

# New Oscillatory Theorems for Third-Order Nonlinear Delay Dynamic Equations on Time Scales

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## Abstract

This paper is concerned with the oscillatory properties of the third-order nonlinear delay dynamic equations of the form

$$\left( r_2(t) \left[ \left( r_1(t) x^\Delta(t) \right)^\Delta \right]^\alpha \right)^\Delta + q(t) f(x[\tau(t)]) = 0, \quad t \in \mathbb{T}$$

on time scales  $\mathbb{T}$ , where  $\alpha \geq 1$  is a quotient of odd positive integers. Applying the inequality technique we present two new sufficient conditions which ensure that every solution of equations is oscillatory or converges to zero. The results obtained improve and complement some known results in the literature.

## Keywords

Oscillatory, Third Order, Delay Dynamic Equation, Generalized Riccati Transformation, Time Scale

## 1. Introduction

Beginning with the landmark contribution work of Hilger [1], the time scales theory, which in order to unify the continuous and discrete analysis, has received significant attention. In the recent years, there has been increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of various equations on time scales; we refer the reader to the papers [2]-[18]. Following this trend, we shall consider oscillation for the third-order nonlinear delay dynamic equation

$$\left( r_2(t) \left[ \left( r_1(t) x^\Delta(t) \right)^\Delta \right]^\alpha \right) + q(t) f(x[\tau(t)]) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0, \quad (1)$$

where  $\alpha \geq 1$  is a quotient of odd positive integers.

Throughout this paper, assume that

(H<sub>1</sub>)  $\mathbb{T}$  is a time scale (*i.e.*, a nonempty closed subset of the real numbers  $\mathbb{R}$ ) which is unbounded above, and  $t_0 \in \mathbb{T}$  with  $t_0 > 0$ . We define the time scale interval of the form  $[t_0, \infty)_{\mathbb{T}}$  by  $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$ .

(H<sub>2</sub>)  $r_1(t), r_2(t), q(t)$  are positive, real-valued rd-continuous functions defined on  $\mathbb{T}$ , and  $r_1(t), r_2(t)$  satisfy

$$\int_{t_0}^{\infty} \frac{1}{r_1(s)} \Delta s = \infty, \quad \int_{t_0}^{\infty} \left( \frac{1}{r_2(s)} \right)^{\frac{1}{\alpha}} \Delta s = \infty.$$

(H<sub>3</sub>)  $\tau: \mathbb{T} \rightarrow \mathbb{T}$  is a strictly increasing and differentiable function such that

$$\tau(t) \leq t, \quad \lim_{t \rightarrow \infty} \tau(t) = \infty \quad \text{and} \quad \tau(\mathbb{T}) = \mathbb{T}.$$

(H<sub>4</sub>)  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function, and there exists a positive number  $K$  such that  $\frac{f(x)}{x^\alpha} \geq K > 0$  for  $x \neq 0$ .

By a solution of (1), we mean a nontrivial function  $x(t)$  satisfying (1) which has the properties  $x(t) \in C_{rd}^1([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$  for  $T_x \geq t_0$ , and

$r_2(t) \left[ \left( r_1(t) x^\Delta(t) \right)^\Delta \right]^\alpha \in C_{rd}^1([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$ . Our attention is restricted to those

solutions of (1) which satisfy  $\sup\{|x(t)|: t \geq T\} > 0$  for all  $T \geq T_x$ . A solution  $x$  of Equation (1) is said to be oscillatory on  $[T_x, \infty)_{\mathbb{T}}$  if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

If  $\alpha = 1, \tau(t) = t$ , then (1) simplifies to the third-order nonlinear dynamic equation

$$\left( r_2(t) \left[ \left( r_1(t) x^\Delta(t) \right)^\Delta \right] \right) + q(t) f(x(t)) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \quad (2)$$

If, furthermore,  $r_1(t) = r_2(t) = 1$ ,  $f(x) = x, \tau(t) = t$ , then (1) reduces to the third-order linear dynamic equation

$$x^{\Delta\Delta\Delta}(t) + q(t)x(t) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \quad (3)$$

If, in addition,  $\alpha = 1$ , then (1) reduces to the nonlinear delay dynamic equation

$$\left( r_2(t) \left[ \left( r_1(t) x^\Delta(t) \right)^\Delta \right] \right) + q(t) f(x[\tau(t)]) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \quad (4)$$

In [11], Erbe *et al.* established some sufficient conditions which ensure that every solution of Equation (2) is oscillatory or converges to zero. In [12], Erbe *et al.* studied the third-order linear dynamic Equation (3), and they obtained Hille and Nehari type oscillation criteria for the Equation (3). In [16], Han *et al.*

extended and improved the results of [11] [12], meanwhile obtained some oscillatory criteria for the Equation (4). In [18], Gao *et al.* considered the third-order nonlinear dynamic Equation (1). By employing the generalized Riccati transformation and the integral averaging technique, they established three sufficient conditions which ensure that every solution of Equation (1) is oscillatory or converges to zero. On this basis, we continue to study Equation (1). If (4.11) in ([18], Theorem 4.3) is not hold, then we obtain two new sufficient conditions which guarantee that every solution of Equation (1) is oscillatory or converges to zero. Our results will improve some previous results. The usual notation and concepts of the time scales calculus, which will be used throughout the paper, can be found in [19] [20].

## 2. Several Lemmas

**Lemma 1** Assume that  $x(t)$  is an eventually positive solution of (1). Then there exists  $T \in [t_0, \infty)_{\mathbb{T}}$  such that either

$$(I) \quad x(t) > 0, x^\Delta(t) > 0, (r_1(t)x^\Delta(t))^\Delta > 0, t \in [T, \infty)_{\mathbb{T}};$$

or

$$(II) \quad x(t) > 0, x^\Delta(t) < 0, (r_1(t)x^\Delta(t))^\Delta > 0, t \in [T, \infty)_{\mathbb{T}}.$$

The proof is similar to that of ([11], Lemma 1).

