$ and $p(s)=1$ for all $s \in S$ by 0 and 1, write $p \perp q$ if the functions p and q are orthogonal, *i.e.*, $p \le q' := 1-q$, and agree that the symbols +, − and · indicate addition, subtraction and multiplication in $\mathbb{R}$. Then a set P of numerical events (*i.e.* functions from S to $[0,1]$) is called an algebra of S-probabilities (or algebra of numerical events), if it satisfies the following axioms:

A1) $0 \in P$;

A2) if $p \in P$ then $p' \in P$;

A3) if $p,q \in P$ and $p \perp q$ then $p+q \in P$;

A4) if $p,q,r \in P$ and $p \perp q \perp r \perp p$ then $p+q+r \in P$.

If only (A1)-(A3) are fulfilled, then P is called a *generalized field of events*, abbreviated by GFE (cf. [3]).

Further, we point out that (A3) is a special case of (A4). It should also be remarked that under the assumption of (A2), Axiom (A3) is equivalent to the following axiom:

A5) If $p,q \in P$ and $p \le q$ then $q-p \in P$.

Let $'$ also denote the complementation of sets and $\bigcup$ and $\bigcap$ stand for the union and intersection of sets, respectively. Thinking of classical event systems, (A2) can be considered as a translation of $A \subseteq B'$ into $p \le 1-q$, and (A4) is motivated by the fact that pairwise orthogonality of a triple A,B,C of events implies $A \subseteq B' \cap C' = (B \bigcup C)'$, which in terms of functions means $p \le 1-(q+r)$.

An algebra of S-probabilities is an orthomodular poset with a full set of states, and algebras of S-probabilities and orthomodular posets with a full sets of states are in one-to-one correspondence (cf. [4]). Boolean algebras are a special class of algebras of S-probabilities and in general represent the situation that one deals with a classical physical system.

Given a set of measurements represented by S-probabilities, our goal is to find out to which kind of structure the obtained measurements belong or give rise to.

If one deals with an algebra of $\mathcal{S}$-probabilities that is not a Boolean algebra one may assume to be concerned with a quantum mechanical phenomenon. GFEs which are not algebras of $\mathcal{S}$-probabilities can also give a clue to the lack of further information or else, they might also not be appropriate to determine the structure of events underlying an experiment.

Several papers have been published on the subject of assessing the situation whether the obtained measurements give rise to or can be embedded into a Boolean algebra by adding further measurements (cf. [2] [3] and [5] [6] [7] [8] [9]). In this paper we will focus on the question which kind of structure will be generated by already achieved measurements without taking into account the possibility to obtain further relevant data.

We will study GFEs and algebras of $\mathcal{S}$-probabilities generated by given measurements dependent on the structure of these quantities. First, we will assume that there will only be two outcomes to measurements, namely that an $\mathcal{S}$-probability might be either 0 or 1, and then we will generalize some of the obtained results to arbitrary numerical events. As for the structure of the generating sets of $\mathcal{S}$-probabilities we will focus on the situations that the $\mathcal{S}$-probabilities and their complements are pairwise incomparable or else orthogonal or that they form chains. In particular, we will show when under these assumptions the resulting GFE or algebra of $\mathcal{S}$-probabilities is a Boolean algebra, which means that one deals with a classical phenomenon, or when there is not enough information available to decide about classicality or non-classicality. To explain our results we provide some examples of physical experiments.

2. $\mathcal{S}$-Probabilities Which Can Only Assume the Values 0 and 1

To illustrate the structure of GFEs and algebras of $\mathcal{S}$-probabilities we listen here some basic facts concerning GFEs, writing $p \wedge q$ and $p \vee q$ for the infimum and supremum of two $\mathcal{S}$-probabilities p and q, respectively (if these exist).

Proposition 2. (cf. [3]) For a GFE P the following hold:

1) If $p \leq q$ for $p, q \in P$ then $q - p \in P$, and if P is an algebra of $\mathcal{S}$-probabilities then $q - p = q \wedge p'$;

2) P is an algebra of $\mathcal{S}$-probabilities if and only if for all $p, q \in P$ with $p \perp q$ we have $p + q = p \vee q$;

3) if $p \vee q$ exists for all $p, q \in P$ then P is a lattice;

4) If P is an algebra of $\mathcal{S}$-probabilities which is a lattice then P is an orthomodular lattice;

5) A lattice-ordered algebra P of $\mathcal{S}$-probabilities is a Boolean algebra, if and only if $p \vee q \leq p + q$ for all $p, q \in P$.

All forthcoming theorems in this section will be exclusively about $\mathcal{S}$-probabilities which only take on the values 0 and 1.

As one can see immediately, a GFE P of numerical events assuming only the values 0 and 1 is an algebra of $\mathcal{S}$-probabilities. We further point out that such an algebra of $\mathcal{S}$-probabilities can be represented by sets (see below) and is therefore also referred to as a so called concrete logic (cf. [10]). (That all algebras of S-probabilities can also be considered as quantum logics can be concluded from the fact that they are orthomodular posets which do serve as quantum logics.)

Definition 3. Let S be a set and M a set of subsets of S (we will link to GFEs further on). We call M a G-system on S if it satisfies the following axioms:

G1) $\varnothing \in M$;

G2) if $A \in M$ then $A' := S \setminus A \in M$;

G3) if $A, B \in M$ and $A \cap B = \varnothing$ then $A \cup B \in M$.

