

On the Extension of the Three-Term Recurrence Relation to Probabilities Distributions without Moments

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Abstract

In this paper, we extend the three-term recurrence relation for orthogonal polynomials associated with a probability distribution having a finite moment of all orders to a class of orthogonal functions associated with an infinitely divisible probability distribution μ having a finite moments of order less or equal to four. An explicit expression of these functions will be given in term of the Lévy-Khintchine function of the measure μ .

Keywords

Three-Term Recurrence Relation, Quantum Decomposition of Random Variables Without Moments, Lévy-Khintchine Function

1. Introduction

It has been known from [1] and [2] that for every probability distribution μ with finite moments of all orders, there exists a family of monic orthogonal polynomials P_n and a pair of sequences $\alpha_0, \alpha_1, \dots$ and $w_0 = 1, w_1, w_2, \dots > 0$ satisfying the *three-term recurrence relation* (or the *tri-diagonal Jacobi relation*)

$$P_{-1} = 0 \quad (1)$$

$$P_0 = 1 \quad (2)$$

$$xP_n = P_{n+1} + \alpha_n P_n + w_n P_{n-1}, \quad n \in \mathbb{N} := \{0, 1, 2, \dots\} \quad (3)$$

The sequences (α_n) and (w_n) are called the *Szego-Jacobi parameters* of μ .

The starting point of the quantum probabilistic approach to the theory of orthogonal polynomials (OP) is an operator interpretation of the tri-diagonal Jacobi relation (3) in terms of Creation, Annihilation and Preservation (CAP)

operators. This allows to associate, in a canonical way, to any random variable with all moments commutation relations that generalize the Heisenberg commutation relations (corresponding to the Gauss-Poisson class). From the mathematical point of view, this approach has led to some new results in the theory of OP.

In order to give this operator interpretation, we shall recall the notion of the interacting Fock probability space associated with the measure μ (See [3] for more details).

Consider an infinite-dimensional separable Hilbert space \mathcal{H} , in which a complete orthonormal basis $\{\Phi_n; n \in \mathbb{N}\}$ is chosen. Let $\mathcal{H}_0 \subset \mathcal{H}$ denote the dense subspace spanned by the complete orthonormal basis $\{\Phi_n\}$.

Given the sequence $\{w_n, n \in \mathbb{N}\}$, we associate linear operators $B^\pm \in L(\mathcal{H}_0)$ given by:

$$B^+ \Phi_n = \sqrt{w_{n+1}} \Phi_{n+1}$$

$$B^- \Phi_0 = 0, \quad B^- \Phi_n = \sqrt{w_n} \Phi_{n-1}, \quad n \geq 1.$$

It is known that B^\pm are mutually adjoint and the linear subspace $\Gamma \subset \mathcal{H}_0$ spanned by the set $\{(B^+)^n \Phi_0, n = 0, 1, 2, \dots\}$ is invariant under the action of B^\pm .

The quadruple $\{\Gamma, (\Phi_n)_n, B^\pm\}$ is called the interacting Fock probability space associated with μ . The operators B^+ and B^- are called the creation operator and the annihilation operators respectively. The linear operator given by

$$N \Phi_n = n \Phi_n, \quad n = 0, 1, 2, \dots$$

is called the number operator. More generally, with the sequence $\{\alpha_n; n = 1, 2, \dots\}$, we associate the preservation operator $\alpha_N \in L(\Gamma)$ by the prescription

$$\alpha_N \Phi_n = \alpha_n \Phi_n, \quad n = 1, 2, \dots$$

Let $L^2_{\mathbb{C}}(\mathbb{R}, \mu)$ be the space of classes of complex valued, square integrable functions w.r.t μ . In the following, we simply denote it by $L^2(\mu)$ and we assume that the sub-space $\mathcal{P}(\mathbb{R}, \mu) \subset L^2(\mu)$ spanned by the polynomial functions is dense in $L^2(\mu)$. So that $(P_n)_n$ is an Hilbertian basis of $L^2(\mu)$. In such case, we consider the isomorphism U from Γ to $L^2(\mu)$ whose restriction on \mathcal{H}_0 given by:

$$U : \Phi_0 \mapsto P_0, \quad \sqrt{w_n!} \Phi_n \mapsto P_n, \quad n \in \mathbb{N}^* = \{1, 2, \dots\},$$

where $w_n! = w_0 w_1 \dots w_n$. Then the U is unitary and we have

$$qP_n = P_{n+1} + \alpha_n P_n + w_n P_{n-1}$$

$$\Leftrightarrow qU(\sqrt{w_n!} \Phi_n) = U(\sqrt{w_{n+1}!} \Phi_{n+1} + \alpha_n \sqrt{w_n!} \Phi_n + w_n \sqrt{w_{n-1}!} \Phi_{n-1})$$

$$\Leftrightarrow \sqrt{w_n!} U(U^{-1} qU) \Phi_n = \sqrt{w_n!} U(\sqrt{w_{n+1}!} \Phi_{n+1} + \alpha_n \Phi_n + \sqrt{w_n!} \Phi_{n-1})$$