**Lemma 2** (see [19], Theorem 1.90) If  $x$  is differentiable, then

$$(x^\gamma)^\Delta = \gamma x^\Delta \int_0^1 [hx^\sigma + (1-h)x]^\gamma dh. \quad (5)$$

**Lemma 3** (see [21], Theorem 41) Assume that  $X$  and  $Y$  are nonnegative real numbers. Then

$$\lambda XY^{\lambda-1} - X^\lambda \leq (\lambda - 1)Y^\lambda \quad \text{for all } \lambda > 1, \quad (6)$$

where the equality holds if and only if  $X = Y$ .

Throughout this paper, for sufficiently large  $T$ , we denote

$$R(t, T) = \int_T^t \left( \frac{1}{r_2(s)} \right)^{\frac{1}{\alpha}} \Delta s.$$

**Lemma 4** Assume that  $x(t)$  is an eventually positive solution of (1) which satisfies case (I) in Lemma 1. Then there exists  $T \in [t_0, \infty)_{\mathbb{T}}$ , such that

$$x^\Delta(t) \geq \frac{R(t, T)}{r_1(t)} r_2^{\frac{1}{\alpha}}(t) (r_1(t)x^\Delta(t))^\Delta, \quad t \in [T, \infty)_{\mathbb{T}}. \quad (7)$$

The proof is similar to that of [18], Lemma 3.4).

**Lemma 5** Assume that  $x(t)$  is an eventually positive solution of (1) which satisfies case (I) in Lemma 1. Furthermore, assume that  $r_1^\Delta(t) \leq 0$  and

$$\int_{t_0}^{\infty} q(s) \tau^\alpha(s) \Delta s = \infty. \quad (8)$$

Then there exists  $T \in [t_0, \infty)_{\mathbb{T}}$  such that  $x(t) > tx^\Delta(t)$ , and  $\frac{x(t)}{t}$  is strictly decreasing on  $[T, \infty)_{\mathbb{T}}$ .

The proof is similar to that of ([16], Lemma 2.3).

**Lemma 6** Assume that  $x(t)$  is an eventually positive solution of (1) which satisfies case (II) in Lemma 1. Furthermore,

$$\int_{t_0}^{\infty} \frac{1}{r_1(t)} \int_t^{\infty} \left[ \frac{1}{r_2(s)} \int_s^{\infty} q(u) \Delta u \right]^{\frac{1}{\alpha}} \Delta s \Delta t = \infty. \tag{9}$$

Then  $\lim_{t \rightarrow \infty} x(t) = 0$ .

**Proof** Assume that  $x(t)$  is an eventually positive solution of (1) which satisfies the case (II) in Lemma 1. Then  $x(t)$  is decreasing and  $\lim_{t \rightarrow \infty} x(t) = l \geq 0$ . We assert that  $l = 0$ . If not, then  $x[\tau(t)] \geq x(t) \geq l > 0$  for  $t \in [t_0, \infty)_{\mathbb{T}}$ . Integrating (1) from  $t$  to  $\infty$ , we get

$$\begin{aligned} & -r_2(t) \left[ (r_1(t)x^\Delta(t))^\Delta \right]^\alpha \\ & \leq -K \int_t^{\infty} q(s) x^\alpha[\tau(s)] \Delta s \leq -Kl^\alpha \int_t^{\infty} q(s) \Delta s, \quad t \in [t_0, \infty)_{\mathbb{T}}. \end{aligned}$$

Hence, we have

$$-(r_1(t)x^\Delta(t))^\Delta \leq -l \left[ \frac{1}{r_2(t)} \int_t^{\infty} Kq(s) \Delta s \right]^{\frac{1}{\alpha}}.$$

Integrating the above inequality from  $t$  to  $\infty$ , we obtain

$$r_1(t)x^\Delta(t) \leq -lK^{\frac{1}{\alpha}} \int_t^{\infty} \left[ \frac{1}{r_2(s)} \int_s^{\infty} q(u) \Delta u \right]^{\frac{1}{\alpha}} \Delta s.$$

Integrating the last inequality again from  $T$  to  $t$ , we have

$$x(t) - x(T) \leq -lK^{\frac{1}{\alpha}} \int_T^t \frac{1}{r_1(s)} \int_s^{\infty} \left[ \frac{1}{r_2(u)} \int_u^{\infty} q(v) \Delta v \right]^{\frac{1}{\alpha}} \Delta u \Delta s.$$

Since condition (9) holds, we obtain  $\lim_{t \rightarrow \infty} x(t) = -\infty$ , which contradicts  $x(t) > 0$ . Hence  $l = 0$ . This completes the proof.

**Lemma 7** (see [22], Theorem 3) Let  $a, b \in \mathbb{T}$  and  $a < b$ , for positive rd-continuous functions  $f, g : [a, b] \rightarrow \mathbb{R}$ , we have

$$\int_a^b |f(s)g(s)| \Delta s \leq \left( \int_a^b |f(s)|^p \Delta s \right)^{\frac{1}{p}} \left( \int_a^b |g(s)|^q \Delta s \right)^{\frac{1}{q}}, \tag{10}$$

where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .

