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Deep Transfers of p-Class Tower Groups

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Abstract

Let p be a prime. For any finite p-group G, the deep transfers $T_{H,G'}: H/H' \to G'/G''$ from the maximal subgroups H of index $(G:H)=p$ in G to the derived subgroup G' are introduced as an innovative tool for identifying G uniquely by means of the family of kernels $\varkappa_d(G) = (\ker(T_{H,G'}))_{(G:H)=p}$. For all finite 3-groups G of coclass $\mathrm{cc}(G)=1$, the family $\varkappa_d(G)$ is determined explicitly. The results are applied to the Galois groups $G = \mathrm{Gal}(F_3^{(\infty)}/F)$ of the Hilbert 3-class towers of all real quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $d>1$, 3-class group $\mathrm{Cl}_3(F) \simeq C_3 \times C_3$, and total 3-principalization in each of their four unramified cyclic cubic extensions E/F. A systematic statistical evaluation is given for the complete range $1<d<10^7$, and a few exceptional cases are pointed out for $1<d<10^8$.

Keywords

Hilbert p-Class Field Towers, p-Class Groups, p-Principalization, Quadratic Fields, Dihedral Fields of Degree $2p$, Finite p-Groups, Two-Step Centralizers, Polarization Principle, Descendant Trees, p-Group Generation Algorithm, p-Multiplicator Rank, Relation Rank, Generator Rank, Deep Transfers, Shallow Transfers, Partial Order and Monotony Principle of Artin Patterns, Parametrized Polycyclic pc-Presentations, Commutator Calculus

1. Introduction

The layout of this paper is the following. Deep transfers of finite p-groups G, with an assigned prime number p, are introduced as an innovative supplement to the (usual) shallow transfers [[1], p. 50], [[2], Equation (4), p. 470] in §2. The family $\varkappa_d(G) = (\ker(T_{H,G'}))_{(G:H)=p}$ of the kernels of all deep transfers of G is called the *deep transfer kernel type* of G and will play a crucial role in this paper. For all finite 3-groups G of coclass $\mathrm{cc}(G)=1$, the deep transfer kernel type

$\varkappa_d(G) = (\ker(T_{H_i,G'}))_{1\le i\le 4}$ is determined explicitly with the aid of commutator calculus in §3 using a parametrized polycyclic power-commutator presentation of G [3] [4] [5]. In the concluding §4, the orders of the deep transfer kernels are sufficient for identifying the Galois group $G_3^{\infty}F := \mathrm{Gal}(F_3^{(\infty)}/F)$ of the maximal unramified pro-3 extension of real quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with 3-class group $\mathrm{Cl}_3(F) \simeq C_3 \times C_3$, and total 3-principalization in each of their four unramified cyclic cubic extensions $E_1,\ldots,E_4$.

2. Shallow and Deep Transfer of *p*-Groups

With an assigned prime number $p \ge 2$, let G be a finite *p*-group. Since our focus in this paper will be on the simplest possible non-trivial situation, we assume that the abelianization G/G' of G is of elementary type (p,p) with rank two. For applications in number theory, concerning *p*-class towers, the Artin pattern has proved to be a decisive collection of information on *G*.

Definition 2.1. *The Artin pattern* $\mathrm{AP}(G) := (\tau(G), \varkappa(G))$ *of G consists of two families*

$$\tau(G) := (H_i / H_i')_{1\le i\le p+1} \text{ and } \varkappa(G) := (\ker(T_{G,H_i}))_{1\le i\le p+1} \tag{2.1}$$

containing the targets and kernels of the Artin transfer homomorphisms $T_{G,H_i} : G/G' \to H_i/H_i'$ [[5], Lem. 6.4, p. 198], [[2], Equation (4), p. 470] from G to its $p+1$ maximal subgroups H_i with $i \in \{1,\ldots,p+1\}$. Since the maximal subgroups form the shallow layer $\mathrm{Lyr}_1(G)$ of subgroups of index $(G:H_i) = p$ of *G*, we shall call the T_{G,H_i} the *shallow transfers* of *G*, and $\varkappa_s(G) := \varkappa(G)$ the *shallow transfer kernel type* (sTKT) of *G*.

We recall [[2], §2.2, pp. 475-476] that the sTKT is usually simplified by a family of non-negative integers, in the following way. For $1 \le i \le p+1$,

$$\varkappa_s(G)_i := \begin{cases} j & \text{if } \ker(T_{G,H_i}) = H_j / G' \text{ for some } j \in \{1,\ldots,p+1\}, \\ 0 & \text{if } \ker(T_{G,H_i}) = G/G'. \end{cases} \tag{2.2}$$

The progressive innovation in this paper, however, is the introduction of the deep Artin transfer.

Definition 2.2. By the *deep transfers* we understand the Artin transfer homomorphisms $T_{H_i,G'} : H_i/H_i' \to G'/G''$ [[5], Lem. 6.1, p. 196], [[6], Dfn. 3.3, p. 69] from the maximal subgroups $H_1,\ldots,H_{p+1}$ to the commutator subgroup G' of G, which forms the deep layer $\mathrm{Lyr}_2(G)$ of the (unique) subgroup of index $(G:G') = p^2$ of G with abelian quotient G/G'. Accordingly, we call the family

$$\varkappa_d(G) = (\#\ker(T_{H_i,G'}))_{1\le i\le p+1} \tag{2.3}$$

the *deep transfer kernel type* (dTKT) of *G*.

We point out that, as opposed to the sTKT, the members of the dTKT are only *cardinalities*, since this will suffice for reaching our intended goals in this paper. This preliminary coarse definition is open to further refinement in subsequent publications (See the proof of Theorem 3.1.).

3. Identification of 3-Groups by Deep Transfers

The drawback of the sTKT is the fact that occasionally several non-isomorphic p-groups G share a common Artin pattern $\mathrm{AP}(G) := (\tau(G), \varkappa_s(G))$ [[7], Thm. 7.2, p. 158]. The benefit of the dTKT is its ability to distinguish the members of such batches of p-groups which have been inseparable up to now. After the general introduction of the dTKT for arbitrary p-groups in §2, we are now going to demonstrate its advantages in the particular situation of the prime $p = 3$ and finite 3-groups G of coclass $cc(G) = 1$, which are necessarily metabelian with second derived subgroup $G'' = 1$ and abelianization $G/G' \simeq C_3 \times C_3$, according to Blackburn [8].