$$\Leftrightarrow U(U^{-1} qU) \Phi_n = U(B^+ \Phi_n + \alpha_N \Phi_n + B^- \Phi_n)$$

$$\Leftrightarrow (U^{-1}qU)\Phi_n = (B^+ + \alpha_N + B^-)\Phi_n$$

$$\Leftrightarrow U^{-1}qU = B^+ + \alpha_N + B^-$$

This means that the field operator $T := B^+ + \alpha_N + B^-$ is the U^{-1} -image of the position operator $q := M_x$ on $L^2(\mu)$ providing, in this way, a new interpretation of the recursion relation driving by OP in term of CAP operators. Since the random variable with distribution μ can be identified, up to stochastic equivalence \equiv , with the position operator q on $L^2(\mu)$, the previous new formulation of the tri-diagonal Jacobi relation in term of the CAP operators is called the *quantum decomposition* of the classical random variable. In fact we have seen that

$$q \equiv T = B^+ + \alpha_N + B^-$$

This shows that any classical random variable has a built in non commutative structure which is intrinsic and canonical, and not artificially put by hands, that is a sum of three non commuting random variables.

This result motivated the apparition of a series of papers [4]-[9] dealing in the same context and provided many applications in the theory of quantum probability. In the paper [4], a similar result was obtained but for the family of random variables having an infinitely divisible distribution (I.D-distribution in the following) and having only the moment of the second order. Here, similarity means that the quantum decomposition can be obtained also for this family of random variables.

Based on the notion of the positive definite kernel and using the Lévy-Khintchine function established for the I.D-distributions, the paper [4] constructed a natural isomorphism U from the Fock space $\Gamma(L^2(\nu))$ over the L^2 -space w.r.t the Lévy measure ν to the space $L^2(\mu)$. Then the U^{-1} -image of the position operator q is the field operator

$$Q = A^+(q) + A^-(q) + \Lambda(q) + E(X)\mathbf{1} \tag{4}$$

where q is the function $(t \mapsto q(t) := (q \cdot 1)(t) = t) \in L^2(\nu)$. See papers [10] and [11] in which the operator Q was widely studied.

In this approach, the construction was not based on the orthogonal polynomials sequence associated with μ . But it required only the infinite divisibility property, where the Lévy-Khinchine function have played an important role. Then one can ask about the analytic form of the relation (4), or equivalently the counterpart of the three-term recurrence relation. The only obscure point is the existence of such an analogue of the sequence of the orthogonal polynomials. Since the hypothesis on moments is not satisfied, such a sequence of orthogonal polynomial does exist. But the isomorphism U provided us a such chaos-decomposition of the space $L^2(\mu)$. For this reason we ask the question if there exist a such analogue for the family of orthogonal polynomial, if it is the case it must be a total family of orthogonal functions in the space $L^2(\mu)$ satisfying a recursion relation similar to the well known for OP.

This paper is organized as follows:

In Section 2, we recall some known facts about the bosonic Fock space and the quantum decomposition of classical random variables without moments, having I.D-distributions, obtained in [12] [4] and [5]. In Section 3, we compute the action of the generalized field operator $Q = U^{-1}qU$ on the n^{th} particle vectors $(Q_{n,\alpha})$. The main result of this paper will be given in Section 4, so that we compute the action of the position operator $q = M_x$ on the orthogonal functions $E_{n,\alpha}$. This provide such a generalization of the tri-diagonal recursion relation for OP. Finally, the explicit form of these functions will be given.

2. Preliminaries

2.1. The Bosonic Fock Space

Let \mathcal{H} be a separable Hilbert space. Let us denote $\mathcal{H}^{\otimes n}$ (resp. $f^{\otimes n}$) the tensor product of n -copies of \mathcal{H} (resp. $f \in \mathcal{H}$) and let u_σ be the unique unitary operator such that

$$u_\sigma : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}, \quad f_1 \otimes \dots \otimes f_n \mapsto f_{\sigma(1)} \otimes \dots \otimes f_{\sigma(n)}$$

where $\sigma \in \mathfrak{S}_n$ is a permutation of n -variables.

Let $\mathcal{H}^{\otimes 0} := \mathbb{C}\Phi$, where Φ is the vacuum vector, let

$$S_n := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} u_\sigma : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}$$

be the orthogonal projection.

We define

$$\Gamma(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n},$$

where $\mathcal{H}^{\otimes n} = S_n(\mathcal{H}^{\otimes n})$.

Let us denote

$$Exp(f) := \sum_{n=0}^{\infty} \frac{f^{\otimes n}}{\sqrt{n!}}, \quad f \in \mathcal{H}.$$

Then $\langle Exp(f), Exp(g) \rangle = e^{\langle f, g \rangle}$. Moreover, the set $\{Exp(f) : f \in \mathcal{H}\}$ is linearly independent dense in $\Gamma(\mathcal{H})$.