It should be remarked that in general M is not an algebra of sets. Moreover, (G3) implies the following axiom:

G4) If A, B, C are pairwise disjoint elements of M then $A \cup B \cup C \in M$.

We further observe that under the assumption of (G2), Axiom (G3) is equivalent to the following axiom:

G5) If $A, B \in M$ and $A \subseteq B$ then $B \setminus A \in M$.

Finally, if for $A, B \in M$ we have $A \cup B \in M$ then $A \cup B = A \vee B$, and if $A \cap B \in M$ then $A \cap B = A \wedge B$.

For every set S and every subset A of S let I_A denote the mapping from S to $[0,1]$ defined by

$$I_A(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{otherwise} \end{cases}$$

($s \in S$). The mapping I_A is called the *indicator function* corresponding to A.

The following theorem is immediate.

Theorem 4. Let S be a set. Then the following holds:

1) If a set P of mappings from S to $[0,1]$ is a GFE then $\{p^{-1}(\{1\}) \mid p \in P\}$ is a G-system on S;

2) if M is a G-system on S then the set $\{I_A \mid A \in M\}$ is a GFE;

3) The correspondence described in (i) and (ii) is one-to-one.

Due to Theorem 4 the study of GFEs P of S-probabilities assuming only the values 0 and 1 can be reduced to arguments about G-systems and indicator functions.

For every set S and any set Q of subsets of S let $\langle Q \rangle$ denote the G-system on S generated by Q, *i.e.* the smallest G-system on S including Q or the intersection of all G-systems on S including Q or the smallest set of subsets of S including Q and satisfying (G1)-(G3).

Analogously, for every set S and any set Q of functions from S to $[0,1]$ let likewise $\langle Q \rangle$ denote the GFE generated by Q, *i.e.* the smallest GFE including Q or the intersection of all GFEs including Q or the smallest set of functions from S to $[0,1]$ including Q and satisfying (A1)-(A3).

As for arbitrary algebras P of S-probabilities, if for a $p \in P$ we have $p \le 1/2$ then p, p, p' are pairwise orthogonal and hence $p + p + p' \le 1$, *i.e.* $p \le 0$ which implies $p = 0$. Analogously, from $p \ge 1/2$ we infer that p, p', p' are pairwise orthogonal and hence $p+p'+p'\le 1$, *i.e.* $p\ge 1$ which implies $p=1$. We will call $p\ne 0,1$ *proper* if neither $p\le 1/2$ nor $p\ge 1/2$. The elements 0 and 1 will also be considered as proper elements of *P*.

Let $[Q]$ denote the algebra of *S*-probabilities generated by a set Q of proper numerical events. $[Q]$ need not always exist. However, if the elements of Q only assume the values 0 and 1, then $[Q]=\langle Q\rangle$. Moreover, if $p\in Q$ with $0<p(s)<1$ for all $s\in S$ is proper and $Q\setminus\{p\}$ consists of *S*-probabilities that can only assume the values 0 and 1, then again $[Q]=\langle Q\rangle$. (This holds, because p is not compatible to any element from $\langle Q\setminus\{p\}\rangle$ apart from 0 and 1).

In general, $[Q]\ne\langle Q\rangle$, because, if $p\perp q\perp r\perp p$ for three proper *S*-probabilities p,q,r with $p+q+r\le 1$, it can happen that $p+q+r$ is not proper anymore and therefore does not belong to $[Q]$. E.g., consider the three pairwise orthogonal *S*-probabilities $p=(0,1/8,5/8)$, $q=(1/8,6/8,1/8)$ and $r=(6/8,1/8,1/8)$ the sum of which equals $(7/8,1,7/8)$.

In the following let MO_n for $n\ge 1$ denote the orthomodular lattice with the Hasse diagram (see **Figure 1**).

Proposition 5. Let S be a set of states and $p,q:S\to\{0,1\}$ two numerical events. Then the following holds:

1) If the four elements p,p',q,q' are pairwise incomparable then

$$\langle\{p,q\}\rangle=[\{p,q\}]=\{0,p,p',q,q',1\}\cong\mathrm{MO}_2.$$

2) If two of the four *S*-probabilities p,p',q,q' are comparable then $\langle\{p,q\}\rangle=[\{p,q\}]$ is a Boolean algebra having at most eight elements.

Proof. Put $A:=p^{-1}(\{1\})$ and $B:=q^{-1}(\{1\})$.

1) This is clear.

2) If e.g. A and $A'$are comparable then $A\in\{\varnothing,S\}$ and $\langle\{A,B\}\rangle=\{\varnothing,B,B',S\}$. If $A\subseteq B$ then $\langle\{A,B\}\rangle$ consists of all unions of some of the pairwise disjoint sets $A,B\setminus A,B'$. □

Proposition 5 will be the initial point for generalizations (see Section 3).

Throughout this and the next section of the paper let $n\ge 2$ be an arbitrary integer and put $N:=\{1,\cdots,n\}$.

Theorem 6. If S is a set of states then $|\langle Q\rangle|\le\min\left(2^{2^{|Q|}},2^{|S|}\right)$ for any arbitrary finite set Q of functions from S to $\{0,1\}$.

