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Positive Radial Solutions for a Class of Semilinear Elliptic Problems Involving Critical Hardy-Sobolev Exponent and Hardy Terms

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Abstract

In this paper, we investigate the solvability of a class of semilinear elliptic equations which are perturbation of the problems involving critical Hardy-Sobolev exponent and Hardy singular terms. The existence of at least a positive radial solution is established for a class of semilinear elliptic problems involving critical Hardy-Sobolev exponent and Hardy terms. The main tools are variational method, critical point theory and some analysis techniques.

Keywords

Hardy Singular Terms, Hardy-Sobolev Exponent, Positive Radial Solution, Perturbation Method, Variational Approach

1. Introduction and Main Results

In this paper, we are concerned with the existence of positive radial solutions for the following semilinear elliptic problem with Hardy-Sobolev exponent and Hardy singular terms:

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = \left[1+\delta h(|x|)\right]\dfrac{|u|^{2^*(s)-2}}{|x|^s}u, & x\in\mathbb{R}^N \\ u>0, & x\in\mathbb{R}^N \\ u\in D_r^{1,2}(\mathbb{R}^N)=\left\{u\in D^{1,2}(\mathbb{R}^N): u \text{ is radial}\right\}, \end{cases} \tag{1.1}$$

where $0<s<2, 2^*(s)=\dfrac{2(N-s)}{N-2}$ is the Hardy-Sobolev critical exponent and $2^*=2^*(0)=\dfrac{2N}{N-2}$ is the Sobolev critical exponent, $\mu<\bar{\mu}\triangleq\dfrac{(N-2)^2}{4}$.

$D^{1,2}(\mathbb{R}^N)(N \geq 3)$ denotes the space of the functions $u \in L^{2^*}(\mathbb{R}^N)$ such that $\nabla u \in L^2(\mathbb{R}^N)$, endowed with scalar product and norm, respectively, given by

$$\langle u,v \rangle = \int_{\mathbb{R}^N} \left(\nabla u \cdot \nabla v - \mu \frac{uv}{|x|^2} \right) \mathrm{d}x,$$

$$\|u\|^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) \mathrm{d}x,$$

that coincides with the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the L^2-norm of the gradient. By Hardy inequality [1], we easily derive that the norm is equivalent to the usual norm:

$$\|u\|_0^2 = \int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x$$

in $D^{1,2}(\mathbb{R}^N)$.

Clearly, $D_r^{1,2}(\mathbb{R}^N)$ is a closed subset of $D^{1,2}(\mathbb{R}^N)$, so $D_r^{1,2}(\mathbb{R}^N)$ is a Hilbert space. By the symmetric criticality principle, in view of [2], we know that the positive radial solutions of problem (1.1) correspond to the nonzero critical points of the functional $I_\delta : D_r^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I_\delta(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) \mathrm{d}x - \frac{1}{2^*(s)} \int_{\mathbb{R}^N} \frac{|u^+|^{2^*(s)}}{|x|^s} \mathrm{d}x$$
$$- \frac{\delta}{2^*(s)} \int_{\mathbb{R}^N} h(|x|) \frac{|u^+|^{2^*(s)}}{|x|^s} \mathrm{d}x,$$

where $u^+ = \max\{u, 0\}$.

The reason why we investigate (1.1) is the presence of the Hardy-Sobolev exponent, the unbounded domain $\mathbb{R}^N$ and the so-called inverse square potential in the linear part, which cause the loss of compactness of embedding $D^{1,2}(\mathbb{R}^N) \to L^{2^*}(\mathbb{R}^N)$, $H^1(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ and $D^{1,2}(\mathbb{R}^N) \to L^2(|x|^{-2}\,\mathrm{d}x)$. Hence, we face a type of triple loss of compactness whose interacting with each other will result in some new difficulties. In last two decades, loss of compactness leads to many interesting existence and nonexistence phenomena for elliptic equations. There are abundant results about this class of problems. For example, by using the concentration compactness principle, the strong maximum principle and the Mountain Pass lemma, Li *et al.* [3] had obtained the existence of positive solutions for singular elliptic equations with mixed Dirichlet-Neumann boundary conditions involving Sobolev-Hardy critical exponents and Hardy terms. Bouchekif and Messirdi [4] obtained the existence of positive solution to the elliptic problem involving two different critical Hardy-Sobolev exponents at the same pole by variational methods and concentration compactness principle. Lan and Tang [5] have obtained some existence results of (1.1) with $\mu = 0$ via an abstract perturbation method in critical point theory. There are some other sufficient conditions, we refer the interested readers to ([6]-[18]) and the references therein.