The *bosonic creation* and *annihilation* operators are defined, on the total set

$$\{S_n(v_1 \otimes \dots \otimes v_n) \in \mathcal{H}^{\otimes n} : v_1, \dots, v_n \in \mathcal{H}\}$$

as follows:

For $u \in \mathcal{H}$,

$$\begin{aligned} A^+(u) : \mathcal{H}^{\otimes n} &\rightarrow \mathcal{H}^{\otimes(n+1)} \\ S_n(v_1 \otimes \dots \otimes v_n) &\mapsto \sqrt{n+1} S_{n+1}(u \otimes v_1 \otimes \dots \otimes v_n), \end{aligned} \tag{5}$$

and

$$A^+(u)\Phi = u$$

$$A^-(u) : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes(n-1)}$$

$$S_n(v_1 \otimes \dots \otimes v_n) \mapsto \frac{1}{\sqrt{n}} \sum_{i=1}^n \langle u, v_i \rangle S_{n-1}(v_1 \otimes \dots \otimes \widehat{v}_i \otimes \dots \otimes v_n), \quad (6)$$

and

$$A^-(u)\Phi = 0,$$

where $\widehat{}$ denotes omission of the corresponding variable. The preservation operator associated with the self adjoint operator T on \mathcal{H} is given by:

$$\Lambda(T) : S_n(v_1 \otimes \dots \otimes v_n) \in \mathcal{H}^{\otimes n} \mapsto \sum_{i=1}^n S_n(v_1 \otimes \dots \otimes T v_i \otimes \dots \otimes v_n) \in \mathcal{H}^{\otimes n} \quad (7)$$

2.2. The Quantum Decomposition of Classical Random Variables with I.D-Distributions

In this section, we recall briefly, what has been obtained in the paper [4] around quantum decomposition of random variables with I.D-distributions and having a finite second order moment.

Let us consider a random variable X with I.D-probability distribution μ having a finite second order moment. It is known (see [13]), that the Fourier transform of μ given by

$$\widehat{\mu}(x) = e^{\Psi(x)}; \quad x \in \mathbb{R}, \quad (8)$$

where Ψ is given by

$$\Psi(x) = i\gamma x - \frac{\sigma^2}{2} x^2 + \int_{\mathbb{R} \setminus \{0\}} \left(e^{ixt} - 1 - \frac{ixt}{1+t^2} \right) \nu(dt); \quad x \in \mathbb{R} \quad (9)$$

such that $\gamma, \sigma \in \mathbb{R}$ and ν is the the Lévy measure of μ . The function Ψ is called the Lévy-Khintchine function or the characteristic exponent associated with μ .

Since the second order moment of μ is finite, the same result will be true for ν , i.e.:

$$\int_{\mathbb{R} \setminus \{0\}} |t|^2 \nu(dt) < +\infty. \quad (10)$$

We suppose also that the gaussian part of μ is null (i.e., $\sigma = 0$). Under these conditions, we have the following results:

The family $\{e_x : \mathbb{R} \rightarrow \mathbb{C}, t \mapsto e^{itx}\}$ of the trigonometric functions is total in $L^2(\mu)$ and the family of the functions

$$f_x(t) := e^{itx} - 1, \quad x \in \mathbb{R} \quad (11)$$

is total in $L^2(\nu)$.

Then by applying the Araki-Woods-Parthasarathy-Schmidt isomorphism in [12] for the infinitely divisible positive definite kernel

$$k(x, y) := e^{\Psi(y-x)}, \quad x, y \in \mathbb{R},$$

we have proved the following theorem (See [4] for more details and descriptions).

Theorem 2.1. *The unique linear operator U given on the exponential vectors*

$\{Exp(f_x), x \in \mathbb{R}\}$ by:

$$U : \Gamma(L^2(\nu)) \rightarrow L^2(\mu)$$

$$Exp(f_x) \mapsto U(Exp(f_x)) := e^{-\psi(x)} e_x \tag{12}$$

is an unitary isomorphism from the Fock space $\Gamma(L^2(\nu))$ to $L^2(\mu)$.

Definition 1. Let q be the multiplication (position) operator in $L^2(\mu)$:

$$(qf)(t) := tf(t); \quad f \in L^2(\mu), t \in \mathbb{R}$$

Define the operator Q on $\Gamma(L^2(\nu))$ by

$$Q := U^* q U$$

where U is the isomorphism defined by (12). Since μ is a finite measure on \mathbb{R} , the operator q is self-adjoint (see [14] Proposition 1, chapter VIII. 3) and

$$e^{iQ} = U^* e^{iq} U, \quad t \in \mathbb{R}$$

The operator Q is called the *generalized field operator*.

It follows from condition (10) that the total set $\{Exp(f_y), y \in \mathbb{R}\}$ is in the domain of Q . Moreover, one has the following theorem:

Theorem 2.2. Let q be the function given by

$$q(t) = t, \quad t \in \mathbb{R}.$$

Then the generalized field operator Q takes the form

$$Q = A_v^+(q) + A_v^-(q) + \Lambda_v(q) + m\mathbf{1}, \tag{13}$$

where $m = E(X)$, the expectation of X , and A_v^+, A_v^-, Λ_v are the creation, annihilation and preservation operators in the Fock space $\Gamma(L^2(\nu))$ given by the prescriptions as in (5)-(7).