For the statement of our main theorem, we need a precise ordering of the four maximal subgroups $H_1, \ldots, H_4$ of the group $G = \langle x, y \rangle$, which can be generated by two elements x, y, according to the Burnside basis theorem. For this purpose, we select the generators x, y such that

$$H_1 = \langle y, G' \rangle, \quad H_2 = \langle x, G' \rangle, \quad H_3 = \langle xy, G' \rangle, \quad H_4 = \langle xy^2, G' \rangle, \tag{3.1}$$

and $H_1 = \chi_2(G)$, provided that G is of nilpotency class $\mathrm{cl}(G) \ge 3$. Here we denote by

$$\chi_2(G) := \{ g \in G \mid (\forall\, h \in G')\, [g, h] \in \gamma_4(G) \} \tag{3.2}$$

the *two-step centralizer* of G' in G, where we let $(\gamma_i(G))_{i \ge 1}$ be the lower central series of $G =: \gamma_1(G)$ with $\gamma_i(G) = [\gamma_{i-1}(G), G]$ for $i \ge 2$, in particular, $\gamma_2(G) = G'$.

The identification of the groups will be achieved with the aid of parametrized polycyclic power-commutator presentations, as given by Blackburn [3], Miech [4], and Nebelung [5]:

$$\begin{aligned} G_a^n(z,w) := \langle\, & x, y, s_2, \ldots, s_{n-1} \mid s_2 = y, x, (\forall_{i=3}^{n})\ s_i = [s_{i-1}, x],\ s_n = 1, [y, s_2] = s_{n-1}^a, \\ & (\forall_{i=3}^{n-1})\, [y, s_i] = 1,\ x^3 = s_{n-1}^w,\ y^3 s_2^3 s_3 = s_{n-1}^z,\ (\forall_{i=2}^{n-3})\ s_i^3 s_{i+1}^3 s_{i+2} = 1,\ s_{n-2}^3 = s_{n-1}^3 = 1 \,\rangle, \end{aligned} \tag{3.3}$$

where $a \in \{0, 1\}$ and $w, z \in \{-1, 0, 1\}$ are bounded parameters, and the *index of nilpotency* $n = \mathrm{cl}(G) + 1 = \mathrm{cl}(G) + \mathrm{cc}(G) = \log_3(\mathrm{ord}(G)) =: \mathrm{lo}(G)$ is an unbounded parameter.

Lemma 3.1. *Let G be an arbitrary group with elements $x, y \in G$. Then the second and third power of the product xy are given by*

1) $(xy)^2 = x^2 y^2 s_2 t_3$, where $s_2 := [y, x]$, $t_3 := [s_2, y]$,

2) $(xy)^3 = x^3 y^3 (s_2 t_3^2 t_4)^2 s_3 u_4^2 u_5 s_2 t_3$, where $s_3 = [s_2, x]$, $t_4 = [t_3, y]$, $u_4 = [s_3, y]$, $u_5 = [u_4, y]$.

If $G \simeq G_a^n(z, w)$, then $(xy)^2 = x^2 y^2 s_2 s_{n-1}^{-a}$ and $(xy)^3 = x^3 y^3 s_2^3 s_3 s_{n-1}^{-2a}$, and the second and third power of xy^2 are given by $(xy^2)^2 = x^2 y^4 s_2^2 s_{n-1}^{-2a}$ and $(xy^2)^3 = x^3 y^6 s_2^6 s_3^2 s_{n-1}^{-2a}$.

Proof. We prepare the calculation of the powers by proving a few preliminary identities:

$yx = 1 \cdot yx = xyy^{-1}x^{-1} \cdot yx = xy \cdot y^{-1}x^{-1}yx = xy \cdot [y, x] = xys_2$, and similarly

$s_2 y = ys_2 \cdot [s_2, y] = ys_2 t_3$ and $s_2 x = xs_2 \cdot [s_2, x] = xs_2 s_3$ and

$t_3 y = y t_3 \cdot [t_3, y] = y t_3 t_4$ and $s_3 y = y s_3 \cdot [s_3, y] = y s_3 u_4$ and $u_4 y = y u_4 \cdot [u_4, y] = y u_4 u_5$. Furthermore,

$yx^2 = yx \cdot x = xys_2 \cdot x = xy \cdot s_2 x = xy \cdot xs_2 s_3 = x \cdot yx \cdot s_2 s_3 = x \cdot xys_2 \cdot s_2 s_3 = x^2 y s_2^2 s_3$,

$s_2 y^2 = s_2 y \cdot y = y s_2 t_3 \cdot y = y s_2 \cdot t_3 y = y s_2 \cdot y t_3 t_4 = y \cdot s_2 y \cdot t_3 t_4 = y \cdot y s_2 t_3 \cdot t_3 t_4 = y^2 s_2 t_3^2 t_4$,

$$s_3 y^2 = s_3 y \cdot y = y s_3 u_4 \cdot y = y s_3 \cdot u_4 y = y s_3 \cdot y u_4 u_5 = y \cdot s_3 y \cdot u_4 u_5 = y \cdot y s_3 u_4 \cdot u_4 u_5 = y^2 s_3 u_4^2 u_5 .$$

Now the second power of xy is

$$(xy)^2 = xyxy = x \cdot yx \cdot y = x \cdot xys_2 \cdot y = x^2 y \cdot s_2 y = x^2 y \cdot y s_2 t_3 = x^2 y^2 s_2 t_3$$

and the third power of xy is

$$\begin{aligned}(xy)^3 &= xy \cdot (xy)^2 = xy \cdot x^2 y^2 s_2 t_3 = x \cdot yx^2 \cdot y^2 s_2 t_3 = x \cdot x^2 y s_2^2 s_3 \cdot y^2 s_2 t_3 \\ &= x^3 y s_2^2 \cdot s_3 y^2 \cdot s_2 t_3 = x^3 y s_2^2 \cdot y^2 s_3 u_4^2 u_5 \cdot s_2 t_3 = x^3 y s_2 \cdot s_2 y^2 \cdot s_3 u_4^2 u_5 s_2 t_3 \\ &= x^3 y s_2 \cdot y^2 s_2 t_3^2 t_4 \cdot s_3 u_4^2 u_5 s_2 t_3 = x^3 y \cdot s_2 y^2 \cdot s_2 t_3^2 t_4 s_3 u_4^2 u_5 s_2 t_3 \\ &= x^3 y \cdot y^2 s_2 t_3^2 t_4 \cdot s_2 t_3^2 t_4 s_3 u_4^2 u_5 s_2 t_3 = x^3 y^3 (s_2 t_3^2 t_4)^2 s_3 u_4^2 u_5 s_2 t_3\end{aligned} .$$

If $G \simeq G_a^n(z,w)$, then $t_4 = u_4 = u_5 = 1$, $t_3 = s_{n-1}^{-a}$, $t_3^3 = s_{n-1}^{-3a} = 1$, and G' is abelian. □

Theorem 3.1. **(3*-groups G of coclass cc*(*G*) = 1.)** *Let G be a finite* 3*-group of coclass* $\mathrm{cc}(G) = 1$ *and order* $\mathrm{ord}(G) = 3^n$ *with an integer exponent* $n \ge 2$ *. Then the shallow and deep transfer kernel type of G are given in dependence on the relational parameters* a, n, w, z *of* $G \simeq G_a^n(z,w)$ *by* **Table 1***.*