### 3. Main Results

**Theorem 1** Assume that (8) and (9) hold,  $r_1^\Delta(t) \leq 0$ . Furthermore, assume that

there exist functions  $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$ , where  $\mathbb{D} \equiv \{(t, s) : t \geq s \geq T\}$  such that

$$H(t, t) = 0, t \geq T; H(t, s) > 0, t > s \geq T, \tag{11}$$

and  $H$  has a nonpositive continuous  $\Delta$ -partial derivative  $H^{\Delta_s}(t, s)$  with respect to the second variable and satisfies

$$H^{\Delta_s}(\sigma(t), s) + \frac{(\delta^\Delta(s))_+}{\delta(s)} H(\sigma(t), s) = -\frac{h(t, s)}{\delta(s)} H^{\alpha+1}(\sigma(t), s), \tag{12}$$

and, for all sufficiently large  $T$ , that there exists  $T_0 \geq T$ ,

$$0 < \inf_{s \geq T_0} \left[ \liminf_{t \rightarrow \infty} \frac{H(\sigma(t), s)}{H(\sigma(t), T_0)} \right] \leq \infty, \tag{13}$$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \frac{h_-^{\alpha+1}(t, s) [\sigma^\alpha(s) r_1(s)]^\alpha}{[s^\alpha \delta(\sigma(s)) R(s, T)]^\alpha} \Delta s < \infty, \tag{14}$$

and a real rd-continuous function  $\Psi : [t_0, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}$  such that

$$\int_T^\infty \frac{s^\alpha \delta(\sigma(s)) R(s, T)}{\delta^{1+\frac{1}{\alpha}}(s) \sigma^\alpha(s) r_1(s)} \Psi_+^{1+\frac{1}{\alpha}}(s) \Delta s = \infty, \tag{15}$$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \left[ H(\sigma(t), s) Q(s) - \frac{h_-^{\alpha+1}(t, s) [\sigma^\alpha(s) r_1(s)]^\alpha}{(\alpha+1)^{\alpha+1} [s^\alpha \delta(\sigma(s)) R(s, T)]^\alpha} \right] \Delta s \geq \Psi(T_0), \tag{16}$$

where  $\delta(t)$  is a positive  $\Delta$ -differentiable function,

$$Q(t) = Kq(t) \delta(\sigma(t)) \left( \frac{\tau(t)}{\sigma(t)} \right)^\alpha.$$

$$h_-(t, s) = \max\{0, -h(t, s)\}, h_+(t, s) = \max\{0, h(t, s)\}, \Psi_+(t) = \max\{0, \Psi(t)\}.$$

Then every solution  $x(t)$  of Equation (1) is either oscillatory or converges to zero.

**Proof** Assume that (1) has a nonoscillatory solution  $x(t)$  on  $[t_0, \infty)_{\mathbb{T}}$ . Without loss generality we may assume that there exists sufficiently large  $T \geq t_0$  such that  $x(t) > 0$  and  $x[\tau(t)] > 0$  for all  $t \in [T, \infty)_{\mathbb{T}}$ . By Lemma 1, we see that  $x(t)$  satisfies either case (I) or case (II).

If case (I) holds, then  $x^\Delta(t) > 0, t \in [T, \infty)_{\mathbb{T}}$ . Define the function  $\omega(t)$  by

$$\omega(t) = \delta(t) r_2(t) \left( \frac{(r_1(t) x^\Delta(t))^\Delta}{x(t)} \right)^\alpha, t \in [T, \infty)_{\mathbb{T}}.$$

Obviously  $\omega(t) > 0$ . Using the product and quotient rule of  $\Delta$ -differential, we obtain

$$\begin{aligned} \omega^\Delta(t) &= \frac{\delta^\Delta(t)}{\delta(t)} \omega(t) - \delta(\sigma(t)) \frac{q(t)f(x[\tau(t)])}{x^\alpha(\sigma(t))} \\ &\quad - \delta(\sigma(t)) r_2(t) \left[ \frac{(r_1(t)x^\Delta(t))^\Delta}{x(t)} \right]^\alpha \frac{(x^\alpha(t))^\Delta}{x^\alpha(\sigma(t))}. \end{aligned}$$

By Lemma 2, we get

$$\begin{aligned} \omega^\Delta(t) &\leq \frac{(\delta^\Delta(t))_+}{\delta(t)} \omega(t) - \delta(\sigma(t)) \frac{q(t)f(x[\tau(t)])}{x^\alpha(\sigma(t))} \\ &\quad - \alpha \delta(\sigma(t)) r_2(t) \left[ \frac{(r_1(t)x^\Delta(t))^\Delta}{x(t)} \right]^\alpha \frac{x^\alpha(t)x^\Delta(t)}{x^\alpha(\sigma(t))x(t)}, \end{aligned}$$

where  $(\delta^\Delta(t))_+ = \max\{0, \delta^\Delta(t)\}$ , and get

$$\begin{aligned} \omega^\Delta(t) &\leq \frac{(\delta^\Delta(t))_+}{\delta(t)} \omega(t) - \delta(\sigma(t)) \frac{q(t)f(x[\tau(t)])}{x^\alpha(\sigma(t))} \\ &\quad - \alpha \delta(\sigma(t)) r_2(t) \left[ \frac{(r_1(t)x^\Delta(t))^\Delta}{x(t)} \right]^\alpha \frac{x^\alpha(t)x^\Delta(t)}{x^\alpha(\sigma(t))x(t)}. \end{aligned}$$

From Lemma 5, we obtain  $\frac{x(\tau(t))}{x(\sigma(t))} \geq \frac{\tau(t)}{\sigma(t)}$ ,  $\frac{x(t)}{x(\sigma(t))} \geq \frac{t}{\sigma(t)}$ , and using (7),

so we obtain

$$\begin{aligned} \omega^\Delta(t) &\leq -K \delta(\sigma(t)) q(t) \left( \frac{\tau(t)}{\sigma(t)} \right)^\alpha + \frac{(\delta^\Delta(t))_+}{\delta(t)} \omega(t) \\ &\quad - \frac{\alpha \delta(\sigma(t)) R(t, T)}{r_1(t)} \left( \frac{t}{\sigma(t)} \right)^\alpha r_2^{\frac{1}{\alpha}}(t) \left[ \frac{(r_1(t)x^\Delta(t))^\Delta}{x(t)} \right]^{\alpha+1}. \end{aligned}$$