3. The Generalized Field Operator

3.1. Notations

We denote by τ the set of all sequences of non negatives integers with finite number of nonzero entries. In the sequel \mathbb{N}^n (resp. \mathbb{C}^n) will be interpreted as subset of the set τ (resp. $l^2(\mathbb{C})$). Throughout the remain of this paper we shall use the following notations:

For $z = (z_1, \dots, z_n, z_{n+1}, \dots) \in l^2(\mathbb{C})$, $\alpha = (\alpha_1, \dots, \alpha_n, 0, 0, \dots)$ and $\beta = (\beta_1, \dots, \beta_n, 0, 0, \dots) \in \tau$,

$$\|z\|^2 = \sum_{k=1}^{\infty} |z_k|^2; \quad z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}; \quad \alpha! = \alpha_1! \dots \alpha_n!$$

$$\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n, \dots); \quad |\alpha| = \sum_{k=1}^{\infty} \alpha_k; \quad e_\alpha = \frac{z^\alpha}{\sqrt{\alpha!}}$$

The support of such element $\alpha \in \tau$ is defined by

$$\pi(\alpha) := \{j \in \mathbb{N}^* : \alpha_j \neq 0\}$$

When $g = (g_1, \dots, g_m, g_{m+1}, \dots)$ is a sequence of elements of an Hilbert space

\mathcal{H} and $\alpha = (\alpha_1, \dots, \alpha_m, 0, 0, \dots) \in \tau$, we denote

$$g^{\otimes \alpha} := g_1^{\otimes \alpha_1} \otimes \dots \otimes g_m^{\otimes \alpha_m} \in \mathcal{H}^{|\alpha|}$$

In particular if $\pi(\alpha) = i_1, \dots, i_s$ and $|\alpha| = n$, so that α takes the form

$$\alpha = (0, \dots, 0, \alpha_{i_1}, 0, \dots, 0, \alpha_{i_2}, 0, \dots, 0, \alpha_{i_s}, 0, 0, \dots) \quad \text{with } \alpha_{i_1} + \dots + \alpha_{i_s} = n,$$

then

$$g^{\otimes \alpha} = \bigotimes_{j \in \pi(\alpha)} g_j^{\otimes \alpha_j} = \bigotimes_{j=1}^s g_{i_j}^{\otimes \alpha_{i_j}}$$

From [15], we recall the following identity which is the analogue of the multinomial Newton formula

$$\left(\sum_{k=1}^{\infty} g_k \right)^{\otimes n} = \sum_{\alpha \in \tau, |\alpha|=n} \frac{n!}{\alpha!} S_n(g^{\otimes \alpha}), \tag{14}$$

which take place whenever the series $\sum_{j=1}^{\infty} g_j$ is convergent.

If \mathcal{H} is separable and $\{g_k\}_{k \geq 1}$ is an Hilbertian basis of it, then the set

$$\left\{ Q_n(\alpha) := \sqrt{\frac{n!}{\alpha!}} S_n(g^{\otimes \alpha}) : |\alpha| = n, n \in \mathcal{H} \right\}$$

is an orthonormal basis of the Hilbert space $\mathcal{H}^{\otimes n}$, with the convention $Q_0 = \Phi$.

Let $\{\delta_j, j = 1, 2, \dots\}$ be the canonic basis of $l^2(\mathbb{C})$. For $\alpha \in \tau$, we denote

$$\alpha + k := \alpha + \delta_k = (\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \alpha_k + 1, \alpha_{k+1}, \dots), \quad k = 1, 2, \dots \tag{15}$$

and

$$\alpha - k := \alpha - \delta_k = (\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \alpha_k - 1, \alpha_{k+1}, \dots), \quad k \in \pi(\alpha) \subset \mathbb{N}^*. \tag{16}$$

Note that if $k \notin \pi(\alpha)$, then $\alpha - k$ can be defined as in (16), however it is not an element of τ , because its k^{th} -entry $(\alpha - k)_k = -1$. In this case, we adapt by convention that

$$Q_n(\alpha - k) = 0. \tag{17}$$

Finally, we recall that

$$(\alpha + k)! = (\alpha_k + 1)\alpha!, \quad k \in \mathbb{N}^* \tag{18}$$

$$(\alpha - k)! = \frac{\alpha!}{\alpha_k}, \quad k \in \pi(\alpha) \tag{19}$$

3.2. Computation of the Action of the Generalized Field Operator on the Basis $(\Phi_n)_n$

In the remain, we take $\mathcal{H} = L^2(\nu)$ and we assume that second order moment

of ν is finite. Let g_1 be the function given by $g_1(t) := \frac{q(t)}{\nu(q)}$, where

$$\nu(q) := \left(\int_{\mathbb{R}} t^2 \nu(dt) \right)^{\frac{1}{2}}.$$

Since the set $\{f_x, x \in \mathbb{R}\}$ is total in $L^2(\nu)$ (See (11)), then