Proof. The shallow TKT $\varkappa_s(G)$ of all 3-groups G of coclass $\mathrm{cc}(G) = 1$ has been determined in [2], where the designations a.*n* of the types were introduced with $n \in \{1,2,3\}$. Here, we indicate a capable mainline vertex of the tree $\mathcal{T}^1(R)$ with root $R = C_3 \times C_3$ [7] by the type a.1* with a trailing asterisk. As usual, type a.3* indicates the unique 3-group $G \simeq \mathrm{Syl}_3 A_9$ with $\tau(G) = (3,3,3),(3,3)^3$. Now we want to determine the deep TKT $\varkappa_d(G)$, using the presentation of $G \simeq G_a^n(z,w)$ in Formula (3.3). For this purpose, we need expressions for the images of the deep Artin transfers $T_i := T_{H_i,G'} : H_i / H_i' \to G'$, for each $1 \le i \le 4$. (Observe that $p = 3$ implies $G'' = 1$ by [8].) Generally, we have to distinguish *outer* transfers, $T_i(g \cdot H_i') = g^3$ if $g \in H_i \setminus G'$ [[2], Equation (4), p. 470], and *inner* transfers,

$T_i(g \cdot H_i') = g^{1+h+h^2} = g^3 \cdot [g, h^3] \cdot [[g,h],h]$ if $g \in G'$ and h is selected in $H_i \setminus G'$ [[2], Equation (6), p. 486].

First, we consider the distinguished two-step centralizer $H_1 = \chi_2(G)$ with $i = 1$. Then $H_1 = \langle y, G' \rangle$ and $H_1' = 1$ if $a = 0$ (H_1 abelian), but $H_1' = \gamma_{n-1}(G) = \langle s_{n-1} \rangle$ if $a = 1$ (H_1 non-abelian) [[2], Equation (3), p. 470]. The outer transfer is determined by $T_1(y \cdot H_1') = y^3 = s_2^{-3} s_3^{-1} s_{n-1}^z$. For the inner transfer, we have $T_1(s_j \cdot H_1') = s_j^{1+y+y^2} = s_j^3 \cdot [s_j, y^3] \cdot [[s_j, y], y] = s_j^3 \cdot 1^3 \cdot [1, y] = s_j^3$ for all $j \ge 3$, but $T_1(s_2 \cdot H_1') = s_2^3 \cdot s_{n-1}^{-3a} \cdot [s_{n-1}^{-a}, y] = s_2^3$ for $j = 2$, since $s_{n-1}^{-a} \in \langle s_{n-1} \rangle = \gamma_{n-1}(G) = \zeta_1(G)$ lies in the centre of G. The first kernel equation $s_2^{-3} s_3^{-1} s_{n-1}^z = 1$ is solvable by either $n = 3$, where $z = 0$, $s_3 = 1$, $s_2^3 = 1$, or $n = 4$, $z = 1$, where $s_2^3 = 1$, $s_{n-1}^z = s_3$. The second kernel equation $s_i^3 = 1$ is solvable by

Table 1. Shallow and deep TKT of 3-groups G with $\mathrm{cc}(G)=1$.

$G \simeq$	n	Type	$\varkappa_s(G)$	$\varkappa_d(G)$
$G_0^n(0,0)$	=2	a.1*	(0,0,0,0)	(3,3,3,3)
$G_0^n(0,0)$	≥3	a.1*	(0,0,0,0)	(9,9,9,9)
$G_1^n(0,0)$	≥5	a.1	(0,0,0,0)	(3,9,3,3)
$G_1^n(0,-1)$	≥5	a.1	(0,0,0,0)	(3,3,9,9)
$G_1^n(0,1)$	≥5	a.1	(0,0,0,0)	(3,3,3,3)
$G_0^n(0,1)$	≥4	a.2	(1,0,0,0)	(9,3,3,3)
$G_0^n(-1,0)$	≥4 even	a.3	(2,0,0,0)	(9,9,3,3)
$G_0^n(1,0)$	≥5	a.3	(2,0,0,0)	(9,9,3,3)
$G_0^n(1,0)$	=4	a.3*	(2,0,0,0)	(27,9,3,3)
$G_0^n(0,1)$	=3	A.1	(1,1,1,1)	(9,3,3,3)

either $i=n-1$ or $i=n-2$. Thus, the deep transfer kernel is given by

$$\ker(T_1)=\begin{cases} H_1=\langle y,s_2\rangle\simeq C_3\times C_3 \text{ if } n=3\,(G \text{ extra special}),\\ H_1=\langle y,s_2,s_3\rangle\simeq C_3\times C_3\times C_3 \text{ if } n=4,\ z=1\,(G\simeq \mathrm{Syl}_3A_9),\\ \gamma_{n-2}(G)=\langle s_{n-2},s_{n-1}\rangle\simeq C_3\times C_3 \text{ if } n=4,\ z\neq 1 \text{ or } n\geq 5,\ a=0,\\ \gamma_{n-2}(G)/\gamma_{n-1}(G)\simeq\langle s_{n-2}\rangle\simeq C_3 \text{ if } n\geq 5,\ a=1\,(H_1 \text{ non-abelian}).\end{cases} \tag{3.4}$$

Second, we put $i=2$. Then $H_2=\langle x,G'\rangle$ and $H_2'=\gamma_3(G)=\langle s_3,\dots,s_{n-1}\rangle$. The outer transfer is determined by $T_2(x\cdot H_2')=x^3=s_{n-1}^w$. The inner transfer is given by $T_2(s_j\cdot H_2')=s_j^{1+x+x^2}=s_j^3\cdot[s_j,x^3]\cdot[[s_j,x],x]=s_j^3s_{j+1}^3s_{j+2}=1$, for all $j\geq 2$, independently of a,n,w,z. Consequently, the deep transfer kernel is given by

$$\ker(T_2)=\begin{cases} H_2/H_2'=\langle x,s_2,\dots,s_{n-1}\rangle/\langle s_3,\dots,s_{n-1}\rangle\simeq\langle x,s_2\rangle\simeq C_3\times C_3 \text{ if } w=0,\\ G'/H_2'=\langle s_2,\dots,s_{n-1}\rangle/\langle s_3,\dots,s_{n-1}\rangle\simeq\langle s_2\rangle\simeq C_3 \text{ if } w=\pm 1.\end{cases} \tag{3.5}$$

Next, we put $i=3$. Then $H_3=\langle xy,G'\rangle$ and $H_3'=\gamma_3(G)=\langle s_3,\dots,s_{n-1}\rangle$. The outer transfer is determined by $T_3(xy\cdot H_3')=(xy)^3=x^3y^3s_2^3s_3s_{n-1}^{-2a}=s_{n-1}^{w+z-2a}$. For the inner transfer, we have
$T_3(s_j\cdot H_3')=s_j^{1+xy+(xy)^2}=s_j^3\cdot[s_j,xy^3]\cdot[[s_j,xy],xy]=s_j^3s_{j+1}^3s_{j+2}=1$, for all $j\geq 3$, independently of a,n,w,z. The first kernel equation $s_{n-1}^{w+z-2a}=1 \Leftrightarrow w+z-2a\equiv 0 \pmod 3$ is solvable by either $a=w=z=0$ or $a=1$, $w=-1$.