Hence, by the definition of  $\omega(t), Q(t)$ , we obtain

$$\omega^\Delta(t) \leq -Q(t) + \frac{(\delta^\Delta(t))_+}{\delta(t)} \omega(t) - \frac{\alpha \delta(\sigma(t)) R(t, T)}{\delta^{\frac{1}{\alpha}}(t) r_1(t)} \left( \frac{t}{\sigma(t)} \right)^\alpha \omega^{\frac{1}{\alpha}}(t). \tag{17}$$

Multiplying both sides of (17), with  $t$  replaced by  $s$ , by  $H(\sigma(t), s)$ , integrating with respect to  $s$  from  $T_0$  to  $\sigma(t)$ ,  $t \geq T_0 \geq T$ , we get

$$\begin{aligned} &\int_{T_0}^{\sigma(t)} H(\sigma(t), s) Q(s) \Delta s \\ &\leq - \int_{T_0}^{\sigma(t)} H(\sigma(t), s) \omega^\Delta(s) \Delta s + \int_{T_0}^{\sigma(t)} \frac{H(\sigma(t), s) (\delta^\Delta(s))_+}{\delta(s)} \omega(s) \Delta s \\ &\quad - \int_{T_0}^{\sigma(t)} \frac{\alpha s^\alpha H(\sigma(t), s) \delta(\sigma(s)) R(s, T)}{\delta^{\frac{1}{\alpha}}(s) \sigma^\alpha(s) r_1(s)} \omega^{\frac{1}{\alpha}}(s) \Delta s. \end{aligned}$$

Integrating by parts and using (11), we obtain

$$\begin{aligned} & \int_{T_0}^{\sigma(t)} H(\sigma(t), s) Q(s) \Delta s \leq H(\sigma(t), T_0) \omega(T_0) \\ & + \int_{T_0}^{\sigma(t)} H^{\Delta s}(\sigma(t), s) \omega(s) \Delta s + \int_{T_0}^{\sigma(t)} \frac{H(\sigma(t), s) (\delta^\alpha(s))_+}{\delta(s)} \omega(s) \Delta s \\ & - \int_{T_0}^{\sigma(t)} \frac{\alpha s^\alpha H(\sigma(t), s) \delta(\sigma(s)) R(s, T)}{\delta^{1+\frac{1}{\alpha}}(s) \sigma^\alpha(s) r_1(s)} \omega^{1+\frac{1}{\alpha}}(s) \Delta s \\ & \leq H(\sigma(t), T_0) \omega(T_0) + \int_{T_0}^{\sigma(t)} \left[ -\frac{h(t, s) H^{\frac{\alpha}{1+\alpha}}(\sigma(t), s)}{\delta(s)} \omega(s) \right. \\ & \left. - \frac{\alpha s^\alpha H(\sigma(t), s) \delta(\sigma(s)) R(s, T)}{\delta^{1+\frac{1}{\alpha}}(s) \sigma^\alpha(s) r_1(s)} \omega^{1+\frac{1}{\alpha}}(s) \right] \Delta s, \end{aligned}$$

and so

$$\begin{aligned} & \int_{T_0}^{\sigma(t)} H(\sigma(t), s) Q(s) \Delta s \\ & \leq H(\sigma(t), T_0) \omega(T_0) + \int_{T_0}^{\sigma(t)} \left[ -\frac{h(t, s) H^{\frac{\alpha}{1+\alpha}}(\sigma(t), s)}{\delta(s)} \omega(s) \right. \\ & \left. - \frac{\alpha s^\alpha H(\sigma(t), s) \delta(\sigma(s)) R(s, T)}{\delta^{1+\frac{1}{\alpha}}(s) \sigma^\alpha(s) r_1(s)} \omega^{1+\frac{1}{\alpha}}(s) \right] \Delta s, \end{aligned} \tag{18}$$

Now set

$$\begin{aligned} X^\lambda &= \frac{\alpha s^\alpha H(\sigma(t), s) \delta(\sigma(s)) R(s, T)}{\delta^{1+\frac{1}{\alpha}}(s) \sigma^\alpha(s) r_1(s)} \omega^\lambda(s), \\ Y^{\lambda-1} &= \frac{h_-(t, s) [\sigma^\alpha(s) r_1(s)]^{\frac{1}{\lambda}}}{\lambda [\alpha s^\alpha \delta(s) R(s, T)]^{\frac{1}{\lambda}}}, \end{aligned}$$

where  $\lambda = \frac{\alpha+1}{\alpha} > 1$ ,  $X \geq 0$  and  $Y \geq 0$ . Using the inequality (6), we obtain

$$\begin{aligned} & \frac{h_-(t, s) H^{\frac{1}{\lambda}}(\sigma(t), s)}{\delta(s)} \omega(s) - \frac{\alpha s^\alpha H(\sigma(t), s) \delta(\sigma(s)) R(s, T)}{\delta^{\lambda-1}(s) \sigma^\alpha(s) r_1(s)} \omega^\lambda(s) \\ & \leq \frac{h_-(t, s) [\sigma^\alpha(s) r_1(s)]^\alpha}{(\alpha+1)^{\alpha+1} [\alpha s^\alpha \delta(\sigma(s)) R(s, T)]^\alpha}. \end{aligned} \tag{19}$$

Combining (18) and (19), we get

$$\begin{aligned} & \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \left[ H(\sigma(t), s) Q(s) - \frac{h_-(t, s) [\sigma^\alpha(s) r_1(s)]^\alpha}{(\alpha+1)^{\alpha+1} [\alpha s^\alpha \delta(\sigma(s)) R(s, T)]^\alpha} \right] \Delta s \\ & \leq \omega(T_0). \end{aligned}$$

