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Controllability of a Stochastic Neutral Functional Differential Equation Driven by a fBm

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Abstract

In this paper, we consider a class of Sobolev-type fractional neutral stochastic differential equations driven by fractional Brownian motion with infinite delay in a Hilbert space. When $\alpha > 1 - H$, by the technique of Sadovskii's fixed point theorem, stochastic calculus and the methods adopted directly from deterministic control problems, we study the approximate controllability of the stochastic system.

Keywords

Fractional Stochastic Neutral Functional Differential Equation, Fractional Brownian Motion, Fractional Calculus, Controllability

1. Introduction

As an important part of mathematical control theory, the research on approximate controllability has attracted more and more attention [1] [2] [3]. Approximate controllability means that the system can be steered to a small neighborhood of the final state. In fact, the approximate controllability of systems has been studied by several authors [4] [5]. During the past three decades, the importance of fractional differential equations and their applications are prominent, especially in modeling several complex phenomena such as anomalous diffusion of particles (see, for examples, [6] [7]). In addition, neutral stochastic differential equations with infinite delay have become very useful in the mathematical models of physical and social sciences [8] [9]. So, it is necessarily and significantly to study fractional order neutral differential equations of Sobolev-type ([10] [11] and references therein).

On the other hand, the properties of long/short-range dependence are widely

used in describing many phenomena in fields like hydrology and geophysics as well as economics and telecommunications. As extension of Brownian motion, fractional Brownian motion (fBm) is a self-similar Gaussian process which has the properties of long/short-range dependence. However, fractional Brownian motion is neither a semimartingale nor a Markov process (except for the case $H = \dfrac{1}{2}$ when it is a Brownian motion). For this reason, there are a few publications leaning the systems which are driven by this type of noise. We refer [12] [13] and references therein for the details of the theory of stochastic calculus for fractional Brownian motion. In [14], authors consider the approximate controllability of a class of Sobolev-type fractional stochastic equation driven by fractional Brownian motion in a Hilbert space.

Motivated by these results, in this paper, we study the approximate controllability of the Sobolev-type fractional stochastic differential equations of the form

$$
\begin{cases}
{}^{c}D_t^{\alpha}\left[Lx(t) - G(t, x_t)\right] = Ax(t) + f(t, x_t) + Bu(t) + \sigma(t)\dfrac{\mathrm{d}}{\mathrm{d}t}B^H(t), & t \in (0, T], \\
x(t) = \phi(t), & t \in (-\infty, 0].
\end{cases}
$$
(1.1)

In the above system, we assume that

- ${}^{c}D^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (1 - H, 1)$,
- A, L are two linear operators on a Hilbert space U,
- B is a bounded linear operator from the Hilbert space V into Hilbert space U,
- The time history $x_t(\theta) = x(t + \theta)$, $t > 0$,
- $u(\cdot)$ is a control function on $L^2\left([0, T], V\right)$,
- $B^H = \left\{B^H(t), t \in [0, T]\right\}$ is a cylindrical fractional Brownian motion with Hurst index $H \in \left(\dfrac{1}{2}, 1\right)$,
- The functions G, f and σ are Borel functions with some suitable conditions.

The paper is organized as follows. In Section 2, we represent some preliminaries for stochastic integral of fractional Brownian motion in Hilbert space. In Section 3, we obtain the approximate controllability results of the Sobolev-type fractional neutral stochastic system (1.1).

2. Preliminaries

In this section, we will introduce some definitions, lemmas and notions which will be used in the next section.

2.1. Fractional Brownian Motion

Let $\left(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathrm{P}\right)$ be a complete filtered probability space. A fractional Brownian motion (fBm) $\beta^H = \left\{\beta^H(t), t \in [0, T]\right\}$ with Hurst index $H \in (0, 1)$ is a mean zero Gaussian process such that $\beta^H(0) = 0$ and

$$E\left(\beta^H(t)\beta^H(s)\right)=\frac{1}{2}\left(s^{2H}+t^{2H}-|t-s|^{2H}\right)$$

for all $t,s\geq 0$. When $H=1/2$, β^H coincides with the standard Brownian motion, and when $H\neq\frac{1}{2}$ it is neither a semi-martingale nor a Markov process. The fBm β^H admits the following integral representation:

$$\beta^H(t)=\int_0^t K_H(t,s)\,\mathrm{d}W(t)$$

for all $t\geq 0$, where $\left\{W(t),0\leq t\leq T\right\}$ is a standard Brownian motion and the kernel $K_H(t,s)$ satisfies

$$\frac{\partial K_H}{\partial t}(t,s)=\kappa_H\left(H-\frac{1}{2}\right)\left(\frac{s}{t}\right)^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}$$

with a normalizing constant $\kappa_H>0$ such that $E\left(\beta_1^H\right)^2=1$. Throughout this paper we assume that $\frac{1}{2}\leq H<1$ is arbitrary but fixed.

Let $\mathcal{H}$ be the completion of the linear space $\mathcal{E}$ generated by the indicator functions $1_{[0,t]},t\in[0,T]$ with respect to the inner product

$$\left\langle 1_{[0,s]},1_{[0,t]}\right\rangle_{\mathcal{H}}=\frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right).$$

The mapping

$$\mathcal{E}\ni\varphi\rightarrow\beta^H(\varphi):=\int_0^T\varphi(s)\,\mathrm{d}\beta^H(s)$$

is an isometry from $\mathcal{E}$ to the Gaussian space generated by β^H and it can be extended to $\mathcal{H}$, which is called the Wiener integral with respect to β^H. Consider the operator K_H^* from $\mathcal{E}$ to $L^2([0,T])$ defined by

$$\left(K_H^*\varphi\right)(s)=\int_s^T\varphi(t)\frac{\partial K_H}{\partial t}(t,s)\,\mathrm{d}t$$

for $\varphi\in\mathcal{E}$. Then, the operator K_H^* is an isometry between $\mathcal{E}$ and $L^2([0,T])$ which can be also extended to the Hilbert space $\mathcal{H}$.