Proof. Assume $Q=\{p_1,\cdots,p_n\}$ and put $A_i:=p_i^{-1}(\{1\})$ for all $i\in N$. Then

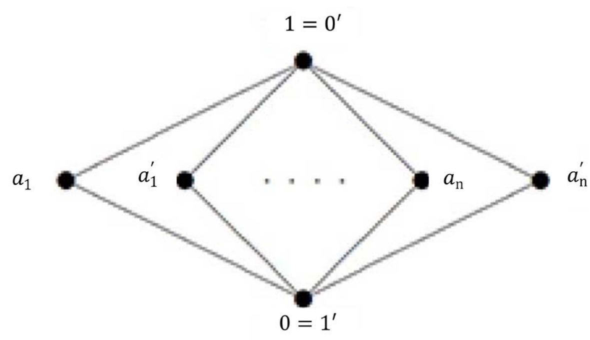

Figure 1. Orthomodular lattice MO_n.

$$\left\{\bigcup_{I\in T}\left(\bigcap_{i\in I}A_i\cap\bigcap_{i\in I'}A_i'\right)\mid T\in 2^{2^N}\right\}$$

is a G-system on S of cardinality $\le 2^{2^n}$ including $\{A_1,\cdots,A_n\}$ since

$$A_j=\bigcup_{j\in I\subseteq N}\left(\bigcap_{i\in I}A_i\cap\bigcap_{i\in I'}A_i'\right)$$

for all $j\in N$. □

Theorem 7. Let S be a set of states and $p_1,\cdots,p_{n-1}:S\to\{0,1\}$ pairwise orthogonal numerical events and put $q_i:=p_i$ for $i=1,\cdots,n-1$ and $q_n:=1-p_1-\cdots-p_{n-1}$. Then

$$\langle\{p_1,\cdots,p_{n-1}\}\rangle=[\{p_1,\cdots,p_{n-1}\}]=\left\{\sum_{i\in I}q_i\mid I\subseteq N\right\}$$

is a Boolean algebra having at most 2^n elements.

$$\left(\sum_{i\in\varnothing}q_i:=0\right)$$

Proof. Put $A_i:=p_i^{-1}(\{1\})$ for $i=1,\cdots,n-1$. Then

$$\langle\{A_1,\cdots,A_{n-1}\}\rangle=[\{A_1,\cdots,A_{n-1}\}]$$

consists of all unions of some of the pairwise disjoint sets

$$A_1,\cdots,A_{n-1},(A_1\cup\cdots\cup A_{n-1})'.$$ □

As a simple numerical example we choose $|S|=4$, $p_1=(0,1,0,0)$ and $p_2=(1,0,0,1)$. Then $q_1=p_1$, $q_2=p_2$, $q_3=(0,0,1,0)$ and we obtain

$$\begin{aligned}\langle\{p_1,p_2\}\rangle=\{&(0,0,0,0),(0,1,0,0),(1,0,0,1),(0,0,1,0),\\&(1,1,0,1),(0,1,1,0),(1,0,1,1),(1,1,1,1)\},\end{aligned}$$

an eight-element Boolean algebra.

Theorem 8. Let S be a set of states and $p_1,\cdots,p_{n-1}:S\to\{0,1\}$ numerical events with $p_1\le\cdots\le p_{n-1}$ and put $p_0:=0$ and $p_n:=1$. Then

$$\langle\{p_1,\cdots,p_{n-1}\}\rangle=[\{p_1,\cdots,p_{n-1}\}]=\left\{\sum_{i\in I}(p_i-p_{i-1})\mid I\subseteq N\right\}$$

is a Boolean algebra having at most 2^n elements.

Proof. $A_i:=p_i^{-1}(\{1\})$ for $i=1,\cdots,n-1$. Then

$$\langle\{A_1,\cdots,A_{n-1}\}\rangle=[\{A_1,\cdots,A_{n-1}\}]$$

consists of all unions of some of the pairwise disjoint sets

$$A_1,A_2\setminus A_1,\cdots,A_{n-1}\setminus A_{n-2},A_{n-1}'.$$ □

Next we take into account that with a given finite set of numerical events also the intersection of some or arbitrary numbers of these events will be numerical events pertaining to an experiment. For ensuring such concurrences of circumstances we can utilize the following feature of GFEs:

If the product of two elements p and q that only take on the values 0 and 1 is an element of the GFE, then $p \wedge q = pq$.

Denoting the power set of a set M by $\mathcal{P}(M)$ we then obtain:

Theorem 9. Let S be a set of states and $p_1, \cdots, p_n : S \to \{0,1\}$ numerical events. Then

$$\left\langle \left\{ \prod_{i \in I} p_i \mid I \subseteq N \right\} \right\rangle = \left[\left\{ \prod_{i \in I} p_i \mid I \subseteq N \right\} \right] = \left\{ \sum_{I \in T} \prod_{i \in I} p_i \prod_{i \in I'} p_i' \mid T \subseteq \mathcal{P}(N) \right\}$$

is a Boolean algebra having at most 2^{2^n} elements.