From (16), we obtain

$$\Psi(T_0) \leq \omega(T_0), \quad T_0 \in [T, \infty)_{\mathbb{T}},$$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} H(\sigma(t), s) Q(s) \Delta s \geq \Psi(T_0). \quad (20)$$

By (18), we get

$$\begin{aligned} & \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} H(\sigma(t), s) Q(s) \Delta s \\ & \leq \omega(T_0) + \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \frac{h_-(t, s) H^{\frac{\alpha}{1+\alpha}}(\sigma(t), s)}{\delta(s)} \omega(s) \Delta s \\ & \quad - \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \frac{\alpha s^\alpha H(\sigma(t), s) \delta(\sigma(s)) R(s, T)}{\delta^{\frac{1+\frac{1}{\alpha}}}(s) \sigma^\alpha(s) r_1(s)} \omega^{\frac{1}{1+\alpha}}(s) \Delta s. \end{aligned} \quad (21)$$

We denote

$$u(t) = \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \frac{h_-(t, s) H^{\frac{\alpha}{1+\alpha}}(\sigma(t), s)}{\delta(s)} \omega(s) \Delta s,$$

$$v(t) = \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \frac{\alpha s^\alpha H(\sigma(t), s) \delta(\sigma(s)) R(s, T)}{\delta^{\frac{1+\frac{1}{\alpha}}}(s) \sigma^\alpha(s) r_1(s)} \omega^{\frac{1}{1+\alpha}}(s) \Delta s,$$

meanwhile noting that (16), we obtain

$$\liminf_{t \rightarrow \infty} [v(t) - u(t)] \leq \omega(T_0) - \Psi(T_0) < \infty.$$

Now we assert that

$$\int_T^\infty \frac{s^\alpha \delta(\sigma(s)) R(s, T)}{\delta^{\frac{1+\frac{1}{\alpha}}}(s) \sigma^\alpha(s) r_1(s)} \omega^{\frac{1}{1+\alpha}}(s) \Delta s < \infty \quad (22)$$

holds. Suppose to the contrary that

$$\int_T^\infty \frac{s^\alpha \delta(\sigma(s)) R(s, T)}{\delta^{\frac{1+\frac{1}{\alpha}}}(s) \sigma^\alpha(s) r_1(s)} \omega^{\frac{1}{1+\alpha}}(s) \Delta s = \infty, \quad (23)$$

by (13), there exists a constant  $\varepsilon > 0$  such that

$$\inf_{s \geq T_0} \left[ \liminf_{t \rightarrow \infty} \frac{H(\sigma(t), s)}{H(\sigma(t), T_0)} \right] > \varepsilon > 0, \quad (24)$$

from (23), there exists  $T_1 \in [T_0, \infty)_{\mathbb{T}}$  for arbitrary real number  $M > 0$  such that

$$\int_{T_1}^{\sigma(t)} \frac{s^\alpha \delta(\sigma(s)) R(s, T)}{\delta^{\frac{1+\frac{1}{\alpha}}}(s) \sigma^\alpha(s) r_1(s)} \omega^{\frac{1}{1+\alpha}}(s) \Delta s \geq \frac{M}{\alpha \varepsilon}, \quad \text{for } t \in [T_1, \infty)_{\mathbb{T}}.$$

Using the integration by parts formula of  $\Delta$ -differential, we obtain



$$\begin{aligned}
 v(t) &= \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \left\{ \alpha H(\sigma(t), s) \left( \int_{T_0}^s \frac{u^\alpha \delta(\sigma(u)) R(u, T)}{\delta^{1+\frac{1}{\alpha}}(u) \sigma^\alpha(u) r_1(u)} W^{1+\frac{1}{\alpha}}(u) \Delta u \right)^{\Delta_s} \right\} \Delta s \\
 &= \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \left\{ -\alpha H^{\Delta_s}(\sigma(t), s) \int_{T_0}^{\sigma(s)} \frac{u^\alpha \delta(\sigma(u)) R(u, T)}{\delta^{1+\frac{1}{\alpha}}(u) \sigma^\alpha(u) r_1(u)} \omega^{1+\frac{1}{\alpha}}(u) \Delta u \right\} \Delta s \\
 &\geq \frac{1}{H(\sigma(t), T_0)} \int_{T_1}^{\sigma(t)} \left\{ -\alpha H^{\Delta_s}(\sigma(t), s) \int_{T_0}^{\sigma(s)} \frac{u^\alpha \delta(\sigma(u)) R(u, T)}{\delta^{1+\frac{1}{\alpha}}(u) \sigma^\alpha(u) r_1(u)} \omega^{1+\frac{1}{\alpha}}(u) \Delta u \right\} \Delta s \\
 &\geq \frac{1}{H(\sigma(t), T_0)} \int_{T_1}^{\sigma(t)} -\alpha H^{\Delta_s}(\sigma(t), s) \frac{M}{\alpha \varepsilon} \Delta s = \frac{M}{\varepsilon} \frac{H(\sigma(t), T_1)}{H(\sigma(t), T_0)}.
 \end{aligned}$$