Lemma 2.1 *For every $\varphi\in\mathcal{H}$, we have*

$$\int_0^T\varphi(s)\,\mathrm{d}\beta^H(s)=\int_0^T\left(K_H^*\varphi\right)(s)\,\mathrm{d}W(s).$$

We now recall that the definition of stochastic integral of fBm in the Hilbert space V. Let $\left\{B^H(t),0\leq t\leq T\right\}$ be a W-valued $\mathscr{F}_t$-adapted fBm defined on $\left(\Omega,\mathscr{F},\left(\mathscr{F}_t\right),\mathrm{P}\right)$ with the representation of the form

$$B^H(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n^H(t)e_n,\quad t\geq 0,$$

where $\left\{e_n\right\}_{n\in N}$ is a complete orthogonal basis in W, and

- $\left\{\beta_n^H,n=1,2,\cdots\right\}$ is a sequence of independent fBms with the same Hurst index $H\in\left(\frac{1}{2},1\right)$,

- $\{\lambda_n; n \in \mathbb{N}\}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \lambda_n e_n$,
- Q is a non-negative self-adjoint trace class operator with finite trace

$$TrQ = \sum_{n=1}^{\infty} \lambda_n < +\infty.$$

Let $\varphi : [0, T] \to L_2^0(W, U)$ such that

$$\sum_{n=1}^{\infty} \left\| K_H^* \left(\varphi Q^{\frac{1}{2}} e_n \right) \right\|_{L_2^0([0,T];U)} < \infty \tag{2.1}$$

where $L_2^0(W, U)$ is the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}} W$ to U with norm $\|\cdot\|_{L_2^0(W,U)}$ defined by

$$\|\xi\|_{L_2^0(W,U)} = \left\| \xi Q^{\frac{1}{2}} \right\|_{H-S}^2 = Tr\left(\xi Q \xi^* \right) = \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n} \xi e_n \right\|^2.$$

Definition 2.1 *Let* $\varphi : [0, T] \to L_2^0(W, U)$ *satisfy* (2.1). *We define the stochastic integral* $\int_0^t \varphi(s) \mathrm{d}B^H(s)$ *by*

$$\int_0^t \varphi(s) \mathrm{d}B^H(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s) Q^{\frac{1}{2}} e_n \mathrm{d}\beta_n^H = \sum_{n=1}^{\infty} \int_0^t \left(K_H^* \left(\varphi Q^{\frac{1}{2}} e_n \right) \right)(s) \mathrm{d}B(s).$$

Lemma 2.2 *Let* $\varphi : [0, T] \to L_2^0(W, U)$ *satisfy* (2.1). *Then, for any* $a, b \in [0, T]$ *with* $a < b$ *we have*

$$\mathrm{E} \left\| \int_a^b \varphi(s) \mathrm{d}B^H(s) \right\|^2 \le cH(2H-1)(b-a)^{2H-1} \sum_{n=1}^{\infty} \int_a^b \left\| \varphi(s) Q^{\frac{1}{2}} e_n \right\|^2 \mathrm{d}s.$$

In addition, $\sum_{n=1}^{\infty} \left\| \varphi(s) Q^{\frac{1}{2}} e_n \right\|$ *is uniformly convergent in* $t \in [0, T]$, *then, we have*

$$\mathrm{E} \left\| \int_a^b \varphi(s) \mathrm{d}B^H(s) \right\|^2 \le cH(2H-1)(b-a)^{2H-1} \sum_{n=1}^{\infty} \int_a^b \left\| \varphi(s) \right\|_{L_2^0(W,U)}^2 \mathrm{d}s.$$

2.2. Some Assumptions

In this subsection, we recall that some notions of fractional calculus and give some assumptions for the stochastic system (1.1). Recall that the fractional integral I^α of order α for a function $f : [0, \infty) \to R$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad t > 0, \alpha > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t$. Moreover, the Caputo derivative $^c D^\alpha$ of order α for a function $f \in \mathbf{C}^n([0, \infty))$ is defined as

$$^{c}D_{t}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{f^{(n)}(s)}{(t-s)^{1+\alpha-n}}\mathrm{d}s=I^{n-\alpha}f^{(n)}(t),\quad t>0,n-1<\alpha<n.$$

If f is an abstract function with values in U, then the integrals appearing in the above definitions are taken in Bochner's sense.

To study the stochastic system (1.1), we need some assumptions. Throughout this paper we assume that U,V,W is three real separable Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_{U}$, $\langle\cdot,\cdot\rangle_{V}$ and $\langle\cdot,\cdot\rangle_{W}$, respectively. We first give some conditions about the three operators L,A,B as follows:

(A1) A and L are two linear operators on U such that $D(A)\subset U$, $D(L)\subset U$, and A is closed,

(A2) $D(L)\subset D(A)$ and L is bijective,

(A3) $L^{-1}:U\to D(U)$ is compact,

(A4) B is a bounded linear operator from V into U.

From the above assumptions (A1)-(A3) and the closed graph theorem it follows that the linear operator $AL^{-1}:U\to U$ is bounded, and AL^{-1} generates a semigroup $\{S(t),t>0\}$ in U. Denote $M=\max\limits_{t>0}\|S(t)\|$, $\|L\|=M_{1}$ and $\|L^{-1}\|=\widetilde{M}_{1}$.

For $x\in U$, we define two families $\{\mathscr{T}_{L}(t),t\geq0\}$ and $\{\mathscr{S}_{L}(t),t\geq0\}$ of operators by

$$\mathscr{T}_{L}(t)x:=\int_{0}^{\infty}L^{-1}\xi_{\alpha}(\theta)S(t^{\alpha}\theta)x\mathrm{d}\theta$$

and

$$\mathscr{S}_{L}(t)x:=\alpha\int_{0}^{\infty}L^{-1}\theta\xi_{\alpha}(\theta)S(t^{\alpha}\theta)x\mathrm{d}\theta,$$

where

$$\theta\mapsto\xi_{\alpha}(\theta)=\frac{1}{\pi\alpha}\sum_{n=1}^{\infty}(-\theta)^{n-1}\frac{1}{n!}\Gamma(n\alpha+1)\sin(n\pi\alpha)$$

is a probability density function defined on $(0,\infty)$.

Lemma 2.3 Feckan, M. *et al.* [15] *The operators $\mathscr{T}_{L}(t)$ and $\mathscr{S}_{L}(t)$ have the following properties:*

- For every $t\geq0$, $\mathscr{T}_{L}(t)$ and $\mathscr{S}_{L}(t)$ are linear and bounded, and moreover for every $x\in U$

$$\|\mathscr{T}_{L}(t)x\|\leq M\widetilde{M}_{1}\|x\|,$$

$$\|\mathscr{S}_{L}(t)x\|\leq\frac{M\widetilde{M}_{1}}{\Gamma(\alpha)}\|x\|. \tag{2.2}$$

- $\mathscr{T}_{L}(t)$ and $\mathscr{S}_{L}(t)$ are strong continuous and compact.