Therefore, the deep transfer kernel is given by

$$\ker(T_3)=\begin{cases} H_3/H_3'\simeq\langle xy,s_2\rangle\simeq C_3\times C_3 \text{ if either } a=w=z=0 \text{ or } a=1,\ w=-1,\\ G'/H_3'\simeq\langle s_2\rangle\simeq C_3 \text{ otherwise.}\end{cases} \tag{3.6}$$

Finally, we put $i=4$. Then $H_4=\langle xy^2,G'\rangle$ and $H_4'=\gamma_3(G)=\langle s_3,\dots,s_{n-1}\rangle$. The outer transfer is determined by $T_4(xy^2\cdot H_4')=(xy^2)^3=x^3y^6s_2^6s_3^2s_{n-1}^{-2a}=s_{n-1}^{w+2z-2a}$. The inner transfer is given by
$T_4(s_j\cdot H_4')=s_j^{1+xy^2+(xy^2)^2}=s_j^3\cdot[s_j,xy^{23}]\cdot[[s_j,xy^2],xy^2]=s_j^3s_{j+1}^3s_{j+2}=1$, for all $j\geq 3$, independently of a,n,w,z. The first kernel equation $s_{n-1}^{w+2z-2a}=1 \Leftrightarrow w+2z-2a\equiv 0 \pmod 3$ is solvable by either $a=w=z=0$ or $a=1$, $w=-1$.

Thus, the deep transfer kernel is given by

$$\ker(T_4) = \begin{cases} H_4 / H_4' \simeq \langle xy^2, s_2 \rangle \simeq C_3 \times C_3 \text{ if either } a = w = z = 0 \text{ or } a = 1, w = -1, \\ G' / H_4' \simeq \langle s_2 \rangle \simeq C_3 \text{ otherwise.} \end{cases} \tag{3.7}$$

These finer results are summarized in terms of coarser cardinalities in **Table 1**.

□

4. Arithmetical Application to 3-Class Tower Groups

4.1. Real Quadratic Fields

As a final highlight of our progressive innovations, we come to a number theoretic application of Theorem 3.1, more precisely, the unambiguous identification of the pro-3 Galois group $G_3^\infty F = \mathrm{Gal}(F_3^{(\infty)} / F)$ of the maximal unramified pro-3 extension $F_3^{(\infty)}$, that is the Hilbert 3-class field tower, of certain real quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with fundamental discriminant $d > 1$, 3-class group $\mathrm{Cl}_3(F)$ of elementary type $(3,3)$, and shallow transfer kernel type a.1, $\varkappa_s(F) = (0,0,0,0)$, in its *ground state* with $\tau(F) \sim (9,9),(3,3)^3$ or in a higher *excited state* with $\tau(F) \sim (3^e,3^e),(3,3)^3$, $e \ge 3$.

The first field of this kind with $d = 62501$ was discovered by Heider and Schmithals in 1982 [9]. They computed the sTKT $\varkappa_s(F) = (0,0,0,0)$ with four total 3-principalizations in the unramified cyclic cubic extensions E_i / F, $1 \le i \le 4$, on a CDC Cyber mainframe. The fact that $d = 62501$ is a triadic irregular discriminant (in the sense of Gauss) with non-cyclic 3-class group $\mathrm{Cl}_3(F) \simeq C_3 \times C_3$ has been pointed out earlier in 1936 by Pall [10] already. The second field of this kind with $d = 152949$ was discovered by ourselves in 1991 by computing $\varkappa_s(F)$ on an AMDAHL mainframe [11]. In 2006, there followed $d = 252977$ and $d = 358285$, and many other cases in 2009 [12] [13].

Generally, there are three contestants for the group $G = G_3^\infty F$, for any assigned state $\tau(F) \sim (3^e,3^e),(3,3)^3$, $e \ge 2$, and the following **Main Theorem** admits their identification by means of the deep transfer kernel type (See their statistical distribution at the end of Section 4.1.).

Theorem 4.1. ***(3-class tower groups G of coclass cc(G) = 1 and type* a.1.)** *Let* $F = \mathbb{Q}(\sqrt{d})$ *be a quadratic field with fundamental discriminant d,* 3*-class group* $\mathrm{Cl}_3(F) \simeq C_3 \times C_3$*, and shallow transfer kernel type a.*1*,* $\varkappa_s(F) = (0,0,0,0)$*.*

Then F is real with $d > 1$, the 3-class tower group $G = G_3^\infty F$ of F has coclass $\mathrm{cc}(G) = 1$, and the relational parameters $n \ge 5$ and $w \in \{-1,0,1\}$ of $G \simeq G_1^n(0,w)$ are given in dependence on the deep transfer kernel type $\varkappa_d(F)$ as follows:

$$\begin{aligned} G &\simeq G_1^{2(e+1)}(0,0) && \text{with} \quad n = 2(e+1), w = 0 && \Leftrightarrow \varkappa_d(F) \sim (3,9,3,3), \\ G &\simeq G_1^{2(e+1)}(0,-1) && \text{with} \quad n = 2(e+1), w = -1 && \Leftrightarrow \varkappa_d(F) \sim (3,3,9,9), \\ G &\simeq G_1^{2(e+1)}(0,1) && \text{with} \quad n = 2(e+1), w = 1 && \Leftrightarrow \varkappa_d(F) \sim (3,3,3,3), \end{aligned} \tag{4.1}$$

*where we suppose that the state of type a.*1 *is determined by the transfer target* *type* $\tau(F) \sim (3^e, 3^e), (3,3)^3$ *with* $e \ge 2$.

Proof. Let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic field with 3-class group $\mathrm{Cl}_3(F) \simeq C_3 \times C_3$, denote by $E_1, \ldots, E_4$ its four unramified cyclic cubic extensions and by $T_{E_i/F} : \mathrm{Cl}_3(F) \to \mathrm{Cl}_3(E_i)$ $(1 \le i \le 4)$ the transfer homomorphisms of 3-classes.

If the 3-principalization is total, that is $\ker(T_{E_i/F}) = \mathrm{Cl}_3(F)$, for each $1 \le i \le 4$, then F must be a *real* quadratic field with positive fundamental discriminant $d > 1$, since the order of the principalization kernels $\ker(T_{E_i/F})$ of an imaginary quadratic field F is bounded from above by $(U_F : N_{E_i/F} U_{E_i}) \cdot E_i : F = 1 \cdot 3 = 3$, according to the Theorem on the Herbrand quotient of the unit groups U_{E_i}.