From (24), there exists  $T_2 \in [T_1, \infty)_{\mathbb{T}}$ , we get  $\frac{H(\sigma(t), T_1)}{H(\sigma(t), T_0)} \geq \varepsilon$  for  $t \in [T_2, \infty)_{\mathbb{T}}$ , so that  $v(t) \geq M$ . Since  $M$  is arbitrary, we obtain

$$\lim_{t \rightarrow \infty} v(t) = \infty. \tag{25}$$

Selecting a sequence  $\{t_n\}_{n=1}^\infty : t_n \in [T_0, \infty)_{\mathbb{T}}$  with  $\lim_{n \rightarrow \infty} t_n = \infty$  satisfying

$$\lim_{n \rightarrow \infty} [v(t_n) - u(t_n)] = \liminf_{t \rightarrow \infty} [v(t) - u(t)] < \infty,$$

then there exists a constant  $M_0 > 0$  such that

$$v(t_n) - u(t_n) \leq M_0 \tag{26}$$

for sufficiently large positive integer  $n$ . From (22), we can easily see

$$\lim_{n \rightarrow \infty} v(t_n) = \infty, \tag{27}$$

(26) implies that

$$\lim_{n \rightarrow \infty} u(t_n) = \infty. \tag{28}$$

From (26) and (27), we obtain

$$\frac{u(t_n)}{v(t_n)} - 1 \geq -\frac{M_0}{v(t_n)} > -\frac{M_0}{2M_0} = -\frac{1}{2},$$

i.e.,

$$\frac{u(t_n)}{v(t_n)} > \frac{1}{2}$$

for sufficiently large positive integer  $n$ , which together with (28) implies

$$\lim_{n \rightarrow \infty} \frac{[u(t_n)]^{\alpha+1}}{[v(t_n)]^\alpha} = \lim_{n \rightarrow \infty} \left[ \frac{u(t_n)}{v(t_n)} \right]^\alpha u(t_n) = \infty. \tag{29}$$

On the other hand, by Lemma 7, we obtain

$$\begin{aligned}
 u(t_n) &= \frac{1}{H(\sigma(t_n), T_0)} \int_{T_0}^{\sigma(t_n)} \frac{h_-(t_n, s) H^{\frac{\alpha}{\alpha+1}}(\sigma(t_n), s)}{\delta(s)} \omega(s) \Delta s \\
 &= \int_{T_0}^{\sigma(t_n)} \frac{h_-(t_n, s) H^{\frac{\alpha}{\alpha+1}}(\sigma(t_n), s)}{H(\sigma(t_n), T_0) \delta(s)} \omega(s) \Delta s \\
 &= \int_{T_0}^{\sigma(t_n)} \left\{ \left[ \frac{\alpha s^\alpha H(\sigma(t_n), s) \delta(\sigma(s)) R(s, T)}{H(\sigma(t_n), T_0) \sigma^\alpha(s) r_1(s)} \right]^{\frac{\alpha}{\alpha+1}} \frac{\omega(s)}{\delta(s)} \right\} \\
 &\quad \times \left\{ \frac{h_-(t_n, s) H^{\frac{\alpha}{\alpha+1}}(\sigma(t_n), s)}{H(\sigma(t_n), T_0)} \left[ \frac{\alpha s^\alpha H(\sigma(t_n), s) \delta(\sigma(s)) R(s, T)}{H(\sigma(t_n), T_0) \sigma^\alpha(s) r_1(s)} \right]^{\frac{\alpha}{\alpha+1}} \right\} \Delta s \\
 &\leq \left\{ \int_{T_0}^{\sigma(t_n)} \frac{\alpha s^\alpha H(\sigma(t_n), s) \delta(\sigma(s)) R(s, T)}{H(\sigma(t_n), T_0) \sigma^\alpha(s) r_1(s)} \left[ \frac{\omega(s)}{\delta(s)} \right]^{\frac{\alpha}{\alpha+1}} \Delta s \right\}^{\frac{\alpha}{\alpha+1}} \\
 &\quad \times \left\{ \int_{T_0}^{\sigma(t_n)} \frac{h_-(t_n, s) H^{\frac{\alpha}{\alpha+1}}(\sigma(t_n), s)}{H^{\frac{\alpha}{\alpha+1}}(\sigma(t_n), T_0)} \left[ \frac{\alpha s^\alpha H(\sigma(t_n), s) \delta(\sigma(s)) R(s, T)}{H(\sigma(t_n), T_0) \sigma^\alpha(s) r_1(s)} \right]^{-\alpha} \Delta s \right\}^{\frac{1}{\alpha+1}} \\
 &= [v(t_n)]^{\frac{\alpha}{\alpha+1}} \left\{ \frac{1}{\alpha^\alpha H(\sigma(t_n), T_0)} \int_{T_0}^{\sigma(t_n)} \frac{h_-(t_n, s) \sigma^{\alpha^2}(s) r_1^\alpha(s)}{[s^\alpha \delta(\sigma(s)) R(s, T)]^\alpha} \Delta s \right\}^{\frac{1}{\alpha+1}}.
 \end{aligned}$$

The above inequality show that

$$\frac{[u(t_n)]^{\alpha+1}}{[v(t_n)]^\alpha} \leq \frac{1}{\alpha^\alpha H(\sigma(t_n), T_0)} \int_{T_0}^{\sigma(t_n)} \frac{h_-(t_n, s) \sigma^{\alpha^2}(s) r_1^\alpha(s)}{[s^\alpha \delta(\sigma(s)) R(s, T)]^\alpha} \Delta s.$$

Hence, (29) implies

$$\lim_{n \rightarrow \infty} \frac{1}{\alpha^\alpha H(\sigma(t_n), T_0)} \int_{T_0}^{\sigma(t_n)} \frac{h_-(t_n, s) \sigma^{\alpha^2}(s) r_1^\alpha(s)}{[s^\alpha \delta(\sigma(s)) R(s, T)]^\alpha} \Delta s = \infty.$$

This contradicts (14). Therefore (22) holds. Noting  $\Psi(T_0) \leq \omega(T_0)$  for  $T_0 \in [T, \infty)_{\mathbb{T}}$ , by using (22), we obtain

$$\int_T^\infty \frac{s^\alpha \delta(\sigma(s)) R(s, T)}{\delta^{\frac{1+\frac{1}{\alpha}}}(s) \sigma^\alpha(s) r_1(s)} \Psi_+^{\frac{1}{\alpha}}(s) \Delta s \leq \int_T^\infty \frac{s^\alpha \delta(\sigma(s)) R(s, T)}{\delta^{\frac{1+\frac{1}{\alpha}}}(s) \sigma^\alpha(s) r_1(s)} \omega^{\frac{1}{\alpha}}(s) \Delta s < \infty.$$

This contradicts (15).