We now introduce the abstract phase space. For a continuous function $h:(-\infty,0]\to(0,\infty)$ satisfying

$$l:=\int_{-\infty}^{0}h(t)\mathrm{d}t<\infty,$$

we define a phase space $\mathscr{B}_{h}$ associated with h as follows

$$\mathscr{B}_h = \left\{ \phi : (-\infty, 0] \to U, \text{for any } a > 0, \left(\mathrm{E} \|\phi(\theta)\|^2 \right)^{1/2} \text{ is bounded} \right.$$

$$\text{and measurable functions on } [-a, 0] \text{ with } \phi(0) = 0$$

$$\left. \text{and } \int_{-\infty}^0 h(s) \sup_{s \le \theta \le 0} \left(\mathrm{E} \|\phi(\theta)\|^2 \right)^{1/2} \mathrm{d}s < \infty \right\}.$$

Clearly, $\left(\mathscr{B}_h, \|\cdot\|_{\mathscr{B}_h} \right)$ is a Banach space if $\mathscr{B}_h$ is endowed with the norm (see, Cui and Yan [16])

$$\|\phi\|_{\mathscr{B}_h} = \int_{-\infty}^0 h(s) \sup_{s \le \theta \le 0} \left(\mathrm{E} \|\phi(\theta)\|^2 \right)^{1/2} \mathrm{d}s$$

for $\phi \in \mathscr{B}_h$.

We present the definition of mild solutions of (1.1).

Definition 2.2 *An U-valued stochastic process* $\{x(t), t \in [-\infty, T]\}$ *is a mild solution of* (1.1) *if the next conditions hold:*

i) $x(t)$ is measurable and $\mathscr{F}_t$-adapted, and x_t is $\mathscr{B}_h$-valued,

ii) $x(t)$ is continuous on $[0, T]$ and the function $(t-s)^{\alpha-1} L^{-1} A \mathscr{S}_L (t-s) G(t, x_s)$ is integrable for each $s \in [0, t]$ such that $x(t)$ satisfies the equation

$$\begin{aligned}
x(t) = {} & \mathscr{T}_L(t) \left(L\phi(0) - G(0, \phi) \right) + L^{-1} G(t, x_t) \\
& + \int_0^t (t-s)^{\alpha-1} L^{-1} A \mathscr{S}_L (t-s) G(t, x_s) \mathrm{d}s \\
& + \int_0^t (t-s)^{\alpha-1} \mathscr{S}_L (t-s) \left[f(s, x_s) + Bu(s) \right] \mathrm{d}s \\
& + \int_0^t (t-s)^{\alpha-1} \mathscr{S}_L (t-s) \sigma(s) \mathrm{d}B^H(s),
\end{aligned} \tag{2.3}$$

iii) $x(t) = \phi(t)$ on $(-\infty, 0]$ such that $\|\phi\|_{\mathscr{B}_h}^2 < \infty$.

Finally, in order to prove our main statement, we need some conditions as follows.

(B1) Let the function $f : [0, T] \times \mathscr{B}_h \to U$ is continuous and there exist some constants $N_f > 0$, $k_f > 0$ such that for $t \in [0, T]$ and $\xi, \eta \in \mathscr{B}_h$

$$\mathrm{E} \|f(t, \xi) - f(t, \eta)\|^2 \le N_f \|\xi - \eta\|_{\mathscr{B}_h}^2$$

for all $t \in [0, T]$ and $k_f = \sup_{t \in [0, T]} \|f(t, 0)\|^2$.

(B2) For the complete orthogonal basis $\{e_n\}_{n \in \mathbb{N}}$ in W, the function $\sigma : [0, T] \to L_2^0(W, U)$ satisfy

$$\sum_{n=1}^{\infty} \left\| \sigma Q^{\frac{1}{2}} e_n \right\|_{L^2([0,T], U)} < \infty$$

and $\sum_{n=1}^{\infty} \left\| \sigma(t) Q^{\frac{1}{2}} e_n \right\|$ is uniformly convergent in $t \in [0, T]$. In addition, there exist some t_0 and $\delta > 0$ such that

$$\int_0^{t_0} \int_0^{t_0} r^{-\delta} s^{-\delta} \|\sigma(r)\|_{L_2^0(W, U)} \|\sigma(s)\|_{L_2^0(W, U)} \mathrm{d}r \mathrm{d}s < \infty.$$

(B3) Let the function $G:[0,T]\times\mathscr{R}_h \to U$ is continuous and satisfies:

(a) there exist some constants $N_G > 0$, $k_G > 0$ for $t, s \in [0,T]$ and $\xi, \eta \in \mathscr{R}_h$ such that the function AG satisfies the Lipschitz condition

$$\mathrm{E}\|AG(t,\xi) - AG(t,\eta)\|^2 \leq N_G \|\xi - \eta\|_{\mathscr{R}_h}^2$$

for all $t \in [0,T]$ and $k_g = \sup_{t \in [0,T]} \|AG(t,0)\|^2$.

(b) there exist constants $\widetilde{N_G} > 0$, $\widetilde{k_g} > 0$ such that

$$\mathrm{E}\|G(t,\xi) - G(t,\eta)\|^2 \leq \widetilde{N_G} \|\xi - \eta\|_{\mathscr{R}_h}^2$$

for all $t \in [0,T]$, $\xi, \eta \in \mathscr{R}_h$ and $\widetilde{k_g} = \sup_{t \in [0,T]} \|G(t,0)\|^2$.

(B4) There is a constant $\mathscr{K} > 0$ such that $\dfrac{\mathcal{K}_1 \mathscr{K}}{\mathcal{K}_2} > 1$, where

$$\mathcal{K}_1 = 1 - 24l^2 \left(\frac{M\widetilde{M}_1 T^\alpha}{\alpha\Gamma(\alpha)}\right)^2 N_f \left(1 + 4\left(\frac{M^2 \widetilde{M}_1^2 M_B T^\alpha}{\alpha\lambda\Gamma(\alpha)^2}\right)^2\right) > 0,$$

$$\mathcal{K}_2 = 4l^2\left(MM_1\widetilde{M}_1\right)^2 \mathrm{E}\|\phi_0\|^2 \left[1 + 12\left(\frac{M^2\widetilde{M}_1^2 M_B^2 T^\alpha}{\alpha\lambda\Gamma(\alpha)^2}\right)^2\right] + 4\|\phi\|_{\mathscr{R}_h}^2$$

$$+ 12l^2\left(\frac{M\widetilde{M}_1}{\Gamma(\alpha)}\right)^2 \left[1 + 4\left(\frac{M^2\widetilde{M}_1^2 M_B^2 T^\alpha}{\alpha\lambda\Gamma(\alpha)^2}\right)^2\right] \times \left(\frac{2k_f T^{2\alpha}}{\alpha^2} + cH(2H-1)N_\sigma\right)$$

$$+ 96l^2\left(\frac{M^2\widetilde{M}_1^2 M_B^2 T^\alpha}{\alpha\lambda\Gamma(\alpha)^2}\right)^2 \left(\mathrm{E}\|z_T\|^2 + cH(2H-1)T^{2H-1}\int_0^T \|\hat{\varphi}(s)\|_{L_2^0(W,U)}^2 \mathrm{d}s\right),$$

$$N_\sigma := \sup_{t\in[0,T]} \int_0^t\int_0^t (t-s)^{\alpha-1}(t-r)^{\alpha-1}|s-r|^{2H-2}\|\sigma(s)\|_{L_2^0(W,U)}\|\sigma(r)\|_{L_2^0(W,U)}\mathrm{d}s\mathrm{d}r < \infty,$$

and $M_B = \|B\|$.