$$\left(\prod_{i \in \varnothing} p_i := 1 \right)$$

Proof. Put $A_i := p_i^{-1}(\{1\})$ for $i \in N$,

$$G := \left\langle \left\{ \bigcap_{i \in I} A_i \mid I \in 2^N \right\} \right\rangle$$

and for every $I \in 2^N$ put

$$B_I := \bigcap_{i \in I} A_i \cap \bigcap_{i \in I'} A_i'.$$

By induction on k we prove that for $k = n, \cdots, 0$ the following holds: $B_I \in G$ for all $I \in 2^N$ with $|I| = k$. For $k = n$ this is clearly true since $B_N \in G$ according to the assumption. Now let $k \in \{0, \cdots, n-1\}$ and assume $B_I \in G$ for all $I \in 2^N$ with $|I| > k$. Further, let $J \in 2^N$ with $|J| = k$. Then

$$\bigcap_{i \in J} A_i = \bigcup_{K \supseteq J} B_K$$

and hence

$$B_J = \left(\bigcap_{i \in J} A_i \right) \setminus \left(\bigcup_{K \supset J} B_K \right) \in G$$

since $B_K, K \supset J$ are pairwise disjoint sets belonging to G according to the induction hypothesis and since

$$\bigcup_{K \supset J} B_K \subseteq \bigcap_{i \in J} A_i.$$

This shows $B_I \in G$ for all $I \in 2^N$. Because

$$A_j = \bigcup_{j \in I \subseteq N} B_I$$

for all $j \in N$, G consists of all unions of some of the pairwise disjoint sets $B_I, I \in 2^N$ and therefore is a Boolean algebra having at most 2^{2^n} elements. □

The following example shows that the upper bound 2^{2^n} will not always be achieved.

Assume $|S| = 3$ and $p_1 = (1,0,1)$, $p_2 = (1,1,0)$. Then $p_1' = (0,1,0)$, $p_2' = (0,0,1)$ and

$$p_1' p_2' = (0,0,0),$$
$$p_1 p_2' = (0,0,1),$$
$$p_1' p_2 = (0,1,0),$$
$$p_1 p_2 = (1,0,0)$$

We therefore obtain

$$\langle\{p_1'p_2', p_1p_2', p_1'p_2, p_1p_2\}\rangle = \{0,1\}^3,$$

an eight-element Boolean algebra.

3. Generating Algebras of Arbitrary *S*-Probabilities

Theorem 10. Let S be a set of states and $p_1,\cdots,p_n : S \to [0,1]$ numerical events $\neq 0,1$. If $p_1,\cdots,p_n,p_1',\cdots,p_n'$ are pairwise incomparable then

$$\langle\{p_1,\cdots,p_n\}\rangle = [\{p_1,\cdots,p_n\}] = \{0,p_1,\cdots,p_n,p_1',\cdots,p_n',1\} \cong \mathrm{MO}_n.$$

Proof. For $q,r \in \{0,p_1,\cdots,p_n,p_1',\cdots,p_n',1\}$, $q \perp r$ is possible only if $0 \in \{q,r\}$. □

Let us point out that $p < q$ for two S-probabilities p and q should always mean that $p \le q$, and at least for one state $s \in S$ we assume $p(s) < q(s)$.

Theorem 11. Let S be a set of states and $p_1,\cdots,p_{n-1} : S \to [0,1]$ numerical events satisfying $p_1+\cdots+p_{n-1} \le 1$ and $p_1,\cdots,p_{n-1},1-p_1-\cdots-p_{n-1} \not\le 1/2$. Put $p_n := 1-p_1-\cdots-p_{n-1}$. Then $|S| \ge n$ and

$$\langle\{p_1,\cdots,p_{n-1}\}\rangle = [\{p_1,\cdots,p_{n-1}\}] = \left\{\sum_{i\in I} p_i \mid I \subseteq N\right\}$$

is a 2^n-element Boolean algebra.

Proof. For every $i \in N$ there exists some $s_i \in S$ with $p_i(s_i) > 1/2$. If $j,k \in N$, $j \neq k$ and $s_j = s_k$ then $1 < p_j(s_j) + p_k(s_j) \le 1$, a contradiction. This shows that $s_1,\cdots,s_n$ are pairwise distinct proving $|S| \ge n$. Now put

$$q_I := \sum_{i\in I} p_i$$

for all $I \in 2^N$ and let $J,K,L \in 2^N$. If $J \subseteq K$ then, obviously, $q_J \le q_K$. If $J \not\subseteq K$ then there exists some $j \in J \setminus K$. If we had $q_J \perp q_{K'}$ then we would conclude $2p_j \le q_J + q_{K'} \le 1$ contradicting $p_j \not\le 1/2$. Hence, $q_J \not\perp q_{K'}$, *i.e.*, $q_J \not\le q_K$. Therefore, $J \subseteq K$ if and only if $q_J \le q_K$. Since $q_{J'} = (q_J)'$, $q_\varnothing = 0$ and $q_N = 1$ we obtain that the mapping $I \mapsto q_I$ is an isomorphism from the 2^n-element Boolean algebra $(2^N, \subseteq, ', \varnothing, N)$ onto $(A, \le, ', 0, 1)$ where $A := \{q_I \mid I \in 2^N\}$, and hence the latter is a 2^n-element Boolean algebra, too. Now the following are equivalent: $q_J \perp q_K$; $q_J \le (q_K)'$; $q_J \le q_{K'}$; $J \subseteq K'$; $J \cap K = \varnothing$. Hence $q_J \perp q_K$ if and only if $J \cap K = \varnothing$. If, therefore, q_J, q_K, q_L are pairwise orthogonal then J,K,L will be pairwise disjoint and hence $q_J + q_K + q_L = q_{J\cup K\cup L} \in A$. This shows that A is an algebra of S-probabilities. If B is a GFE containing $p_1,\cdots,p_{n-1}$ then $p_1,\cdots,p_n \in B$. Thus $A \subseteq B$ completing the proof of the theorem. □

Remark. If $S = \{s_1,\cdots,s_n\}$ and $p_1,\cdots,p_{n-1} : S \to [0,1]$ are numerical events satisfying $p_1+\cdots+p_{n-1} \le 1$ and $p_1(s_1),\cdots,p_{n-1}(s_{n-1}),(1-p_1-\cdots-p_{n-1})(s_n) > 1/2$ then the assumptions of Theorem 11 are satisfied.