By the Artin reciprocity law of class field theory [1] [14], the principalization type $\varkappa(F) = (0000)$ of the field F corresponds to the shallow transfer kernel type $\varkappa_s(G) = (0000)$ of the 3-class tower group $G = \mathrm{Gal}(F_3^{(\infty)} / F)$ of F, and the abelian type invariants $\mathrm{Cl}_3(F) \simeq 1^2$ of the 3-class group of F correspond to the abelian quotient invariants $G / G' \simeq 1^2$ of G.

According to [2], a finite 3-group G with $G / G' \simeq 1^2$ and $\varkappa_s(G) = (0000)$ must be of coclass $\mathrm{cc}(G) = 1$. **Table 1** shows that either $G \simeq G_0^n(0,0)$ of type a.1* with $n \ge 2$ or $G \simeq G_1^n(0,w)$ of type a.1 with $n \ge 5$ and $-1 \le w \le 1$.

For a real quadratic field F, the relation rank $d_2(G) = \dim_{\mathbb{F}_3} H_2(G, \mathbb{F}_3)$ of the 3-class tower group $G = G_3^{(\infty)} F$ is bounded by $d_2(G) \le 3$ [[15], Thm. 1.3, pp. 75-76]. Consequently, G cannot be a non-abelian mainline vertex $G_0^n(0,0)$ with $n \ge 3$ of the coclass-1 tree $\mathcal{T}^1(R)$ with root $R = C_3 \times C_3$, since all these vertices have the relation rank 4. According to [[12], Thm. 4.1 (1), p. 486], G cannot be the abelian root $R = G_0^2(0,0)$ either, and we must have $G \simeq G_1^n(0,w)$ with $n \ge 5$ and $w \in \{-1, 0, 1\}$.

Now the claim is a consequence of Theorem 3.1 and **Table 1**. □

Table 2 shows that the *ground state* $\tau(F) = (9,9), (3,3)^3$ of the sTKT $\varkappa_s(F) = (0,0,0,0)$ has the nice property that the smallest three discriminants already realize three different 3-class tower groups $G = G_3^\infty F \simeq \langle 729, i \rangle$ with $i \in \{99, 100, 101\}$, identified by their dTKT $\varkappa_d(F) = \varkappa_d(G)$.

In **Table 3**, we see that the *first excited state* $\tau(F) = (27,27), (3,3)^3$ of the sTKT $\varkappa_s(F) = (0,0,0,0)$ does not behave so well: although the smallest two discriminants [12] [13] [16] [17] already realize two different 3-class tower groups $G = G_3^\infty F \simeq \langle 6561, i \rangle$ with $i \in \{2225, 2227\}$, we have to wait for the seventh occurrence until $\langle 6561, 2226 \rangle$ is realized, as the dTKT $\varkappa_d(F) = \varkappa_d(G)$ shows. The counter 7 is a typical example of a *statistic delay*.

The *second excited state* $\tau(F) = (81,81), (3,3)^3$ of the sTKT

Table 2. Deep TKT of 3-class tower groups G with $\tau(G) = (9,9), (3,3)^3$.

G	$\varkappa_d(G)$	MD
$\langle 729, 99 \rangle \simeq G_1^6(0,0)$	(3,9,3,3)	62,501
$\langle 729, 100 \rangle \simeq G_1^6(0,-1)$	(3,3,9,9)	152,949
$\langle 729, 101 \rangle \simeq G_1^6(0,1)$	(3,3,3,3)	252,977

Table 3. Deep TKT of 3-class tower groups G with $\tau(G) = (27,27),(3,3)^3$.

G	$\varkappa_d(G)$	MD	further discriminants
$\langle 6561, 2225\rangle \simeq G_1^8(0,0)$	(3,9,3,3)	10,399,596	16,613,448
$\langle 6561, 2226\rangle \simeq G_1^8(0,-1)$	(3,3,9,9)	27,780,297	
$\langle 6561, 2227\rangle \simeq G_1^8(0,1)$	(3,3,3,3)	2,905,160	14,369,932, 15,019,617, 21,050,241

$\varkappa_s(F) = (0,0,0,0)$, however, is well-behaved again: the smallest three discriminants already realize three different 3-class tower groups $G = G_3^\infty F \simeq G_1^{10}(0,w)$ with $w \in \{0,-1,1\}$, identified by their dTKT $\varkappa_d(F) = \varkappa_d(G)$. (For logarithmic orders ≥ 9, no SmallGroup identifiers exist.) See **Table 4**.

In all tables, the shortcut MD means the *minimal discriminant* [[7], Dfn. 6.2, p. 148].

The diagram in **Figure 1** visualizes the initial eight branches of the coclass tree $\mathcal{T}^1(R)$ with abelian root $R = \langle 9,2\rangle \simeq C_3 \times C_3$. Basic definitions, facts, and notation concerning general descendant trees of finite *p*-groups are summarized briefly in [[18], §2, pp. 410-411] [19]. They are discussed thoroughly in the broadest detail in the initial sections of [20]. Descendant trees are crucial for recent progress in the theory of *p*-class field towers [15] [21] [22], in particular for describing the mutual location of the second *p*-class group $\mathrm{G}_p^2 F$ and the *p*-class tower group $\mathrm{G}_p^\infty F$ of a number field *G*. Generally, the vertices of the coclass tree in the figure represent isomorphism classes of finite 3-groups. Two vertices are connected by a directed edge $G \to H$ if H is isomorphic to the last lower central quotient $G/\gamma_c(G)$, where $c = \mathrm{cl}(G) = n-1$ denotes the nilpotency class of *G*, and $|G| = 3|H|$, that is, $\gamma_c(G) \simeq C_3$ is cyclic of order 3. See also [[18], §2.2, p. 410-411] and [[20], §4, p. 163-164].