3. Main Results

In this section, we will show the approximate controllability of the stochastic system (1.1). We need to establish the existence of the solution for the stochastic control system and to show that the corresponding linear part is approximate controllability.

Definition 3.1 *The system* (1.1) *is called to be approximately controllable on* $[0,T]$ *if*

$$\overline{\mathfrak{R}(T)} = U \tag{3.1}$$

with $\mathfrak{R}(t) = \left\{x(t) = x(t,u) : u \in L^2\left([0,T], V\right)\right\}$.

Consider the corresponding linear fractional deterministic control system to (1.1)

$$\begin{cases} {}^{c}D_t^{\alpha}\left[Lx(t)\right] = Ax(t) + Bu(t), & t \in [0,T] \\ x(0) = \phi(0), \end{cases} \tag{3.2}$$

and define the relevant operators

$$\Gamma_0^T = \int_0^T (T-s)^{\alpha-1}\mathscr{S}_L(T-s)BB^*\mathscr{S}_L^*(T-s)\mathrm{d}s \tag{3.3}$$

and

$$R\left(\alpha,\Gamma_0^T\right) = \left(\alpha I + \Gamma_0^T\right)^{-1}, \tag{3.4}$$

where B^* and $\mathscr{S}_L^*(T-s)$ denote the adjoint operators of B and $\mathscr{S}_L(T-s)$, respectively. It is clear that the operator Γ_0^T is a linear bounded operator. The fact that the linear Sobolev-type fractional control system (3.2) is approximately controllable on $[0,T]$ is equivalent to the next hypothesis (see, for example, Mahmudov and Denker [17]):

- $\alpha R\left(\alpha,\Gamma_0^T\right) \to 0$ in the strong operator topology, as $\alpha \to 0^+$.

Lemma 3.1 (Guendouzi and Idrissi [18]) *For any* $z_T \in L^2(\Omega;U)$, *there exists* $\hat{\varphi}(t) \in L^2\left(\Omega, L^2\left(0,T;L_2^0\right)\right)$ *such that*

$$z_T = Ez_T + \int_0^T \hat{\varphi}(s)\mathrm{d}B^H(s). \tag{3.5}$$

For any $\lambda > 0$ and $z_T \in L^2(\Omega;U)$, we now define the control function u^{λ} as follows.

$$\begin{aligned} u^{\lambda}(t) &= B^*\mathscr{S}_L^*(T-t)\left(\lambda I + \Gamma_0^T\right)^{-1}\left[-\mathscr{T}_L(T)\left(L\phi(0) - G(0,\phi)\right) - L^{-1}G(T,x_T)\right] \\ &\quad + B^*\mathscr{S}_L^*(T-t)\left(\lambda I + \Gamma_0^T\right)^{-1}\left[Ez_T + \int_0^T \hat{\varphi}(s)\mathrm{d}B^H(s)\right] \\ &\quad - B^*\mathscr{S}_L^*(T-t)\int_0^t \left(\lambda I + \Gamma_s^T\right)^{-1}(T-s)^{\alpha-1}A\mathscr{S}_L(T-s)G(s,x_s)\mathrm{d}s \\ &\quad - B^*\mathscr{S}_L^*(T-t)\int_0^t \left(\lambda I + \Gamma_s^T\right)^{-1}(T-s)^{\alpha-1}\mathscr{S}_L(T-s)f(s,x_s)\mathrm{d}s \\ &\quad - B^*\mathscr{S}_L^*(T-t)\int_0^t \left(\lambda I + \Gamma_s^T\right)^{-1}(T-s)^{\alpha-1}\mathscr{S}_L(T-s)\sigma(s)\mathrm{d}B^H(s). \end{aligned} \tag{3.6}$$

Theorem 3.1 (Daher [19]) *Let* Φ *be a condensing operator on a Banach space* X, *that is,* Φ *is continuous and takes bounded sets into bounded sets, and* $\mu(\Phi(B)) \leq \mu(B)$ *for every bounded set* B *of* X *with* $\mu(B) > 0$. *If* $\Phi(N) \subset N$ *for a convex, closed and bounded set* N *of* X, *then* Φ *has a fixed point in* X (*where* $\mu(\cdot)$ *denotes Kuratowski's measure of noncompactness*).

Define the space

$$\mathscr{P}_a = \left\{x : x \in C\left((-\infty,T],U\right) \text{ with } x_0 = \phi \in \mathscr{P}_h\right\}$$

and let $\|\cdot\|_a$ be a seminorm defined by

$$\|x\|_a = \|\phi\|_{\mathscr{P}_h} + \sup_{t \in [0,T]}\left(E\|x(s)\|^2\right)^{\frac{1}{2}}, x \in \mathscr{P}_a, \tag{3.7}$$

where $C\left((-\infty,T],U\right)$ denotes the space of all continuous U-valued stochastic process $\left\{\xi(t), t \in (-\infty,T]\right\}$.

Lemma 3.2 (Li and Liu [20]) *Assume that* $x \in \mathscr{P}_a$, *then for all* $t \in [0,T], x_t \in \mathscr{P}_h$. *Moreover,*

$$l\left(\mathrm{E}\left\|x(t)\right\|^2\right)^{1/2} \le l \sup_{s\in[0,t]}\left(\mathrm{E}\left\|x(s)\right\|^2\right)^{1/2} + \left\|x_0\right\|_{\mathscr{B}_h}, \tag{3.8}$$

where $l = \int_{-\infty}^{0} h(s)\mathrm{d}s$ is given in Section 2.