In the present paper, we investigate the existence of positive radial solutions of problem (1.1). There are several difficulties in facing this problem by means of variational methods. In addition to the lack of compactness, there are more intrinsic obstructions involving the nature of its critical points. Based on a suitable use of an abstract perturbation method in critical point theory discussed in [5] [13] [14], we show that the semilinear elliptic problem with Hardy-Sobolev exponent and Hardy singular terms has at least a positive radial solution.

In this paper, we assume that h satisfies one of the following conditions:

(H) $h \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N), h(x) = h(|x|) = h(r), r = |x|$, and

$$\int_1^\infty r^{-\alpha+N-s-1} h(r)\,\mathrm{d}r < \infty$$

for some $\alpha < N - s$.

(H') $h \in C^2(\mathbb{R}^N), h(x) = h(|x|) = h(r), r = |x|$, $h(r)$ is T-periodic and

$$\int_0^T h(r)\,\mathrm{d}r = 0.$$

The main results read as follows.

Theorem 1 Let (H) hold, and assume that $h(0) = 0$ and $h \not\equiv 0$. Then for $|\delta|$ small, problem (1.1) has a positive radial solution u_δ.

Remark 1 It is easy to check that the following function $h(r)$ satisfies the conditions of **Theorem 1**,

$$h(r) = \frac{2r}{\mathrm{e}^r}.$$

Theorem 2 If assumption (H) holds, and suppose that $h \in C^2(\mathbb{R}^N)$ and $h(0)h''(0) > 0$. Then for $|\delta|$ small, problem (1.1) has a positive radial solution u_δ.

Remark 2 It is easy to check that the following function $h(r)$ satisfies the conditions of **Theorem 2**,

$$h(r) = \frac{1-2r}{\mathrm{e}^r}.$$

Theorem 3 Assume that (H) holds, and suppose

$$\int_0^\infty h(r)\left(1+r^{2-s}\right)^{-\frac{2(N-s)}{2-s}} r^{N-s-1}\,\mathrm{d}r \neq 0$$

and $\int_0^\infty h(0)h(r)\left(1+r^{2-s}\right)^{-\frac{2(N-s)}{2-s}} r^{N-s-1}\,\mathrm{d}r \le 0.$

Then for $|\delta|$ small, problem (1.1) has a positive radial solution u_δ.

Remark 3 It is easy to check that the following function $h(r)$ satisfies the conditions of **Theorem 3** for all $N \ge 3$ and $0 < s < 2$,

$$h(r) = \frac{r}{\mathrm{e}^r},$$

in fact,

$$\int_0^\infty h(r)\left(1+r^{2-s}\right)^{-\frac{2(N-s)}{2-s}} r^{N-s-1}\mathrm{d}r \neq 0$$

and $\int_0^\infty h(0)h(r)\left(1+r^{2-s}\right)^{-\frac{2(N-s)}{2-s}} r^{N-s-1}\mathrm{d}r = 0;$

We can also give the following example for $N=3$ and $s=1$,

$$h(r)=\frac{1-100r}{\mathrm{e}^r},$$

in fact, with the help of computers, we can get

$$\int_0^\infty \frac{1-100r}{\mathrm{e}^r}\left(1+r^{2-1}\right)^{-\frac{2(3-1)}{2-1}} r^{3-1-1}\mathrm{d}r \approx -4.06 \neq 0$$

and $\int_0^\infty h(0)\frac{1-100r}{\mathrm{e}^r}\left(1+r^{2-1}\right)^{-\frac{2(3-1)}{2-1}} r^{3-1-1}\mathrm{d}r \approx -4.06 < 0.$

Theorem 4 Suppose that assumption (H') holds, and satisfies the condition that $h(0)h''(0)>0$. Then problem (1.1) has a positive radial solution u_δ, provided $|\delta| \ll 1$.