For example, consider a four-element set S,

$$p_1 = (0.6, 0.1, 0.2, 0.2),$$
$$p_2 = (0.1, 0.7, 0.1, 0.1),$$
$$p_3 = (0.2, 0.1, 0.6, 0.1).$$

Then $p_4 = (0.1, 0.1, 0.1, 0.6)$ and we obtain

$$\langle\{p_1, p_2, p_3\}\rangle = \left\{\sum_{i\in I} p_i \mid I \subseteq \{1,2,3,4\}\right\},$$

a 16-element Boolean algebra.

Remark. A special case of Theorem 7, which refers to *S*-probabilities that can only assume the values 0 and 1, can be deduced from Theorem 11 as follows: Let $S = \{s_1, \cdots, s_n\}$ and

$$p_i(s_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

($1 \le i < n$; $j \in N$). Then $p_1, \cdots, p_{n-1}, 1 - p_1 - \cdots - p_{n-1} \not\le 1/2$.

Theorem 12. Let S be a set of states and $p_1, \cdots, p_{n-1} : S \to [0,1]$ numerical events with $0 < p_1 < \cdots < p_{n-1} < 1$. Put $p_0 := 1$ and $p_n := 1$ and assume $p_i - p_{i-1} \not\le 1/2$ for all $i \in N$. Then $|S| \ge n$ and

$$\langle\{p_1, \cdots, p_{n-1}\}\rangle = [\{p_1, \cdots, p_{n-1}\}] = \left\{\sum_{i\in I} (p_i - p_{i-1}) \mid I \subseteq N\right\}$$

is a 2^n-element Boolean algebra.

Proof. Put $q_i := p_i - p_{i-1}$ for all $i \in N$. Then for every $i \in N$ there exists some $s_i \in S$ with $q_i(s_i) > 1/2$. If $j, k \in N$, $j < k$ and $s_j = s_k$ then

$$\frac{1}{2} < q_k(s_k) \le 1 - p_{k-1}(s_j) \le 1 - p_j(s_j) \le 1 - q_j(s_j) < 1 - \frac{1}{2} = \frac{1}{2},$$

a contradiction. This shows that $s_1, \cdots, s_n$ are pairwise distinct proving $|S| \ge n$. Now put

$$r_I := \sum_{i\in I} q_i$$

for all $I \in 2^N$ and let $J, K, L \in 2^N$. If $J \subseteq K$ then, obviously, $r_J \le r_K$. Next assume $J \not\subseteq K$. Then there exists some $j \in J \setminus K$. If we had $r_J \perp r_K'$ then we would conclude $2q_j \le r_J + r_K' \le 1$ contradicting $q_j \not\le 1/2$. Hence, $r_J \not\perp r_K'$, *i.e.*, $r_J \not\le r_K$. Consequently, $J \subseteq K$ if and only if $r_J \le r_K$. Since $r_{J'} = (r_J)'$, $r_\varnothing = 0$ and $r_N = 1$ we obtain that the mapping $I \mapsto r_I$ is an isomorphism from the 2^n-element Boolean algebra $(2^N, \subseteq, ', \varnothing, N)$ onto $(A, \le, ', 0, 1)$ where $A := \{r_I \mid I \in 2^N\}$, and hence the latter is a 2^n-element Boolean algebra, too. Now the following are equivalent: $r_J \perp r_K$; $r_J \le (r_K)'$; $r_J \le r_{K'}$; $J \subseteq K'$; $J \cap K = \varnothing$. Hence $r_J \perp r_K$ if and only if $J \cap K = \varnothing$. If, therefore, r_J, r_K, r_L are pairwise orthogonal then J, K, L are pairwise disjoint and accordingly $r_J + r_K + r_L = r_{J\cup K\cup L} \in A$. This shows that A is an algebra of *S*-probabilities. If B is a GFE containing $p_1, \cdots, p_{n-1}$ then $p_0, \cdots, p_n \in B$ and $q_i = (p_{i-1} + p_i')' \in B$ for all $i \in N$. Thus $A \subseteq B$ completing the proof of the theorem. □

Remark. If $S=\{s_1,\cdots,s_n\}$ and $p_1,\cdots,p_{n-1}:S\to[0,1]$ are numerical events satisfying $0<p_1<\cdots<p_{n-1}<1$ and $(p_i-p_{i-1})(s_i)\nleq 1/2$ for $i\in N$ where $p_0:=0$ and $p_n:=1$ then the assumptions of Theorem 12 are satisfied.

To illustrate this remark, again we choose $|S|=4$ and consider the numerical events

$$p_1=(0.7,0.1,0.1,0.2),$$

$$p_2=(0.8,0.7,0.2,0.3),$$

$$p_3=(0.9,0.8,0.8,0.4).$$

With $p_0:=(0,0,0,0)$, $p_4:=(1,1,1,1)$ we then obtain

$$p_1-p_0=(0.7,0.1,0.1,0.2),$$

$$p_2-p_1=(0.1,0.6,0.1,0.1),$$

$$p_3-p_2=(0.1,0.1,0.6,0.1),$$

$$p_4-p_3=(0.1,0.2,0.2,0.6)$$

and

$$\langle\{p_1,p_2,p_3\}\rangle=\left\{\sum_{i\in I}(p_i-p_{i-1})\,|\,I\subseteq\{1,2,3,4\}\right\},$$

a 16-element Boolean algebra.