Theorem 3.2 *Assume the conditions* (B1)-(B4) *hold, then for each* $\lambda > 0$ *there exists a mild solution of* (1.1) *on* $(-\infty, T]$, *provided that*

$$4l^2\left(\widetilde{M_1}^2 \widetilde{N_G} + \left(\frac{\widetilde{M_1}^2 MT^\alpha}{\alpha\Gamma(\alpha)}\right)^2 N_G\right) < 1.$$

Proof. Define the operator $\Phi : \mathscr{B}_a \to \mathscr{B}_a$ by

$$\Phi x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \mathscr{T}_L(t)\big(L\phi(0) - G(0,\phi)\big) + L^{-1}G(t,x_t) \\ \quad + \int_0^t (t-s)^{\alpha-1} L^{-1}A\mathscr{S}_L(t-s)G(s,x_s)\mathrm{d}s \\ \quad + \int_0^t (t-s)^{\alpha-1} \mathscr{S}_L(t-s)\big[f(s,x_s) + Bu^\lambda(s)\big]\mathrm{d}s \\ \quad + \int_0^t (t-s)^{\alpha-1} \mathscr{S}_L(t-s)\sigma(s)\mathrm{d}B^H(s), & t \in (0,T], \end{cases} \tag{3.9}$$

for $x \in \mathscr{B}_a$.

We will show that Φ has a fixed point which is a mild solution for system (1.1). For $\phi \in \mathscr{B}_h$, define

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \mathscr{T}_L(t)\big(L\phi(0)\big), & t \in (0,T]. \end{cases} \tag{3.10}$$

Then, $\tilde{\phi}(t) \in \mathscr{B}_a$. Let $x(t) = \tilde{\phi}(t) + y(t), t \in (-\infty, T]$. It is easy to check that $x(t)$ satisfies (1.1) if and only if $y_0 = 0$ and

$$\begin{aligned} y(t) = &-\mathscr{T}_L(t)G(0,\phi) + L^{-1}G\big(t, y_t + \tilde{\phi}_t\big) \\ &+ \int_0^t (t-s)^{\alpha-1} L^{-1}A\mathscr{S}_L(t-s)G\big(s, y_s + \tilde{\phi}_s\big)\mathrm{d}s \\ &+ \int_0^t (t-s)^{\alpha-1} \mathscr{S}_L(t-s)\big[f\big(s, y_s + \tilde{\phi}_s\big) + Bu^\lambda(s)\big]\mathrm{d}s \\ &+ \int_0^t (t-s)^{\alpha-1} \mathscr{S}_L(t-s)\sigma(s)\mathrm{d}B^H(s). \end{aligned} \tag{3.11}$$

Denote $\mathscr{B}_b = \{y \in \mathscr{B}_a, y_0 = 0 \in \mathscr{B}_h\}$ and let $\|\cdot\|_b$ be the seminorm in $\mathscr{B}_b$, defined by

$$\|y\|_b = \|y_0\|_{\mathscr{B}_h} + \sup_{s\in[0,T]}\left(\mathrm{E}\left\|y(s)\right\|^2\right)^{1/2} = \sup_{s\in[0,T]}\left(\mathrm{E}\left\|y(s)\right\|^2\right)^{1/2}. \tag{3.12}$$

For $r \ge 0$ we set $B_r = \{y \in \mathscr{B}_b, \|y\|_b^2 \le r\}$. Then, B_r is a bounded closed convex set in $\mathscr{B}_b$ for each *r*. According to Lemma 3.2, we get

$$\begin{aligned} \left\|y_t + \tilde{\phi}_t\right\|_{\mathscr{B}_h}^2 &\le 2\left(\left\|y_t\right\|_{\mathscr{B}_h}^2 + \left\|\tilde{\phi}_t\right\|_{\mathscr{B}_h}^2\right) \\ &\le 4\left(l^2 \sup_{s\in[0,t]}\mathrm{E}\left\|y(s)\right\|^2 + \left\|y_0\right\|_{\mathscr{B}_h}^2 + l^2 \sup_{s\in[0,t]}\mathrm{E}\left\|\tilde{\phi}(s)\right\|^2 + \left\|\tilde{\phi}_0\right\|_{\mathscr{B}_h}^2\right) \\ &\le 4\left(l^2 r + l^2 M^2 M_1^2 \widetilde{M_1}^2 \mathrm{E}\left\|\phi(0)\right\|^2 + \left\|\phi\right\|_{\mathscr{B}_h}^2\right) \equiv r' \end{aligned} \tag{3.13}$$

for $y \in B_r$. Define the mapping $\Psi : \mathscr{R}_b \to \mathscr{R}_b$ by

$$
\begin{aligned}
\Psi y(t) = &-\mathscr{T}_L(t)G(0,\phi)+L^{-1}G\big(t,y_t+\tilde{\phi}_t\big) \\
&+ \int_0^t (t-s)^{\alpha-1} L^{-1}A\mathscr{S}_L(t-s)G\big(s,y_s+\tilde{\phi}_s\big)\mathrm{d}s \\
&+ \int_0^t (t-s)^{\alpha-1} \mathscr{S}_L(t-s)\Big[f\big(s,\tilde{\phi}_s+y_s\big)+Bu^\lambda(s)\Big]\mathrm{d}s \\
&+ \int_0^t (t-s)^{\alpha-1} \mathscr{S}_L(t-s)\sigma(s)\mathrm{d}B^H(s)
\end{aligned}
\tag{3.14}
$$

for $t \in [0,T]$. It is evident that the operator Φ has a fixed point if and only if the operator Ψ has a fixed point. Now, we divide Ψ into $\Psi = \Psi_1 + \Psi_2$, where

$$
\begin{aligned}
\Psi_1 y(t) = &-\mathscr{T}_L(t)G(0,\phi)+L^{-1}G\big(t,y_t+\tilde{\phi}_t\big) \\
&+ \int_0^t (t-s)^{\alpha-1} L^{-1}A\mathscr{S}_L(t-s)G\big(s,y_s+\tilde{\phi}_s\big)\mathrm{d}s,
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
\Psi_2 y(t) = &\int_0^t (t-s)^{\alpha-1} \mathscr{S}_L(t-s)\Big[f\big(s,\tilde{\phi}_s+y_s\big)+Bu^\lambda(s)\Big]\mathrm{d}s \\
&+ \int_0^t (t-s)^{\alpha-1} \mathscr{S}_L(t-s)\sigma(s)\mathrm{d}B^H(s).
\end{aligned}
\tag{3.16}
$$

Now, we need to prove the operator Ψ_1 is a contraction map and Ψ_2 is compact.