The vertices of the tree diagram in **Figure 1** are classified by using various symbols:

1) big contour squares □ represent abelian groups,

2) big full discs • represent metabelian groups with at least one abelian maximal subgroup,

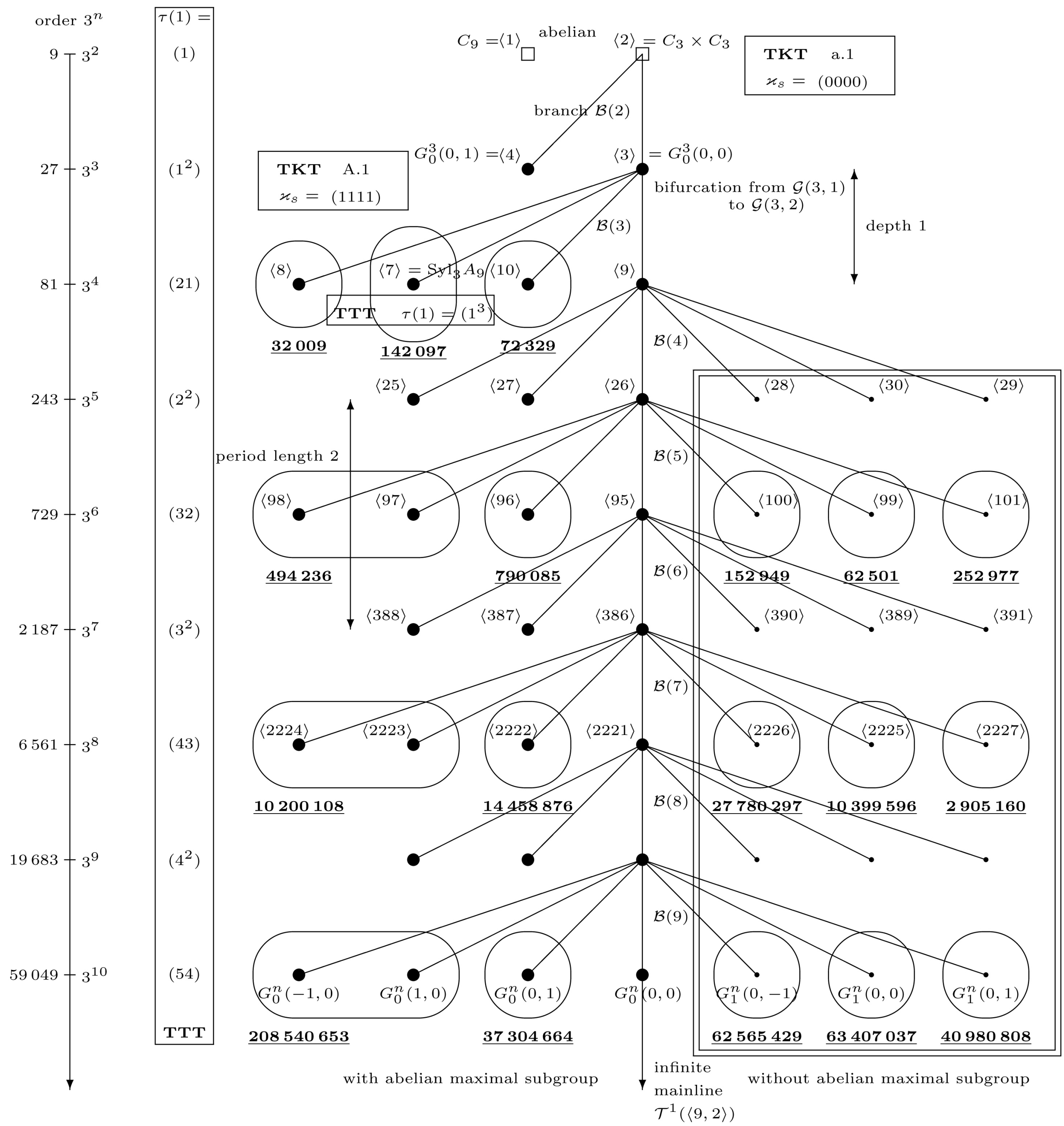
3) small full discs • represent metabelian groups without abelian maximal subgroups.

The groups of particular importance are labelled by a number in angles, which is the identifier in the SmallGroups Library [23] [24] of MAGMA [25]. We omit the orders, which are given on the left hand scale. The sTKT $\varkappa_s$ [[2] Thm. 2.5, Tbl. 6-7], in the bottom rectangle concerns all vertices located vertically above. The first component $\tau(1)$ of the TTT [[26] [27], Dfn. 3.3, p. 288] in the left rectangle concerns vertices G on the same horizontal level containing an abelian maximal subgroup. It is given in logarithmic notation. The periodicity with length 2 of branches, $\mathcal{B}(j) \simeq \mathcal{B}(j+2)$ for $j \geq 4$, sets in with branch $\mathcal{B}(4)$, having a root of order 3^4.

3-class tower groups $G = G_3^\infty F$ with coclass $\mathrm{cc}(G) = 1$ of real quadratic

Table 4. Deep TKT of 3-class tower groups G with $\tau(G) = (81,81),(3,3)^3$.

G	$\varkappa_d(G)$	MD
$G_1^{10}(0,0)$	(3,9,3,3)	63,407,037
$G_1^{10}(0,-1)$	(3,3,9,9)	62,565,429
$G_1^{10}(0,1)$	(3,3,3,3)	40,980,808



TKT	a.3	a.3	a.2	a.1*	a.1	a.1	a.1
$\varkappa_s =$	(2000)	(2000)	(1000)	(0000)	(0000)	(0000)	(0000)

Figure 1. Distribution of minimal discriminants for $G_3^\infty F$ on the coclass-1 tree $\mathcal{T}^1(\langle 9,2\rangle)$

fields $F=\mathbb{Q}(\sqrt{d})$ are located as arithmetically realized vertices on the tree diagram in **Figure 1**. The minimal fundamental discriminants *d*, *i.e.* the MDs, are indicated by underlined boldface integers adjacent to the oval surrounding the realized vertex [6] [24] [25].

The double contour rectangle surrounds the vertices which became distinguishable by the progressive innovations in the present paper and were inseparable up to now.

In **Table 5**, we give the isomorphism type of the 3-class tower group $G=G_3^{\infty}F$ of all real quadratic fields $F=\mathbb{Q}(\sqrt{d})$ with 3-class group $\mathrm{Cl}_3(F)\simeq C_3\times C_3$ and shallow transfer kernel type a.1, $\varkappa_s=(0,0,0,0)$, in its *ground state* $\tau(F)=(9,9),(3,3)^3$, for the complete range $1<d<10^7$ of 150 fundamental discriminants *d*. It was determined by means of Theorem 4.1, applied to the results of computing the (restricted) deep transfer kernel type $\varkappa_d(F)=(\#\ker(T_{F_3^{(1)}/E_i}))_{2\le i\le 4}$, consisting of the orders of the 3-principalization kernels of those unramified cyclic cubic extensions E_i, $2\le i\le 4$, in the Hilbert 3-class field $F_3^{(1)}$ of *F* whose 3-class group $\mathrm{Cl}_3(E_i)$ is of type $(3,3)$. These trailing three components of the TTT $\tau(F)=(9,9),(3,3)^3$ were called its *stable part* in [[6], Dfn. 5.5, p. 84]. The computations were done with the aid of the computational algebra system MAGMA [25]. The 3-principalization kernel of the remaining extension E_1 with 3-class group $\mathrm{Cl}_3(E_1)$ of type $(9,9)$ does not contain essential information and can be omitted. This leading component of the TTT $\tau(F)=(9,9),(3,3)^3$ was called its *polarized part* in [[6], Dfn. 5.5, p. 84]. For more details on the concepts *stabilization* and *polarization*, see [[6], §6, pp. 90-95].