If case (II) holds, from (9), by Lemma 6  $\lim_{t \rightarrow \infty} x(t) = 0$ . The proof is complete.

**Theorem 2** Assume that (8), (9), (12), (13), (15) and  $r_1^\Delta(t) \leq 0$  hold, where  $H, h$  and  $\delta$  are defined in Theorem 1. Furthermore, assume that there is a real rd-continuous function  $\Psi : [t_0, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}$  such that

$$\liminf_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} H(\sigma(t), s) Q(s) \Delta s < \infty, \tag{30}$$

$$\liminf_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \left[ H(\sigma(t), s) Q(s) - \frac{h_-^{\alpha+1}(t, s) [\sigma^\alpha(s) r_1(s)]^\alpha}{(\alpha+1)^{\alpha+1} [s^\alpha \delta(\sigma(s)) R(s, T)]^\alpha} \Delta s \right] \geq \Psi(T_0), \tag{31}$$

for  $T_0 \in [T, \infty)_{\mathbb{T}}$ , where  $Q(t), \Psi(t)$  are defined in Theorem 1. Then every solution  $x(t)$  of Equation (1) is either oscillatory or converges to zero.

**Proof** Assume that (1) has a nonoscillatory solution  $x(t)$  on  $[t_0, \infty)_{\mathbb{T}}$ . Without loss generality we may assume that there exists sufficiently large  $T \geq t_0$  such that  $x(t) > 0$  and  $x[\tau(t)] > 0$  for all  $t \in [T, \infty)_{\mathbb{T}}$ . By Lemma 1, we see that  $x(t)$  satisfies either case (I) or case (II).

If case (I) holds, proceeding as in the proof of Theorem 1, we get

$$\frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \left[ H(\sigma(t), s) Q(s) - \frac{h_-^{\alpha+1}(t, s) [\sigma^\alpha(s) r_1(s)]^\alpha}{(\alpha+1)^{\alpha+1} [s^\alpha \delta(\sigma(s)) R(s, T)]^\alpha} \right] \Delta s \leq \omega(T_0),$$

From (31), we obtain

$$\Psi(T_0) \leq \omega(T_0), \quad T_0 \in [T, \infty)_{\mathbb{T}};$$

$$\liminf_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} H(\sigma(t), s) Q(s) \Delta s \geq \Psi(T_0); \tag{32}$$

and

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} H(\sigma(t), s) Q(s) \Delta s \\ & - \liminf_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \frac{h_-^{\alpha+1}(t, s) [\sigma^\alpha(s) r_1(s)]^\alpha}{(\alpha+1)^{\alpha+1} [s^\alpha \delta(\sigma(s)) R(s, T)]^\alpha} \Delta s \\ & \geq \liminf_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \left[ H(\sigma(t), s) Q(s) - \frac{h_-^{\alpha+1}(t, s) [\sigma^\alpha(s) r_1(s)]^\alpha}{(\alpha+1)^{\alpha+1} [s^\alpha \delta(\sigma(s)) R(s, T)]^\alpha} \right] \Delta s \\ & \geq \Psi(T_0). \end{aligned} \tag{33}$$

Using (30) and (33), we get

$$\liminf_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \frac{h_-^{\alpha+1}(t, s) [\sigma^\alpha(s) r_1(s)]^\alpha}{(\alpha+1)^{\alpha+1} [s^\alpha \delta(\sigma(s)) R(s, T)]^\alpha} \Delta s < \infty.$$

Thus, there exists a sequence  $\{t_n\}_{n=1}^\infty : t_n \in [T_0, \infty)_{\mathbb{T}}$  with  $\lim_{n \rightarrow \infty} t_n = \infty$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{H(\sigma(t_n), T_0)} \int_{T_0}^{\sigma(t_n)} \frac{h_{-}^{\alpha+1}(t, s) [\sigma^\alpha(s) r_1(s)]^\alpha}{(\alpha + 1)^{\alpha+1} [s^\alpha \delta(\sigma(s)) R(s, T)]^\alpha} \Delta s < \infty.$$

We define  $u(t)$  and  $v(t)$  also, as in the proof of Theorem 1. From (18) and (32), we obtain

$$\limsup_{t \rightarrow \infty} [v(t) - u(t)] \leq \omega(T_0) - \liminf_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} H(\sigma(t), s) Q(s) \Delta s < \infty.$$

For the above sequence  $\{t_n\}_{n=1}^\infty$ , we get

$$\lim_{n \rightarrow \infty} [v(t_n) - u(t_n)] \leq \limsup_{t \rightarrow \infty} [v(t) - u(t)] < \infty.$$

Similar to the proof of Theorem 1, we get (22). The rest of the proof is similar to that of Theorem 1, and hence is omitted. The proof is complete.

**Remark 1** From Theorems 1 and 2, we can obtain different sufficient conditions for the oscillation of Equation (1) with different choices of the functions  $\delta$  and  $H$ .

For example,  $H(t, s) = (t - s)^m$  or  $H(t, s) = \left(\ln \frac{t+1}{s+1}\right)^m$ .

**Remark 2** The theorems in this paper are new even for the cases of  $\mathbb{T} = \mathbb{R}$  and  $\mathbb{T} = \mathbb{Z}$ .