$\mathcal{E} := \{g_1\} \cup \{f_x, x \in \mathbb{Q}\}$ is also total. Then by the Gram-Schmidt procedure, we construct an Hilbertian basis of it, that is denoted by

$$g = (g_1, g_2, \dots, g_k, \dots) \tag{20}$$

Lemma 3.1. *If the 4th-moment of ν is finite then $g_k \in \text{Dom}(q) \subset L^2(\nu)$ for all $k = 1, 2, \dots$.*

Proof. We have

$$\int_{\mathbb{R}} |(qg_1)(t)|^2 \nu(dt) = \int_{\mathbb{R}} \left| \frac{t^2}{\nu(q)} \right|^2 \nu(dt) = \frac{1}{(\nu(q))^2} \int_{\mathbb{R}} t^4 \nu(dt) < +\infty$$

Then $g_1 \in \text{Dom}(q)$. Since $g_k \in \text{Span}\{g_1, f_{x_j}, j = 1, 2, \dots\}$ for all $k = 2, 3, \dots$, then it is sufficient to prove that $f_x \in \text{Dom}(q)$.

We have

$$\int_{\mathbb{R}} |(qf_x)(t)|^2 \nu(dt) = \int_{\mathbb{R}} t^2 |e^{ixt} - 1|^2 \nu(dt) \leq \int_{\mathbb{R}} 4t^2 \nu(dt) < +\infty,$$

where we have used the condition (10).

$$qg_j = \sum_{k=1}^{\infty} \langle g_k, qg_j \rangle g_k$$

□

Proposition 3.1. *Let $\{R_n(\alpha), n \in \mathbb{N}, \alpha \in \tau : |\alpha| = n, n \in \mathbb{N}\}$ be the orthogonal basis of $\Gamma(L^2(\nu))$ given by*

$$R_n(\alpha) := \sqrt{\frac{n!}{\alpha!}} Q_n(\alpha) = \frac{n!}{\alpha!} S_n(g^{\otimes \alpha}); \quad R_0 := \Phi,$$

where $g = (g_1, g_2, \dots, g_k, \dots)$ is the basis given by (20). Then we have

$$A^+(q)R_n(\alpha) = \frac{\nu(q)(\alpha_1 + 1)}{\sqrt{n+1}} R_{n+1}(\alpha + 1) \tag{21}$$

$$A^-(q)R_n(\alpha) = \nu(q)\sqrt{n}R_{n-1}(\alpha - 1) \tag{22}$$

$$\Lambda(q)R_n(\alpha) = s_\alpha R_n(\alpha) + \sum_{j \in \pi(\alpha)} \sum_{k \geq 1, k \neq j}^{\infty} (\alpha_k + 1) \langle g_k, qg_j \rangle R_n(\alpha - j + k) \tag{23}$$

where $s_\alpha = \sum_{j \in \pi(\alpha)} \alpha_j \langle g_j, qg_j \rangle$.

Remark 1. *Note that the relation (22) still true in the case when $1 \notin \pi(\alpha)$ with convention that $R_{n-1}(\alpha - 1) = 0$.*

Proof. From (5), we have

$$\begin{aligned} A^+(q)R_n(\alpha) &= A^+(\nu(q)g_1)R_n(\alpha) \\ &= \nu(q)A^+(g_1)R_n(\alpha) \\ &= \nu(q)\frac{n!}{\alpha!}A^+(g_1)S_n(g^{\otimes \alpha}) \\ &= \nu(q)\frac{n!\sqrt{n+1}}{\alpha!}S_{n+1}(g_1 \otimes g^{\otimes \alpha}) \\ &= \nu(q)\frac{n!\sqrt{n+1}}{\alpha!}S_{n+1}(g^{\otimes(\alpha+1)}) \end{aligned}$$

$$\begin{aligned}
 &= \nu(q) \frac{n! \sqrt{n+1} (\alpha+1)! (n+1)!}{\alpha! (n+1)! (\alpha+1)!} S_{n+1} (g^{\otimes(\alpha+1)}) \\
 &= \nu(q) \frac{n! \sqrt{n+1} (\alpha+1)!}{\alpha! (n+1)!} R_{n+1} (\alpha+1) \\
 &= \nu(q) \frac{n! \sqrt{n+1} (\alpha_1+1) \alpha!}{\alpha! (n+1)!} R_{n+1} (\alpha+1) \\
 &= \frac{\nu(q) (\alpha_1+1)}{\sqrt{n+1}} R_{n+1} (\alpha+1).
 \end{aligned}$$

This prove (21).