A systematic statistical evaluation of **Table 5** shows that, with respect to the complete range $1<d<10^7$, the group $G\simeq\langle 729,99\rangle$ occurs most often with a clearly elevated relative frequency of 44%, whereas $G\simeq\langle 729,100\rangle$ and $G\simeq\langle 729,101\rangle$ share the common lower percentage of 28%, although the automorphism group $\mathrm{Aut}(G)$ of all three groups has the same order. However, the proportion $44:28:28$ for the upper bound 10^7 is obviously not settled yet, because there are remarkable fluctuations, as **Table 6** shows. According to Boston, Bush and Hajir [28] [29], we have to expect an asymptotic limit $33:33:33$ of the proportions for $d\to\infty$.

4.2. Totally Real Dihedral Fields

In fact, we have computed much more information with MAGMA than mentioned at the end of the previous Section 4.1. To understand the actual scope of our numerical results it is necessary to recall that each unramified cyclic cubic relative extension E_i/F, $1\le i\le 4$, gives rise to a dihedral absolute extension $E_i/\mathbb{Q}$ of degree 6, that is an S_3-extension [[12], Prp. 4.1, p. 482]. For the trailing three fields E_i, $2\le i\le 4$, in the stable part of the TTT $\tau(F)=(9,9),(3,3)^3$, *i.e.* with $\mathrm{Cl}_3(E_i)$ of type $(3,3)$, we have constructed the unramified cyclic cubic extensions $\tilde{E}_{i,j}/E_i$, $1\le j\le 4$, and determined the Artin pattern $\mathrm{AP}(E_i)$ of

Table 5. Statistics of 3-class tower groups G with $\tau(G) = (9,9),(3,3)^3$.

No.	d	G	No.	d	G	No.	d	G
1	62,501	⟨729,99⟩	51	3,995,004	⟨729,101⟩	101	7,313,928	⟨729,99⟩
2	152,949	⟨729,100⟩	52	4,045,265	⟨729,101⟩	102	7,391,212	⟨729,99⟩
3	252,977	⟨729,101⟩	53	4,183,205	⟨729,100⟩	103	7,406,249	⟨729,101⟩
4	358,285	⟨729,101⟩	54	4,196,840	⟨729,100⟩	104	7,415,841	⟨729,101⟩
5	531,437	⟨729,99⟩	55	4,199,901	⟨729,101⟩	105	7,447,697	⟨729,100⟩
6	586,760	⟨729,101⟩	56	4,220,977	⟨729,100⟩	106	7,502,501	⟨729,100⟩
7	595,009	⟨729,100⟩	57	4,233,608	⟨729,99⟩	107	7,601,081	⟨729,101⟩
8	726,933	⟨729,99⟩	58	4,252,837	⟨729,100⟩	108	7,623,320	⟨729,101⟩
9	801,368	⟨729,100⟩	59	4,409,313	⟨729,100⟩	109	7,630,645	⟨729,100⟩
10	940,593	⟨729,100⟩	60	4,429,612	⟨729,101⟩	110	7,634,065	⟨729,100⟩
11	966,489	⟨729,99⟩	61	4,533,032	⟨729,99⟩	111	7,643,993	⟨729,100⟩
12	1,177,036	⟨729,99⟩	62	4,586,797	⟨729,100⟩	112	7,683,308	⟨729,101⟩
13	1,192,780	⟨729,101⟩	63	4,662,917	⟨729,100⟩	113	7,704,653	⟨729,100⟩
14	1,313,292	⟨729,99⟩	64	4,680,701	⟨729,99⟩	114	7,713,961	⟨729,99⟩
15	1,315,640	⟨729,99⟩	65	4,766,309	⟨729,99⟩	115	7,804,828	⟨729,100⟩
16	1,358,556	⟨729,100⟩	66	4,782,664	⟨729,99⟩	116	7,936,316	⟨729,100⟩
17	1,398,829	⟨729,101⟩	67	4,783,697	⟨729,100⟩	117	8,037,645	⟨729,100⟩
18	1,463,729	⟨729,101⟩	68	4,965,009	⟨729,100⟩	118	8,101,277	⟨729,101⟩
19	1,580,709	⟨729,100⟩	69	5,039,692	⟨729,99⟩	119	8,235,965	⟨729,101⟩
20	1,595,669	⟨729,100⟩	70	5,048,988	⟨729,99⟩	120	8,248,953	⟨729,99⟩
21	1,722,344	⟨729,99⟩	71	5,111,669	⟨729,100⟩	121	8,263,020	⟨729,99⟩
22	1,751,909	⟨729,101⟩	72	5,119,637	⟨729,99⟩	122	8,320,764	⟨729,99⟩
23	1,831,097	⟨729,99⟩	73	5,154,385	⟨729,100⟩	123	8,375,228	⟨729,99⟩
24	1,942,385	⟨729,101⟩	74	5,226,941	⟨729,100⟩	124	8,501,541	⟨729,101⟩
25	2,021,608	⟨729,99⟩	75	5,226,941	⟨729,99⟩	125	8,523,385	⟨729,101⟩
26	2,042,149	⟨729,101⟩	76	5,350,569	⟨729,100⟩	126	8,578,617	⟨729,99⟩
27	2,076,485	⟨729,99⟩	77	5,353,240	⟨729,99⟩	127	8,623,704	⟨729,101⟩
28	2,185,465	⟨729,101⟩	78	5,362,136	⟨729,101⟩	128	8,637,717	⟨729,99⟩
29	2,197,669	⟨729,101⟩	79	5,400,712	⟨729,101⟩	129	8,674,397	⟨729,99⟩
30	2,314,789	⟨729,99⟩	80	5,478,321	⟨729,99⟩	130	8,723,237	⟨729,99⟩
31	2,409,853	⟨729,99⟩	81	5,827,564	⟨729,99⟩	131	8,737,913	⟨729,101⟩
32	2,433,221	⟨729,101⟩	82	5,891,701	⟨729,101⟩	132	8,748,764	⟨729,99⟩
33	2,539,129	⟨729,101⟩	83	5,909,217	⟨729,99⟩	133	8,816,389	⟨729,99⟩
34	2,555,249	⟨729,100⟩	84	5,982,269	⟨729,101⟩	134	8,957,485	⟨729,101⟩
35	2,710,072	⟨729,100⟩	85	6,105,693	⟨729,100⟩	135	8,993,409	⟨729,100⟩
36	2,851,877	⟨729,99⟩	86	6,155,861	⟨729,99⟩	136	9,006,397	⟨729,101⟩
37	2,954,929	⟨729,99⟩	87	6,337,340	⟨729,99⟩	137	9,051,665	⟨729,99⟩
38	3,005,369	⟨729,101⟩	88	6,429,997	⟨729,100⟩	138	9,058,892	⟨729,101⟩
39	3,197,864	⟨729,100⟩	89	6,618,085	⟨729,99⟩	139	9,130,973	⟨729,99⟩