Remark. A special case of Theorem 8, which refers to $\mathcal{S}$-probabilities that can only assume the values 0 and 1, can be deduced from Theorem 12 as follows: Let $S=\{s_1,\cdots,s_n\}$ and

$$p_i(s_j)=\begin{cases}1 & \text{if } j\le i\\ 0 & \text{otherwise}\end{cases}$$

($1\le i<n$; $j\in N$). Put $p_0:=0$ and $p_n:=1$. Then

$$(p_i-p_{i-1})(s_j)=\begin{cases}1 & \text{if } j=i\\ 0 & \text{otherwise}\end{cases}$$

($i,j\in N$) and therefore $p_i-p_{i-1}\nleq 1/2$ for all $i\in N$.

Though products of arbitrary numerical events do not suggest themselves for a physical interpretation but only for $\mathcal{S}$-probabilities with values 0 and 1 where they represent the meet of events, we can prove an analogue to Theorem 9 for a wider class of numerical events.

Theorem 13. Let S be a set of states and $p_1,\cdots,p_n:S\to[0,1]$ numerical events and assume

$$\prod_{i\in I}p_i\prod_{i\in I'}p_i'\nleq\frac{1}{2}$$

for all $I\subseteq N$. Then

$$\left\langle\left\{\prod_{i\in I}p_i\,|\,I\subseteq N\right\}\right\rangle=\left[\left\{\prod_{i\in I}p_i\,|\,I\subseteq N\right\}\right]=\left\{\sum_{I\in T}\prod_{i\in I}p_i\prod_{i\in I'}p_i'\,|\,T\subseteq\mathcal{P}(N)\right\}$$

is a 2^{2^n}-element Boolean algebra.

$$\left(\prod_{i\in\varnothing} p_i := 1\right)$$

Proof. Put

$$q_I := \prod_{i\in I} p_i \prod_{i\in I'} p_i'$$

for all $I \in 2^N$ and

$$r_T := \sum_{I\in T} q_I$$

for all $T \in 2^{2^N}$. Let $U,V,W \in 2^{2^N}$. If $U \subseteq V$ then, obviously, $r_U \le r_V$. Assume $U \not\subseteq V$. Then there exists some $J \in U \setminus V$. If we had $r_U \perp r_{V'}$ then we would conclude $2q_J \le r_U + r_{V'} \le 1$ contradicting $q_J \not\le 1/2$. Hence, $r_U \not\perp r_{V'}$, *i.e.*, $r_U \not\le r_V$. Therefore, $U \subseteq V$ if and only if $r_U \le r_V$. Now we have

$$\sum_{I\in 2^N} q_I = \sum_{I\in 2^N} \prod_{i\in I} p_i \prod_{i\in I'} p_i' = \prod_{i\in N}\left(p_i + p_i'\right) = 1.$$

Since $r_{U'} = \left(r_U\right)'$, $r_\varnothing = 0$ and $r_{2^N} = 1$ we obtain that the mapping $T \mapsto r_T$ is an isomorphism from the 2^{2^n}-element Boolean algebra $\left(2^{2^N}, \subseteq, ', \varnothing, 2^N\right)$ onto $\left(A, \le, ', 0, 1\right)$ where $A := \left\{r_T \mid T \in 2^{2^N}\right\}$ and accordingly the latter is a 2^{2^n}-element Boolean algebra, too. Henceforth the following are equivalent: $r_U \perp r_V$; $r_U \le \left(r_V\right)'$; $r_U \le r_{V'}$; $U \subseteq V'$; $U \cap V = \varnothing$. It follows that $r_U \perp r_V$ if and only if $U \cap V = \varnothing$. If, therefore, r_U, r_V, r_W are pairwise orthogonal then U,V,W are pairwise disjoint and consequently $r_U + r_V + r_W = r_{U\cup V\cup W} \in A$. This shows that A is an algebra of S-probabilities. Now let B be a GFE containing

$$\prod_{i\in I} p_i$$

for all $I \in 2^N$. By induction on k we prove that for $k = n,\cdots,0$ the following holds: $q_I \in B$ for all $I \in 2^N$ with $|I| = k$. For $k = n$ this is clearly true since $q_N \in B$ according to the assumption. Let $k \in \{0,\cdots,n-1\}$ and assume $q_I \in G$ for all $I \in 2^N$ with $|I| > k$. Let $J \in 2^N$ with $|J| = k$. Then

$$\sum_{K\supseteq J} q_K = \sum_{K\supseteq J} \prod_{i\in K} p_i \prod_{i\in K'} p_i' = \prod_{i\in J} p_i \prod_{i\in J'}\left(p_i + p_i'\right) = \prod_{i\in J} p_i$$

and hence

$$q_J = \prod_{i\in J} p_i - \sum_{K\supset J} q_K \in B$$

since $q_K, K \supset J$ are pairwise orthogonal elements of B according to the induction hypothesis and since

$$\sum_{K\supset J} q_K \le \prod_{i\in J} p_i.$$

This proves $q_I \in B$ for all $I \in 2^N$ and because $q_I, I \in 2^N$ are pairwise orthogonal we obtain $r_T \in B$ for all $T \in 2^{2^N}$, *i.e.* $A \subseteq B$ completing the proof of the theorem. □