Step I. Ψ_1 is a contraction map. For $y, \overline{y} \in \mathscr{R}_b$, we have

$$
\begin{aligned}
&\mathrm{E}\big\|\Psi_1 y(t) - \Psi_1 \overline{y}(t)\big\|^2 \\
&\leq 2\mathrm{E}\big\|L^{-1}G\big(t,y_t+\tilde{\phi}_t\big)-L^{-1}G\big(t,\overline{y}_t+\tilde{\phi}_t\big)\big\|^2 \\
&\quad + 2\mathrm{E}\left\|\int_0^t (t-s)^{\alpha-1} L^{-1}A\mathscr{S}_L(t-s)\Big[G\big(s,y_s+\tilde{\phi}_s\big)-G\big(s,\overline{y}_s+\tilde{\phi}_s\big)\Big]\mathrm{d}s\right\|^2 \\
&\leq 2\widetilde{M_1}^2\widetilde{N_G}\big\|y_t-\overline{y}_t\big\|_{\mathscr{R}_b}^2 + \left(\frac{\widetilde{M_1}^2 MT^\alpha}{\alpha\Gamma(\alpha)}\right)^2 N_G\big\|y_t-\overline{y}_t\big\|_{\mathscr{R}_b}^2 \\
&\leq 4l^2\left(\widetilde{M_1}^2\widetilde{N_G} + \left(\frac{\widetilde{M_1}^2 MT^\alpha}{\alpha\Gamma(\alpha)}\right)^2 N_G\right)\sup_{t\in[0,T]}\big\|y_t-\overline{y}_t\big\|^2.
\end{aligned}
\tag{3.17}
$$

It follows that Ψ_1 is a contraction map with the assumption

$$
4l^2\left(\widetilde{M_1}^2\widetilde{N_G} + \left(\frac{\widetilde{M_1}^2 MT^\alpha}{\alpha\Gamma(\alpha)}\right)^2 N_G\right)<1.
$$

Step II. We claim that Ψ_2 is compact. In [14], we have proved that Ψ_2 maps bounded sets into bounded sets of $\mathscr{R}_b$ and Ψ_2 maps bounded sets into equicontinuous sets of $\mathscr{R}_b$. It is enough to prove that Ψ_2 maps B_r into a precompact set in B_r. Define an operator $\Psi_2^{\varepsilon,\delta}$ on B_r by

$$
\begin{aligned}
\Psi_2^{\varepsilon,\delta} y(t) = &\,\alpha\int_0^{t-\varepsilon}\int_\delta^\infty (t-s)^{\alpha-1} L^{-1}\theta\xi_\alpha(\theta)S\big((t-s)^\alpha\theta\big)\mathrm{d}\theta f\big(s,y_s+\tilde{\phi}_s\big)\mathrm{d}s \\
&+ \alpha\int_0^{t-\varepsilon}\int_\delta^\infty (t-s)^{\alpha-1} L^{-1}\theta\xi_\alpha(\theta)S\big((t-s)^\alpha\theta\big)\mathrm{d}\theta Bu^\lambda(s)\mathrm{d}s \\
&+ \alpha\int_0^{t-\varepsilon}\int_\delta^\infty (t-s)^{\alpha-1} L^{-1}\theta\xi_\alpha(\theta)S\big((t-s)^\alpha\theta\big)\mathrm{d}\theta\sigma(s)\mathrm{d}B^H(s).
\end{aligned}
\tag{3.18}
$$

Since $S(t), t > 0$ is a compact operator, the set $\left\{ \Psi_2^{\varepsilon,\delta} y(t), y \in B_r \right\}$ is precompact in U for every $\varepsilon \in (0,t), \delta > 0$. For each $y \in B_r$

$$
\begin{aligned}
&\mathrm{E} \left\| \Psi_2^{\varepsilon,\delta} y(t) - \Psi_2^{\varepsilon,\delta} y(t) \right\|^2 \\
&\leq 6\alpha^2 \mathrm{E} \left\| \int_0^t \int_0^\delta (t-s)^{\alpha-1} L^{-1} \theta \xi_\alpha(\theta) S\left((t-s)^\alpha \theta\right) \mathrm{d}\theta f\left(s, y_s + \tilde{\phi}_s\right) \mathrm{d}s \right\|^2 \\
&\quad + 6\alpha^2 \mathrm{E} \left\| \int_{t-\varepsilon}^t \int_\delta^\infty (t-s)^{\alpha-1} L^{-1} \theta \xi_\alpha(\theta) S\left((t-s)^\alpha \theta\right) \mathrm{d}\theta f\left(s, y_s + \tilde{\phi}_s\right) \mathrm{d}s \right\|^2 \\
&\quad + 6\alpha^2 \mathrm{E} \left\| \int_0^t \int_0^\delta (t-s)^{\alpha-1} L^{-1} \theta \xi_\alpha(\theta) S\left((t-s)^\alpha \theta\right) \mathrm{d}\theta B u^\lambda(s) \mathrm{d}s \right\|^2 \\
&\quad + 6\alpha^2 \mathrm{E} \left\| \int_{t-\varepsilon}^t \int_\delta^\infty (t-s)^{\alpha-1} L^{-1} \theta \xi_\alpha(\theta) S\left((t-s)^\alpha \theta\right) \mathrm{d}\theta B u^\lambda(s) \mathrm{d}s \right\|^2 \\
&\quad + 6\alpha^2 \mathrm{E} \left\| \int_0^t \int_0^\delta (t-s)^{\alpha-1} L^{-1} \theta \xi_\alpha(\theta) S\left((t-s)^\alpha \theta\right) \mathrm{d}\theta \sigma(s) \mathrm{d}B^H(s) \right\|^2 \\
&\quad + 6\alpha^2 \mathrm{E} \left\| \int_{t-\varepsilon}^t \int_\delta^\infty (t-s)^{\alpha-1} L^{-1} \theta \xi_\alpha(\theta) S\left((t-s)^\alpha \theta\right) \mathrm{d}\theta \sigma(s) \mathrm{d}B^H(s) \right\|^2 \\
&:= 6\alpha^2 \sum_{i=1}^6 J_i.
\end{aligned} \tag{3.19}
$$

By using Hölder inequality and the assumption (B1) we have

$$
\begin{aligned}
J_1 &\leq \widetilde{M_1}^2 M^2 \int_0^t (t-s)^{\alpha-1} \mathrm{d}s \int_0^t (t-s)^{\alpha-1} \mathrm{E} \left\| f\left(s, y_s + \tilde{\phi}_s\right) \right\|^2 \mathrm{d}s \left(\int_0^\delta \theta \xi_\alpha(\theta) \mathrm{d}\theta \right)^2 \\
&\leq \left(\frac{M \widetilde{M_1} t^\alpha}{\alpha} \right)^2 \left(N_f r' + k_f \right) \left(\int_0^\delta \theta \xi_\alpha(\theta) \mathrm{d}\theta \right)^2,
\end{aligned} \tag{3.20}
$$

and

$$
\begin{aligned}
J_2 &\leq \widetilde{M_1}^2 M^2 \int_{t-\varepsilon}^t (t-s)^{\alpha-1} \mathrm{d}s \int_{t-\varepsilon}^t (t-s)^{\alpha-1} \mathrm{E} \left\| f\left(s, y_s + \tilde{\phi}_s\right) \right\|^2 \mathrm{d}s \left(\int_0^\infty \theta \xi_\alpha(\theta) \mathrm{d}\theta \right)^2 \\
&\leq \left(\frac{M \widetilde{M_1} \varepsilon^\alpha}{\alpha} \right)^2 \left(N_f r' + k_f \right) \left(\int_0^\infty \theta \xi_\alpha(\theta) \mathrm{d}\theta \right)^2.
\end{aligned} \tag{3.21}
$$