From (5), we have

$$\begin{aligned}
 A^-(q)R_n(\alpha) &= A^-(\nu(q)g_1)R_n(\alpha) \\
 &= \nu(q)A^-(g_1)R_n(\alpha) \\
 &= \nu(q)\frac{n!}{\alpha!}A^-(g_1)S_n(g^{\otimes\alpha}) \\
 &= \nu(q)\frac{n!}{\alpha!}\frac{1}{\sqrt{n}}\sum_{j \in \pi(\alpha)}\alpha_j\langle g_1, g_j \rangle S_{n-1}(g^{\otimes(\alpha-j)})
 \end{aligned} \tag{24}$$

Here, we have two cases:

If $1 \in \pi(\alpha)$, then (24), becomes

$$\begin{aligned}
 A^-(q)R_n(\alpha) &= \nu(q)\frac{n!}{\alpha!}\frac{1}{\sqrt{n}}\alpha_1\langle g_1, g_1 \rangle S_{n-1}(g^{\otimes(\alpha-1)}) \\
 &= \nu(q)\frac{n!}{\alpha!}\frac{1}{\sqrt{n}}\alpha_1\frac{(\alpha-1)!}{(n-1)!}R_{n-1}(\alpha-1) \\
 &= \nu(q)\sqrt{n}R_{n-1}(\alpha-1)
 \end{aligned} \tag{25}$$

If $1 \notin \pi(\alpha)$, then $\langle g_1, g_j \rangle = 0$ for all $j \in \pi(\alpha)$. Therefore (24) gives

$$A^-(q)R_n(\alpha) = 0.$$

But in view of (17), we have $R_{n-1}(\alpha-1) = 0$ which gives that the relation (25) sill true. Hence (22) is proved.

Now, it remains to justify (23). From (7), we get

$$\Lambda(q)R_n(\alpha) = \frac{n!}{\alpha!} \sum_{j \in \pi(\alpha)} \alpha_j S_n(qg_j \otimes g^{\otimes(\alpha-j)}). \tag{26}$$

Since $qg_j \in L^2(\nu)$, then it can be written as follows:

$$qg_j = \sum_{k=1}^{\infty} \langle g_k, qg_j \rangle g_k$$

Using the fact that S_n is bounded, the Equation (26) becomes

$$\begin{aligned}
 \Lambda(q)R_n(\alpha) &= \frac{n!}{\alpha!} \sum_{j \in \pi(\alpha)} \alpha_j \sum_{k=1}^{\infty} \langle g_k, qg_j \rangle S_n(g_k \otimes g^{\otimes(\alpha-j)}) \\
 &= \sum_{j \in \pi(\alpha)} \sum_{k=1}^{\infty} \langle g_k, qg_j \rangle \frac{\alpha_j (\alpha-j+k)!}{\alpha!} \frac{n!}{(\alpha-j+k)!} S_n(g_k \otimes g^{\otimes(\alpha-j)}) \tag{27} \\
 &= \sum_{j \in \pi(\alpha)} \sum_{k=1}^{\infty} (\alpha-j+k)! \frac{\alpha_j}{\alpha!} \langle g_k, qg_j \rangle R_n(\alpha-j+k)
 \end{aligned}$$

But we have for $j \in \pi(\alpha)$,

$$(\alpha - j + k)! = \begin{cases} \frac{(\alpha_k + 1)\alpha!}{\alpha_j}, & k \neq j; \\ \alpha!, & k = j. \end{cases} \tag{28}$$

Then (27) becomes

$$\begin{aligned} \Lambda(q)R_n(\alpha) &= \sum_{j \in \pi(\alpha)} \sum_{k \geq 1, k \neq j} \frac{(\alpha_k + 1)\alpha!}{\alpha_j} \frac{\alpha_j}{\alpha!} \langle g_k, qg_j \rangle R_n(\alpha - j + k) \\ &\quad + \sum_{j \in \pi(\alpha)} \alpha! \frac{\alpha_j}{\alpha!} \langle g_j, qg_j \rangle R_n(\alpha) \\ &= \sum_{j \in \pi(\alpha)} \sum_{k \geq 1, k \neq j} (\alpha_k + 1) \langle g_k, qg_j \rangle R_n(\alpha - j + k) \\ &\quad + \left(\sum_{j \in \pi(\alpha)} \alpha_j \langle g_j, qg_j \rangle \right) R_n(\alpha) \end{aligned}$$

This ends the proof. \square

Corollary 3.1.1 *The action of the generalized field operator Q on the basis $\{R_n(\alpha); n, \alpha\}$ is given as follows:*

$$\begin{aligned} QR_n(\alpha) &= \frac{\nu(q)(\alpha_1 + 1)}{\sqrt{n+1}} R_{n+1}(\alpha + 1) + (m + s_\alpha) R_n(\alpha) + \nu(q)\sqrt{n} R_{n-1}(\alpha - 1) \\ &\quad + \sum_{j \in \pi(\alpha)} \sum_{k \geq 1, k \neq j} (\alpha_k + 1) \langle g_k, qg_j \rangle R_n(\alpha - j + k) \end{aligned} \tag{29}$$

Proof. A straightforward computations. \square

4. Orthogonal Functions and Generalization of the Three-Term Recurrence Relation

In this section, we give the action of the multiplication operator q on the functions

$$E_{n,\alpha} := U(R_n(\alpha)), n \in \mathbb{N}, \alpha \in \tau, |\alpha| = n \in \mathbb{N}.$$

Then we deduce the generalization of the three-term recurrence relation in term of the orthogonal functions $E_{n,\alpha}$.