Remark. A special case of Theorem 9, which refers to S-probabilities that can only assume the values 0 and 1, can be deduced from Theorem 13 as follows: Let $S = \{s_I \mid I \subseteq N\}$ and

$$p_i(s_J) = \begin{cases} 1 & \text{if } i \in J \\ 0 & \text{otherwise} \end{cases}$$

($i \in N$; $J \subseteq N$). Then

$$\left(\prod_{i \in I} p_i\right)(s_J) = \begin{cases} 1 & \text{if } I \subseteq J \\ 0 & \text{otherwise} \end{cases}$$

and

$$\left(\prod_{i \in I} p_i \prod_{i \in I'} p_i'\right)(s_J) = \begin{cases} 1 & \text{if } J = I \\ 0 & \text{otherwise} \end{cases}$$

for all $I, J \subseteq N$ and hence

$$\prod_{i \in I} p_i \prod_{i \in I'} p_i' \not\leq \frac{1}{2}$$

for all $I \subseteq N$.

4. Examples from Physics

The first example will illustrate the well known quantum mechanical features adherent to Stern-Gerlach experiments by means of *S*-probabilities.

We consider electrons from an electronic source which have two states, namely “spin up” and “spin down”. Representing the spin as a two-dimensional vector within an orthogonal *x*, *y*-coordinate system one can project the spin-vector to the *x*- and *y*-axis, this way obtaining an *x*- and *y*-component of the spin-vector, which on their parts then have two states, namely “spin up” and “spin down”, in symbols $x\uparrow$, $x\downarrow$ and $y\uparrow$, $y\downarrow$, respectively.

We assume that the electronic beam is transmitted through three consecutive Stern-Gerlach apparatuses A_1, A_2, A_3 each of which splits a component of the beam into two, namely one with “spin up” and one with “spin down” and then blocks a component from further output: A_1 splits the *x*-component into beams with $x\uparrow$ and $x\downarrow$ and then blocks the beam with $x\downarrow$, A_2 splits the *y*-component into $y\uparrow$, $y\downarrow$ and blocks $y\downarrow$, A_3 splits *x*-components again into $x\uparrow$, $x\downarrow$ and blocks out $x\uparrow$. Let us denote the state of the beam between the source and A_1 by s_1, let s_2 refer to the state of the beam between A_1 and A_2, denote the state between A_2 and A_3 by s_3 and after A_3 by s_4 (see **Figure 2**).

Taking measurements of the spin one observes that a second measurement

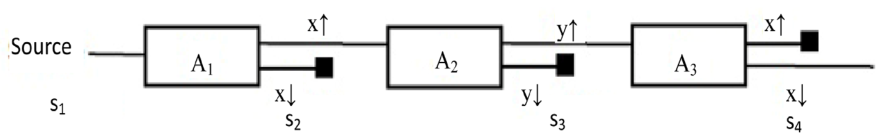

Figure 2. Stern-Gerlach experiment.

always erases the first, which with the experiment at hand means that if a component is blocked by one apparatus it will show up again after having been transmitted through a second apparatus.

Now, let p be the probability that the beam consists of electrons with $x\uparrow$ and $y\downarrow$ and q the probability that this is the case for electrons with $x\downarrow$ and $y\downarrow$. According to our arrangement we then obtain

$$\left(p(s_1),p(s_2),p(s_3),p(s_4)\right)=(1,1,0,0),$$

$$\left(q(s_1),q(s_2),q(s_3),q(s_4)\right)=(1,0,0,1).$$

Since p,q,p',q' are pairwise incomparable the GFE generated by $\{p,q\}$ is isomorphic to MO_2, as one can ascertain by Proposition 5. (The probabilities (1, 1, 1, 1) and (0, 0, 0, 0) can be interpreted as observing something and nothing, respectively.) Having found MO_2 underscores the fact that we are not concerned with a classical mechanical problem and that there is a logical aspect associated with the proceeding of the experiment (due to an underlying quantum logic).

The second example is based on a simple experiment described in [1] and serves to demonstrate the difference between generating GFEs and algebras of $\mathcal{S}$-probabilities:

Consider a coin with faces H (heads) and T (tails) in a box with a window in the upper side such that one can look in. The box is carried from one table (table I)—its state there should be s_1—to a second table (table II), where the box is assumed to be in state s_2. During the process of transporting the box the face of the coin may change. Let p be the relative frequency that the face changes from a given position on table I, let us say it should always be H, to T. If measurements show that due to a certain asymmetry of the coin the relative frequency b that the coin shows T on table II is very high, certainly strictly above 1/2, we will obtain $\left(p(s_1),p(s_2)\right)=(0,b)$, a proper numerical event. Now by Proposition 5 with $q=0$, $\left[\{p\}\right]$ is a Boolean algebra, what we will definitely expect. But next let us change the assumption that $b>1/2$ to $b=1/2$, what cannot actually be measured (only assumed). Then $\left(p(s_1),p(s_2)\right)=(0,1/2)$ is not a proper $\mathcal{S}$-probability anymore, so that we can only consider the GFE $P:=\left\langle\{p\}\right\rangle$ instead of $\left[\{p\}\right]$. As one can easily verify, P is a lattice with six elements and not a Boolean algebra. Whether there is any indication for a quantum process cannot be said, what we can only derive from this result is that more information would be necessary or that the process can't be properly described by the structure of observed numerical events.