Since

$$
\begin{aligned}
\mathrm{E} \left\| u^\lambda(s) \right\|^2 &\leq \frac{4\left(M_B M \widetilde{M_1} \right)^2}{\lambda^2 \Gamma(\alpha)^2} \Bigg\{ \left(M \widetilde{M_1} M_1 \right)^2 \mathrm{E} \left\| \phi(0) \right\|^2 + 2\mathrm{E} \left\| z_T \right\|^2 \\
&\quad + \left(M \widetilde{M_1} M_1 \right)^2 \widetilde{N_G} \left\| \phi \right\|_{\mathscr{A}_h}^2 + \widetilde{M_1}^2 \left(\widetilde{N_G} r' + \widetilde{k_g} \right) \\
&\quad + 2cH(2H-1) T^{2H-1} \int_0^T \left\| \hat{\varphi}(s) \right\|_{L_2^0(W,U)}^2 \mathrm{d}s \\
&\quad + \mu_{AG} + \mu_f + cH(2H-1) \left(\frac{M \widetilde{M_1}}{\Gamma(\alpha)} \right)^2 N_\sigma \Bigg\} \equiv \frac{4(M_B M \widetilde{M_1})^2}{\lambda^2 \Gamma(\alpha)^2} N_u,
\end{aligned} \tag{3.22}
$$

where

$$
\mu_{AG} = 2\left(\frac{M \widetilde{M_1} T^\alpha}{\alpha \Gamma(\alpha)} \right)^2 \left(N_G r' + k_g \right) \text{ and } \mu_f = 2\left(\frac{M \widetilde{M_1} T^\alpha}{\alpha \Gamma(\alpha)} \right)^2 \left(N_f r' + k_f \right), \tag{3.23}
$$

it follows that

$$J_3 \leq \widetilde{M}_1^2 M^2 \int_0^t (t-s)^{\alpha-1} \, ds \int_0^t (t-s)^{\alpha-1} \mathrm{E} \left\| Bu^\lambda(s) \right\|^2 \, ds \left(\int_0^\delta \theta \xi_\alpha(\theta) \, d\theta \right)^2$$

$$\leq \left(\frac{M \widetilde{M}_1 M_B t^\alpha}{\alpha} \right)^2 \frac{4 \left(M_B M \widetilde{M}_1 \right)^2}{\lambda^2 \Gamma(\alpha)^2} N_u \left(\int_0^\delta \theta \xi_\alpha(\theta) \, d\theta \right)^2, \quad (3.24)$$

and

$$J_4 \leq \widetilde{M}_1^2 M^2 \int_{t-\varepsilon}^t (t-s)^{\alpha-1} \, ds \int_{t-\varepsilon}^t (t-s)^{\alpha-1} \mathrm{E} \left\| Bu^\lambda(s) \right\|^2 \, ds \left(\int_0^\infty \theta \xi_\alpha(\theta) \, d\theta \right)^2$$

$$\leq \left(\frac{M \widetilde{M}_1 \varepsilon^\alpha}{\alpha} \right)^2 \frac{4 \left(M_B M \widetilde{M}_1 \right)^2}{\lambda^2 \Gamma(\alpha)^2} N_u \left(\int_0^\infty \theta \xi_\alpha(\theta) \, d\theta \right)^2. \quad (3.25)$$

For the last parts J_5, J_6 when $\alpha > 1 - H$, we have

$$\int_0^T \int_0^T (T-s)^{\alpha-1} (T-t)^{\alpha-1} |s-t|^{2H-2} \, ds dt = \frac{B(\alpha, 2H-1)}{2(\alpha + H - 1)} T^{2(\alpha+H-1)}, \quad (3.26)$$

which imply that

$$J_5 = \mathrm{E} \left\| \sum_{n=1}^\infty \int_0^t \int_0^\delta (t-s)^{\alpha-1} L^{-1} \theta \xi_\alpha(\theta) S\left((t-s)^\alpha \theta \right) d\theta \sigma(s) Q^{\frac{1}{2}} e_n d\beta_n^H(s) \right\|^2$$

$$= \sum_{n=1}^\infty \mathrm{E} \left\| \int_0^t \int_0^\delta (t-s)^{\alpha-1} L^{-1} \theta \xi_\alpha(\theta) S\left((t-s)^\alpha \theta \right) d\theta \sigma(s) Q^{\frac{1}{2}} e_n d\beta_n^H(s) \right\|^2 \quad (3.27)$$

$$\leq cH(2H-1) \left(M \widetilde{M}_1 \right)^2 \int_0^t \int_0^t (t-u)^{\alpha-1} (t-v)^{\alpha-1} |u-v|^{2H-2}$$

$$\times \left\| \sigma(u) \right\|_{L_2^0(W,U)} \left\| \sigma(v) \right\|_{L_2^0(W,U)} du dv \left(\int_0^\delta \theta \xi_\alpha(\theta) \, d\theta \right)^2,$$

and

$$J_6 = \mathrm{E} \left\| \sum_{n=1}^\infty \int_{t-\varepsilon}^t \int_\delta^\infty (t-s)^{\alpha-1} L^{-1} \theta \xi_\alpha(\theta) S\left((t-s)^\alpha \theta \right) d\theta \sigma(s) Q^{\frac{1}{2}} e_n d\beta_n^H(s) \right\|^2$$

$$= \sum_{n=1}^\infty \mathrm{E} \left\| \int_{t-\varepsilon}^t \int_\delta^\infty (t-s)^{\alpha-1} L^{-1} \theta \xi_\alpha(\theta) S\left((t-s)^\alpha \theta \right) d\theta \sigma(s) Q^{\frac{1}{2}} e_n d\beta_n^H(s) \right\|^2 \quad (3.28)$$

$$\leq cH(2H-1) \left(M \widetilde{M}_1 \right)^2 \int_{t-\varepsilon}^t \int_{t-\varepsilon}^t (t-u)^{\alpha-1} (t-v)^{\alpha-1} |u-v|^{2H-2}$$

$$\times \left\| \sigma(u) \right\|_{L_2^0(W,U)} \left\| \sigma(v) \right\|_{L_2^0(W,U)} du dv \left(\int_0^\infty \theta \xi_\alpha(\theta) \, d\theta \right)^2.$$

Then, for each $y \in B_r$,

$$\mathrm{E} \left\| \Psi_2^{\varepsilon,\delta} y(t) - \Psi_2^{\varepsilon,\delta} y(t) \right\|^2 \to 0 \text{ as } \varepsilon \to 0, \, \delta \to 0.$$

Therefore, there are relatively compact sets arbitrary close to the set $\left\{ \Psi_2 y(t), y \in B_r \right\}$ is precompact in B_r. By Arzela-Ascoli's theorem, Ψ_2 is compact. By Sadakovskii's fixed point theorem (Theorem 3.1), the operator Ψ has a fixed point which is a solution to the system (1.1).