**Example 1** Consider the third-order nonlinear delay dynamic equation

$$\left( t \left( \left( \frac{1}{t^2} x^\Delta(t) \right)^\Delta \right)^{\frac{5}{3}} \right)^\Delta + \frac{1}{8} \left( x \left( \frac{t}{2} \right) \right)^{\frac{5}{3}} \left( 1 + \ln \left( 1 + x^2 \left( \frac{t}{2} \right) \right) \right) = 0, \quad t \in \overline{2^{\mathbb{Z}}}, \quad t \geq t_0 := 2. \quad (34)$$

Here  $\alpha = \frac{5}{3}$ ,  $r_1(t) = \frac{1}{t^2}$ ,  $r_2(t) = t$ ,  $q(t) = \frac{1}{8}$ ,  $f(x) = x^{\frac{5}{3}} (1 + \ln(1 + x^2))$

and  $\tau(t) = \frac{t}{2} < t$ .

The conditions (H<sub>1</sub>)-(H<sub>3</sub>) are clearly satisfied, (H<sub>4</sub>) holds with  $K = 1$ .

$\frac{f(x)}{x^\alpha} = 1 + \ln(1 + x^2) \geq K > 0$ .  $r_1^\Delta(t) = -\frac{3}{4t^3} < 0$ , and

$$\int_{t_0}^\infty q(s) \tau^\alpha(s) \Delta s = \int_2^\infty \frac{1}{8} \left( \frac{s}{2} \right)^{\frac{5}{3}} \Delta s = 2^{\frac{5}{3}} \int_2^\infty s^{-1} \Delta s = \infty,$$

$$\begin{aligned} & \int_{t_0}^\infty \frac{1}{r_1(t)} \int_t^\infty \left[ \frac{1}{r_2(s)} \int_s^\infty q(u) \Delta u \right]^\alpha \Delta s \Delta t \\ &= \int_2^\infty t^2 \int_t^\infty \left[ \frac{1}{s} \int_s^\infty \frac{1}{8} \Delta u \right]^{\frac{3}{5}} \Delta s \Delta t \\ &= \left( \frac{1}{1 - 2^{-\frac{5}{3}}} \right)^{\frac{3}{5}} \int_2^\infty t^2 \int_t^\infty \frac{1}{8} \Delta s \Delta t \\ &= \left( \frac{1}{1 - 2^{-\frac{5}{3}}} \right)^{\frac{3}{5}} \frac{1}{1 - 2^{-\frac{3}{5}}} \int_2^\infty t^{\frac{7}{5}} \Delta t = \infty, \end{aligned}$$

so (8), (9) hold. For larger enough  $t > T$ , we have

$$R(t, T) = \int_T^t \left( \frac{1}{r_2(s)} \right)^{\frac{1}{\alpha}} \Delta s = \int_T^t s^{-\frac{3}{5}} \Delta s = \frac{t^{\frac{2}{5}} - T^{\frac{2}{5}}}{\frac{2}{5} - 1} > 1.$$

Let  $\delta(t) = t$ , since  $\sigma(t) = 2t$ , we have

$$Q(t) = Kq(t) \delta(\sigma(t)) \left( \frac{\tau(t)}{\sigma(t)} \right)^{\alpha} = \frac{1}{2^{\frac{7}{5}} t^{\frac{3}{5}}}.$$

Let  $H(t, s) = (t - s)^2$ , that there exists a function  $h(t, s) = -\frac{4(t - s)^2}{(2t - s)^4}$  such

that

$$H^{\Delta_s}(\sigma(t), s) + \frac{(\delta^{\Delta}(s))_+}{\delta(s)} H(\sigma(t), s) = -\frac{h(t, s)}{\delta(s)} H^{\frac{\alpha}{\alpha+1}}(\sigma(t), s).$$

It follows that

$$0 < \inf_{s \geq T_0} \left[ \liminf_{t \rightarrow \infty} \frac{H(\sigma(t), s)}{H(\sigma(t), T_0)} \right] = \inf_{s \geq T_0} \left[ \liminf_{t \rightarrow \infty} \frac{(2t - s)^2}{(2t - T_0)^2} \right] = 1 < \infty,$$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \frac{h^{\alpha+1}(t, s) [\sigma^{\alpha}(s) r_1(s)]^{\alpha}}{[s^{\alpha} \delta(\sigma(s)) R(s, T)]^{\alpha}} \Delta s \leq \frac{2^{\frac{10}{9}} \cdot T_0^{-4}}{1 - 2^{-4}} < \frac{2^6}{T_0^4} < \infty,$$

so (12), (13) and (14) hold. Let  $\Psi(t) = \frac{1}{2t}$ , we have

$$\int_T^{\infty} \frac{s^{\alpha} \delta(\sigma(s)) R(s, T)}{\delta^{\frac{1+\frac{1}{\alpha}}}(s) \sigma^{\alpha}(s) r_1(s)} \Psi_+^{\frac{1+\frac{1}{\alpha}}}(s) \Delta s = \frac{1}{2^{\frac{49}{15}} \left( \frac{2}{2^5 - 1} \right)} \cdot \int_T^{\infty} \left( s^{\frac{1}{5}} - s^{-\frac{1}{5}} \cdot T^{\frac{2}{5}} \right) \Delta s = \infty,$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \left[ H(\sigma(t), s) Q(s) - \frac{h^{\alpha+1}(t, s) [\sigma^{\alpha}(s) r_1(s)]^{\alpha}}{(\alpha + 1)^{\alpha+1} [s^{\alpha} \delta(\sigma(s)) R(s, T)]^{\alpha}} \right] \Delta s$$

$$\geq \frac{2^{\frac{7}{3}}}{\left( 1 - 2^{-\frac{2}{3}} \right) T_0^{\frac{2}{3}}} - \frac{2^{\frac{10}{9}}}{\left( \frac{8}{3} \right)^{\frac{8}{3}} (1 - 2^{-4}) T_0^4} > \frac{1}{2T_0^{\frac{2}{3}}} > \frac{1}{2T_0} = \Psi(T_0).$$

Then, by Theorem 1, every solution  $x(t)$  of Equation (34) is either oscillatory or converges to zero. But the results in [18] cannot be applied in (34).

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