As also already discussed in [1], but from a different point of view, we next consider the states s_1, s_2 of a photon linearly polarized along two orthogonal axes (and propagating in a direction orthogonal to the plane E spanned by these axes). Fixing an orthogonal x,y-coordinate system and introducing a new x_α, y_α-coordinate system in E by rotating the x-and y-axes by an angle α one obtains that the transmission probability in the direction of the x_α-coordinate will be $\cos^2(\alpha)=:p_\alpha(s_1)$ and in the direction of the y_α-coordinate

$1-\cos^2(\alpha) =: p_\alpha(s_2)$.

If we pick only one angle $\alpha \neq \pi/4$ for a measurement besides considering the $\mathcal{S}$-probability $(0,0)$ which can be interpreted as observing nothing and $(1,1)$ which can be dealt as seeing everything, this will not render enough information to decide about classicality and non-classicality, because then according to Theorem 10 we would obtain a four-element Boolean algebra, namely MO_1 (contrary to the well known fact that one deals with a quantum situation.) Picking $\alpha = \pi/4$ for an only measurement yields $p_{\pi/4} = (1/2, 1/2)$, hence $p'_{\pi/4} = p_{\pi/4}$ accounting for a GFE which is a three-element chain and no Boolean algebra indeed. But still, with a practical experiment the position of the polarizer cannot be assumed to be π/4 with 100% security so one has to take into account at least two angles α different from 0, π/2 and π/4, which give rise to M_n (or theoretically) M_∞. If one furthermore adds $p_{\pi/4}$ to these sets of numerical events one obtains GFEs which will suggest the non-classicality of the experiment at hand.

5. Conclusions and Suggestions

In 1991 the physicist E. Beltrametti and the mathematician M. Maczyński introduced the notion of a numerical event, *i.e.* $\mathcal{S}$-probability. Their goal was to provide an approach to quantum mechanics devoid of the necessity to know something about the structure of events pertaining to a physical experiment (cf. [1]). (If one first described events and next measured them in various states then one would have to know the logical structure of events from the beginning or assumed axiomatically.)

In the years that followed, properties of algebras of $\mathcal{S}$-probabilities and generalizations of them have been thoroughly studied, mainly from the algebraic point of view (cf. [2] [3] and [5] [6] [7] [8] [9]). In particular, many characterizations were found for an algebra of $\mathcal{S}$-probabilities to be a Boolean algebra or to be embeddable into a Boolean algebra, both cases in which classicality of the system can be assumed. However, for those characterizations the existence of certain $\mathcal{S}$-probabilities has to be secured which in general means that further measurements have to be carried out.

In our approach we refrained from possibly requiring further information by measurements and investigated the structures that arise when generated by the $\mathcal{S}$-probabilities on hand. We studied various relations between the given $\mathcal{S}$-probabilities like incompatibility, orthogonality or a special order. Beginning with numerical events that can only assume the values 0 and 1 and then generalizing our findings to arbitrary $\mathcal{S}$-probabilities we gained some insight into the logical structure induced by a given set of measurements and this way obtained a clue for classicality or non-classicality.

What remains to be investigated is the influence of the cardinality of $\mathcal{S}$, the number of the possible states of a system, the assumption of a more complex relationship between the given $\mathcal{S}$-probabilities, like certain correlations between the multidimensional probabilities or more general kinds of their partial order, and the application of the prospective results to concrete physical problems.

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References

[1] Beltrametti, E.G. and Maczyński, M.J. (1991) On a Characterization of Classical and Nonclassical Probabilities. *Journal of Mathematical Physics*, **32**, 1280-1286. https://doi.org/10.1063/1.529326

[2] Beltrametti, E.G. and Maczyński, M.J. (1993) On the Characterization of Probabilities: A Generalization of Bell's Inequalities. *Journal of Mathematical Physics*, **34**, 4919-4929. https://doi.org/10.1063/1.530333

[3] Dorninger, D. (2012) On the Structure of Generalized Fields of Events. *Contributions to General Algebra*, **20**, 29-34.

[4] Maczyński, M.J. and Traczyk, T. (1973) A Characterization of Orthomodular Partially Ordered Sets Admitting a Full Set of States, Bulletin de l'Académie Polonaise des Sciences. *Série des Sciences Mathématiques*, *Astronomiques et Physiques*, **21**, 3-8.

[5] Dorfer, G., Dorninger, D. and Länger, H. (2010) On Algebras of Multidimensional Probabilities. *Mathematica Slovaca*, **60**, 571-582. https://doi.org/10.2478/s12175-010-0032-8

[6] Dorfer, G., Dorninger, D. and Länger, H. (2010) On the Structure of Numerical Event Spaces. *Kybernetica*, **46**, 971-981.

[7] Dorninger, D. and Länger, H. (2009) On a Characterization of Physical Systems by Spaces of Numerical Events. *ARGESIM Report*, **35**, 601-607.

[8] Dorninger, D. and Länger, H. (2013) Testing for Classicality of a Physical System. *International Journal of Theoretical Physics*, **52**, 1141-1147. https://doi.org/10.1007/s10773-012-1429-7

[9] Dorninger, D. and Länger, H. (2014) Probability Measurements Characterizing the Classicality of a Physical System. *Reports on Mathematical Physics*, **73**, 127-135. https://doi.org/10.1016/S0034-4877(14)60036-7

[10] Pták, P. (2000) Concrete Quantum Logics. *International Journal of Theoretical Physics*, **39**, 827-837. https://doi.org/10.1023/A:1003626929648