Since U is unitary from $\Gamma(L^2(\nu))$ to $L^2(\mu)$ and $\{R_n(\alpha)\}$ is an orthogonal basis of $\Gamma(L^2(\nu))$, the family $\{E_{n,\alpha} = U(R_n(\alpha)), n, \alpha\}$ is an orthogonal basis of $L^2(\mu)$.

Theorem 4.1 *Let $L_n^2(\mu) := U(L^2(\nu)^{\otimes n})$ and let D_n be the diagonal operator from $L_n^2(\mu)$ to itself given by*

$$D_n(E_{n,\alpha}) := \sum_{j \in \pi(\alpha)} \sum_{k \geq 1, k \neq j} (\alpha_k + 1) \langle g_k, qg_j \rangle E_{n,\alpha - j + k}.$$

Then for all $n \in \mathbb{N}, \alpha \in \tau, |\alpha| = n$, we have

$$qE_{n,\alpha} = \frac{\nu(q)(\alpha_1 + 1)}{\sqrt{n+1}} E_{n+1,\alpha+1} + (m + s_\alpha) E_{n,\alpha} + \nu(q)\sqrt{n} E_{n-1,\alpha-1} + D_n(E_{n,\alpha}) \tag{30}$$

Remark 2. Since U is unitary and the basis $\{R_n(\alpha), n \in \mathbb{N}, \alpha \in \tau, |\alpha| = n\}$ is

orthogonal, then $\{E_{n,\alpha} = U(R_n(\alpha)), n \in \mathbb{N}^*, \alpha \in \tau, |\alpha| = n\}$ is an orthogonal basis of $L^2(\mu)$. Moreover, the chaos decomposition of the Fock space $\Gamma(L^2(\nu))$ induces the following chaos-decomposition of the space $L^2(\mu)$

$$L^2(\mu) = \bigoplus_{n=0}^{\infty} L_n^2(\mu).$$

Now comparing the relation (30) with (3), the only difference is the apparition of a corrective expression $D_n(E_{n,\alpha})$ in (30) which is in the n^{th} chaos. In the case when it is null, (30) will be exactly the well-known tri-diagonal recurrence relation (3). In this sense the relation (30) can be interpreted as a generalization of the three term recurrence relation. Here, the monic orthogonal polynomial sequence is replaced by a double-entries sequence of orthogonal functions parameterized by $n \in \mathbb{N}$ and $\alpha \in \tau$. In addition to the infinite divisibility property, this generalization require only the existence of the second and fourth order moments of μ .

Proof. From relation (29), we deduce that

$$\begin{aligned} qE_{n,\alpha} &= qU(R_n(\alpha)) = U(U^*qU)(R_n(\alpha)) = UQ_{Cb}R_n(\alpha) \\ &= U\left(\frac{\nu(q)(\alpha_1+1)}{\sqrt{n+1}}R_{n+1}(\alpha+1) + (m+s_\alpha)R_n(\alpha) + \nu(q)\sqrt{n}R_{n-1}(\alpha-1) \right. \\ &\quad \left. + \sum_{j \in \pi(\alpha)} \sum_{k \geq 1, k \neq j} (\alpha_k+1)\langle g_k, qg_j \rangle R_n(\alpha-j+k) \right) \\ &= \frac{\nu(q)(\alpha_1+1)}{\sqrt{n+1}}U(R_{n+1}(\alpha+1)) + (m+s_\alpha)U(R_n(\alpha)) \\ &\quad + \nu(q)\sqrt{n}U(R_{n-1}(\alpha-1)) + \sum_{j \in \pi(\alpha)} \sum_{k \geq 1, k \neq j} (\alpha_k+1)\langle g_k, qg_j \rangle U(R_n(\alpha-j+k)) \\ &= \frac{\nu(q)(\alpha_1+1)}{\sqrt{n+1}}E_{n+1,\alpha+1} + (m+s_\alpha)E_{n,\alpha} + \nu(q)\sqrt{n}E_{n-1,\alpha-1} \\ &\quad + \sum_{j \in \pi(\alpha)} \sum_{k \geq 1, k \neq j} (\alpha_k+1)\langle g_k, qg_j \rangle E_{n,\alpha-j+k} \\ &= \frac{\nu(q)(\alpha_1+1)}{\sqrt{n+1}}E_{n+1,\alpha+1} + (m+s_\alpha)E_{n,\alpha} + \nu(q)\sqrt{n}E_{n-1,\alpha-1} + D_n(E_{n,\alpha}) \end{aligned}$$

Proposition 4.2. *We assume that μ is continuous w.r.t the Lebesgue measure with Radon-Nikodym derivative ρ . Then for all $n \in \mathbb{N}$ and $\alpha \in \tau$, one has*

$$E_{n,\alpha} = \frac{\sqrt{n!}}{\alpha! \rho} \mathcal{F}^{-1}\left(e^{\psi(\cdot)}(\nu(\cdot))^\alpha\right), \tag{31}$$

where,

$$\nu(x) := (\langle g_1, f_x \rangle, \langle g_2, f_x \rangle, \langle g_3, f_x \rangle, \dots) \in l^2(\mathbb{C})$$