Continued

40	3,197,944	$\langle 729,101\rangle$	90	6,658,973	$\langle 729,100\rangle$	140	9,185,153	$\langle 729,101\rangle$
41	3,258,120	$\langle 729,101\rangle$	91	6,792,365	$\langle 729,99\rangle$	141	9,195,769	$\langle 729,101\rangle$
42	3,323,065	$\langle 729,99\rangle$	92	6,806,152	$\langle 729,99\rangle$	142	9,328,597	$\langle 729,99\rangle$
43	3,342,785	$\langle 729,99\rangle$	93	6,882,737	$\langle 729,99\rangle$	143	9,379,849	$\langle 729,100\rangle$
44	3,644,357	$\langle 729,99\rangle$	94	6,927,452	$\langle 729,101\rangle$	144	9,380,744	$\langle 729,99\rangle$
45	3,658,421	$\langle 729,100\rangle$	95	6,953,513	$\langle 729,99\rangle$	145	9,419,704	$\langle 729,99\rangle$
46	3,692,717	$\langle 729,99\rangle$	96	6,974,609	$\langle 729,99\rangle$	146	9,511,580	$\langle 729,100\rangle$
47	3,721,565	$\langle 729,99\rangle$	97	7,010,133	$\langle 729,101\rangle$	147	9,615,813	$\langle 729,100\rangle$
48	3,799,597	$\langle 729,100\rangle$	98	7,019,717	$\langle 729,99\rangle$	148	9,645,393	$\langle 729,99\rangle$
49	3,821,244	$\langle 729,99\rangle$	99	7,075,740	$\langle 729,101\rangle$	149	9,801,773	$\langle 729,99\rangle$
50	3,869,909	$\langle 729,99\rangle$	100	7,263,365	$\langle 729,99\rangle$	150	9,834,557	$\langle 729,99\rangle$

Table 6. Proportions of 3-class tower groups $G \simeq \langle 729,i\rangle$ with $i \in \{99,100,101\}$.

G	for $d < 10^6 \times$	1	2	3	4	5	6	7	8	9	10
$\langle 729,99\rangle$		36%	38%	41%	43%	40%	42%	45%	41%	43%	44%
$\langle 729,100\rangle$		36%	29%	24%	24%	31%	31%	30%	32%	29%	28%
$\langle 729,101\rangle$		27%	33%	35%	33%	29%	27%	25%	27%	28%	28%

E_i, in particular, the 3-principalization type of E_i in the fields $\tilde{E}_{i,j}$. The dihedral fields E_i of degree 6 share a common polarization $\tilde{E}_{i,1} = F_3^{(1)}$, the Hilbert 3-class field of *F*, which is contained in the relative 3-genus field $(E_i / F)^*$, whereas the other extensions $\tilde{E}_{i,j}$ with $2 \le j \le 4$ are non-abelian over *F*, for each $2 \le i \le 4$. Our computational results suggest the following conjecture concerning the infinite family of totally real dihedral fields E_i for varying real quadratic fields *F*.

Conjecture 4.1. (*3-class tower groups $\mathcal{G}$ of totally real dihedral fields.*) *Let $F = \mathbb{Q}(\sqrt{d})$ be a real quadratic field with fundamental discriminant $d > 1$, 3-class group $\mathrm{Cl}_3(F) \simeq C_3 \times C_3$, and shallow transfer kernel type* a.1, *$\varkappa_s(F) = (0,0,0,0)$, in the ground state with transfer target type $\tau(F) \sim (9,9),(3,3)^3$. Let E_2, E_3, E_4 be the three unramified cyclic cubic relative extensions of F with 3-class group $\mathrm{Cl}_3(E_i)$ of type $(3,3)$.*

Then $E_i / \mathbb{Q}$ is a totally real dihedral extension of degree 6, for each $2 \le i \le 4$, and the connection between the component $\varkappa_d(F)_i = \#\ker(T_{F_3^{(1)}/E_i})$ of the deep transfer kernel type $\varkappa_d(F)$ of F and the 3-class tower group $\mathcal{G}_i = G_3^{\infty} E_i = \mathrm{Gal}((E_i)_3^{(\infty)} / E_i)$ of E_i is given in the following way:

$$\begin{aligned} \varkappa_d(F)_i = 3 &\iff \mathcal{G}_i \simeq \langle 243,27\rangle \quad \text{with } \varkappa_s(\mathcal{G}_i) = (1,0,0,0), \\ \varkappa_d(F)_i = 9 &\iff \mathcal{G}_i \simeq \langle 243,26\rangle \quad \text{with } \varkappa_s(\mathcal{G}_i) = (0,0,0,0). \end{aligned} \tag{4.2}$$

Remark 4.1. The conjecture is supported by all $3 \cdot 150 = 450$ totally real dihedral fields E_i which were involved in the computation of **Table 5**. A provable argument for the truth of the conjecture is the fact that

$\varkappa_d(F)_i = \#\ker(T_{F_3^{(1)}/E_i}) = \#\varkappa_s(E_i)_1 = \#\varkappa_s(\mathcal{G}_i)_1$, for $2 \le i \le 4$, but it does not explain why the sTKT $\varkappa_s(\mathcal{G}_i)$ is a.2 with a fixed point if $\varkappa_d(F)_i = 3$. It is interesting that a dihedral field E_i of degree 6 is satisfied with a non-σ group, such as $\langle 243,27\rangle$, as its 3-class tower group. On the other hand, it is not surprising that a mainline group, such as $\langle 243,26\rangle$ with sTKT a.1* and relation rank $d_2 = 4$, is possible as $\mathcal{G}_i = G_3^\infty E_i$, since the upper Shafarevich bound for the relation rank of the 3-class tower group of a totally real dihedral field E_i of degree 6 with $\mathrm{Cl}_3(E_i) \simeq C_3 \times C_3$ is given by $\rho + r_1 + r_2 - 1 = 2 + 6 + 0 - 1 = 7 > 4$ [[15], Thm. 1.3, p. 75].

Assuming an asymptotic limit $33:33:33$ of the proportion of the real quadratic 3-class tower groups $G \in \{\langle 729,99\rangle, \langle 729,100\rangle, \langle 729,101\rangle\}$ for the ground state of sTKT a.1, we can also conjecture an asymptotic limit $33:66$ of the corresponding totally real dihedral 3-class tower groups
$\mathcal{G}_i \in \{\langle 243,26\rangle, \{\langle 243,27\rangle\}$, since the restricted dTKTs $(9,3,3)$, $(3,9,9)$, $(3,3,3)$ together contain three times the 9 and six times the 3 in Equation (4.2).

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