Theorem 3.3 *Assume that the conditions of Theorem* 3.2 *and* $(\mathbb{H}_0)$ *hold. In addition, the functions f is uniformly bounded on its domain. Then, the fractional control system* (1.1) *is approximately controllable on* $[0,T]$.

Proof. Let x^λ be a fixed point of the operator Φ^λ. Using the stochastic Fubini theorem, we can get

$$
\begin{aligned}
x^\lambda(T) = z_T &- \lambda\left(\lambda I + \Gamma_0^T\right)^{-1}\left[-\mathscr{T}_L(T)\left(L\phi(0) - G(0,\phi)\right)\right. \\
&\left. - L^{-1}G(t,x_t) + \mathrm{E}z_T + \int_0^T \hat{\varphi}(s)\mathrm{d}B^H(s)\right] \\
&+ \lambda\int_0^T \left(\lambda I + \Gamma_s^T\right)^{-1}(T-s)^{\alpha-1}L^{-1}A\mathscr{S}_L(T-s)G(s,x_s)\mathrm{d}s \quad (3.29) \\
&+ \lambda\int_0^T \left(\lambda I + \Gamma_s^T\right)^{-1}(T-s)^{\alpha-1}\mathscr{S}_L(T-s)f\left(s,x_s^\lambda\right)\mathrm{d}s \\
&+ \lambda\int_0^T \left(\lambda I + \Gamma_s^T\right)^{-1}(T-s)^{\alpha-1}\mathscr{S}_L(T-s)\sigma(s)\mathrm{d}B^H(s).
\end{aligned}
$$

It follows from the property of f, G that there exists $C > 0$ such that $\left\|f\left(s,x_s^\lambda\right)\right\|^2 \leq C$ and $\left\|AG\left(s,x_s^\lambda\right)\right\|^2 \leq C$. Then there is a subsequence denoted by $\left\{f\left(s,x_s^\lambda\right), AG\left(s,x_s^\lambda\right)\right\}$ weakly converging to $\left\{f(s), G\right\}$. Thus, from the above equation, we obtain

$$
\mathrm{E}\left\|x^\lambda(T) - z_T\right\|^2
$$

$$
\leq 8\left\|\lambda\left(\lambda I + \Gamma_0^T\right)^{-1}\left[\mathrm{E}z_T - \mathscr{T}_L(T)\left(L\phi(0) - G(0,\phi)\right)\right]\right\|^2
$$

$$
+ 8\mathrm{E}\left\|\lambda\left(\lambda I + \Gamma_0^T\right)^{-1}\left[L^{-1}G(t,x_t)\right]\right\|^2
$$

$$
+ 8cH(2H-1)T^{2H-1}\mathrm{E}\int_0^T \left\|\lambda\left(\lambda I + \Gamma_0^T\right)^{-1}\hat{\varphi}(s)\right\|_{L_2^0(W,U)}^2 \mathrm{d}s
$$

$$
+ 8\mathrm{E}\left(\int_0^T (T-s)^{\alpha-1}\left\|\lambda\left(\lambda I + \Gamma_s^T\right)^{-1}L^{-1}A\mathscr{S}_L(T-s)\left[G\left(s,x_s^\lambda\right) - G(s)\right]\right\|\mathrm{d}s\right)^2
$$

$$
+ 8\mathrm{E}\left(\int_0^T (T-s)^{\alpha-1}\left\|\lambda\left(\lambda I + \Gamma_s^T\right)^{-1}L^{-1}A\mathscr{S}_L(T-s)G(s)\right\|\mathrm{d}s\right)^2
$$

$$
+ 8\mathrm{E}\left(\int_0^T (T-s)^{\alpha-1}\left\|\lambda\left(\lambda I + \Gamma_s^T\right)^{-1}\mathscr{S}_L(T-s)\left[f\left(s,x_s^\lambda\right) - f(s)\right]\right\|\mathrm{d}s\right)^2
$$

$$
+ 8\mathrm{E}\left(\int_0^T (T-s)^{\alpha-1}\left\|\lambda\left(\lambda I + \Gamma_s^T\right)^{-1}\mathscr{S}_L(T-s)f(s)\right\|\mathrm{d}s\right)^2
$$

$$
+ 8cH(2H-1)\int_0^T\int_0^T (T-s)^{\alpha-1}(T-t)^{\alpha-1}|s-t|^{2H-2}
$$

$$
\times\left\|\lambda\left(\lambda I + \Gamma_s^T\right)^{-1}\mathscr{S}_L(T-s)\sigma(s)\right\|_{L_2^0(W,U)}^2\left\|\lambda\left(\lambda I + \Gamma_t^T\right)^{-1}\mathscr{S}_L(T-t)\sigma(t)\right\|_{L_2^0(W,U)}^2 \mathrm{d}s\mathrm{d}t.
$$

$$
(3.30)
$$

On the other hand, by assumption $(\mathbb{H}_0)$ for all $0 \leq s \leq T$, the operator $\lambda\left(\lambda I + \Gamma_s^T\right)^{-1} \to 0$ strongly as $\lambda \to 0+$, and moreover $\left\|\lambda\left(\lambda I + \Gamma_s^T\right)^{-1}\right\| < 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of $\mathscr{S}_L(t)$, we can get $\mathrm{E}\left\|x^\lambda(T) - z_T\right\|^2 \to 0$ as $\lambda \to 0+$. This gives the approximate controllability of (1.1).

4. Conclusion

We consider the following Sobolev-type fractional neutral stochastic differential

equations driven by fractional Brownian motion with infinite delay:

$$\begin{cases} {}^{c}D_t^{\alpha}\left[Lx(t) - G(t, x_t)\right] = Ax(t) + f(t, x_t) + Bu(t) + \sigma(t)\dfrac{\mathrm{d}}{\mathrm{d}t}B^H(t), & t \in (0, T], \\ x(t) = \phi(t), & t \in (-\infty, 0]. \end{cases}$$

where $u(\cdot)$ is a control function. Inspired by [14], we show the existence of solution and approximate controllability of (1.1).

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