Proof. Since $\{g_k, k \in \mathbb{N}^*\}$ is an Hilbertian basis of $L^2(\nu)$ and $f_x \in L^2(\nu)$,

$$f_x = \sum_{k=1}^{\infty} \langle g_k, f_x \rangle g_k, \quad x \in \mathbb{R},$$

where the series converge in $L^2(\nu)$. It follows, from the multinomial Newton formula (14), that

$$\begin{aligned} f_x^{\otimes n} &= \left(\sum_{k=1}^{\infty} \langle g_k, f_x \rangle g_k \right)^{\otimes n} \\ &= \sum_{\alpha \in \tau, |\alpha|=n} \frac{n!}{\alpha!} S_n \left((\nu(x))^\alpha g^{\otimes \alpha} \right) \\ &= \sum_{\alpha \in \tau, |\alpha|=n} (\nu(x))^\alpha \frac{n!}{\alpha!} S_n \left(g^{\otimes \alpha} \right) \\ &= \sum_{\alpha \in \tau, |\alpha|=n} (\nu(x))^\alpha R_n(\alpha), \end{aligned}$$

where

$$\nu(x) := (\nu_1, \nu_2, \dots) = (\langle g_1, f_x \rangle, \langle g_2, f_x \rangle, \langle g_3, f_x \rangle, \dots) \in l^2(\mathbb{C})$$

This implies that

$$\text{Exp}(f_x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} f_x^{\otimes n} = \sum_{n=0}^{\infty} \sum_{\alpha \in \tau, |\alpha|=n} \frac{(\nu(x))^\alpha}{\sqrt{n!}} R_n(\alpha)$$

From the definition of U , we get

$$\begin{aligned} e_x &= e^{\psi(x)} U(\text{Exp}(f_x)) \\ &= e^{\psi(x)} \sum_{n=0}^{\infty} \sum_{\alpha \in \tau, |\alpha|=n} \frac{(\nu(x))^\alpha}{\sqrt{n!}} U(R_n(\alpha)) \\ &= e^{\psi(x)} \sum_{n=0}^{\infty} \sum_{\alpha \in \tau, |\alpha|=n} \frac{(\nu(x))^\alpha}{\sqrt{n!}} E_{n,\alpha}, \end{aligned}$$

which is the decomposition of e_x in the basis $\{E_{n,\alpha}, n \in \mathbb{N}, \alpha \in \tau\}$. Then

$$\begin{aligned} \langle E_{n,\alpha}, e_x \rangle &= \left\langle E_{n,\alpha}, e^{\psi(x)} \sum_{m=0}^{+\infty} \sum_{\beta \in \tau, |\beta|=m} \frac{(\nu(x))^\beta}{\sqrt{m!}} E_{m,\beta} \right\rangle \\ &= e^{\psi(x)} \frac{(\nu(x))^\alpha}{\sqrt{n!}} \|E_{n,\alpha}\|^2 \\ &= e^{\psi(x)} \frac{(\nu(x))^\alpha}{\sqrt{n!}} \frac{n!}{\alpha!} \\ &= \frac{\sqrt{n!}}{\alpha!} e^{\psi(x)} (\nu(x))^\alpha. \end{aligned}$$

On the other hand, we have

$$\langle E_{n,\alpha}, e_x \rangle = \int_{\mathbb{R}} \overline{E_{n,\alpha}(t)} e^{ixt} \mu(dt) = \int_{\mathbb{R}} \overline{E_{n,\alpha}(t)} e^{ixt} \rho(t) dt.$$

This implies that

$$\int_{\mathbb{R}} \rho(t) \overline{E_{n,\alpha}(t)} e^{ixt} dt = \frac{\sqrt{n!}}{\alpha!} e^{\psi(x)} (\nu(x))^\alpha$$

or equivalently

$$\mathcal{F}(\overline{\rho E_{n,\alpha}})(x) = \frac{\sqrt{n!}}{\alpha!} e^{w(x)} (\nu(x))^\alpha,$$

where \mathcal{F} denotes the Fourier transform on $L^1(\nu)$. Note that the function $\rho E_{n,\alpha}$ belongs to the space $L^1(\nu)$. It follows that

$$\overline{\rho E_{n,\alpha}} = \mathcal{F}^{-1} \left(\frac{\sqrt{n!}}{\alpha!} e^{w(\cdot)} (\nu(\cdot))^\alpha \right)$$

□

which is equivalent to

$$E_{n,\alpha} = \frac{\sqrt{n!}}{\alpha! \rho} \overline{\mathcal{F}^{-1} \left(e^{w(\cdot)} (\nu(\cdot))^\alpha \right)}.$$

5. Conclusion

The infinite-divisibility of the distribution μ gives rise to the Kolmogorov isomorphism U , which was the principal bridge between the Fock space $\Gamma(L^2(\nu))$ and $L^2(\mu)$ transforming, in such canonical way, the quantum decomposition identity to the tri-diagonal recurrence relation.

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