Remark 4 It is easy to check that the following function $h(r)$ satisfies the conditions of **Theorem 4**,

$$h(r)=\mathrm{e}^{\sin\left(\frac{7\pi}{4}+r\right)}\cos\left(\frac{7\pi}{4}+r\right),$$

in fact,

$$h(0)=\mathrm{e}^{\sin\left(\frac{7\pi}{4}+0\right)}\cos\left(\frac{7\pi}{4}+0\right)=\frac{\sqrt{2}}{2}\mathrm{e}^{-\frac{\sqrt{2}}{2}}>0,$$

and by a direct computation, we have

$$h''(0)=\sqrt{2}\mathrm{e}^{-\frac{\sqrt{2}}{2}}>0.$$

Theorem 5 Let h satisfy (H'), and suppose that $h(0)=0$ and $h \not\equiv 0$. Then problem (1.1) has a positive radial solution u_δ, provided $|\delta| \ll 1$.

Remark 5 It is easy to check that the following function $h(r)$ satisfies the conditions of **Theorem 5**,

$$h(r)=\sin 2r.$$

This paper is organized as follows. After a first section we devoted to studying the unperturbed problem $-\Delta u-\mu\frac{u}{|x|^2}=\frac{|u|^{2^*(s)-2}}{|x|^s}u$. The main results are proved in Section 3. In the following discussion, we denote various positive constants as C or $C_i\,(i=0,1,2,3,\cdots)$ for convenience. $o(t)$ denote $\frac{o(t)}{t}\to 0$ as $t\to 0^+$. This idea is essentially introduced in [5] [13].

2. The Case $\delta = 0$

In this section, we will study the unperturbed problem

$$\begin{cases} -\Delta u-\mu\dfrac{u}{|x|^2}=\dfrac{|u|^{2^*(s)-2}}{|x|^s}u, & x\in\mathbb{R}^N; \\ u\in D_r^{1,2}(\mathbb{R}^N),\ u>0, & x\in\mathbb{R}^N. \end{cases} \tag{2.1}$$

It is well-known that the nontrivial solutions of problem (2.1) are equivalent to the nonzero critical points of the energy functional

$$I_0(u)=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2-\mu\frac{u^2}{|x|^2}\right)\mathrm{d}x-\frac{1}{2^*(s)}\int_{\mathbb{R}^N}\frac{|u^+|^{2^*(s)}}{|x|^s}\mathrm{d}x,\quad u\in D_r^{1,2}(\mathbb{R}^N).$$

Obviously, the energy functional $I_0(u)$ is well-defined and is of C^2 with derivatives given by

$$\langle I_0'(u),v\rangle=\int_{\mathbb{R}^N}\left(\nabla u\cdot\nabla v-\mu\frac{uv}{|x|^2}\right)\mathrm{d}x-\int_{\mathbb{R}^N}\frac{|u^+|^{2^*(s)-1}}{|x|^s}v\mathrm{d}x,\quad u,v\in D_r^{1,2}(\mathbb{R}^N);$$

$$\langle I_0''(u)v,w\rangle=\int_{\mathbb{R}^N}\left(\nabla v\cdot\nabla w-\mu\frac{vw}{|x|^2}\right)\mathrm{d}x-\int_{\mathbb{R}^N}\frac{(2^*(s)-1)|u^+|^{2^*(s)-2}}{|x|^s}vw\mathrm{d}x$$
$$u,v,w\in D_r^{1,2}(\mathbb{R}^N).$$

For all $\varepsilon>0$, it is well known that the function

$$z_\varepsilon(r)=\left(\frac{2\varepsilon^{\frac{(2-s)\sqrt{\bar\mu-\mu}}{\sqrt{\bar\mu}}}(N-s)(\bar\mu-\mu)}{\sqrt{\bar\mu}}\right)^{\frac{\sqrt{\bar\mu}}{2-s}}\Bigg/\left(r^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}\left(\varepsilon^{\frac{(2-s)\sqrt{\bar\mu-\mu}}{\sqrt{\bar\mu}}}+r^{\frac{(2-s)\sqrt{\bar\mu-\mu}}{\sqrt{\bar\mu}}}\right)^{\frac{N-2}{2-s}}\right)$$

solves the equation (2.1) and satisfies

$$\int_{\mathbb{R}^N}\left(|\nabla z_\varepsilon|^2-\mu\frac{z_\varepsilon^2}{|x|^2}\right)\mathrm{d}x=\int_{\mathbb{R}^N}\frac{|z_\varepsilon|^{2^*(s)}}{|x|^s}\mathrm{d}x.$$

Let

$$U(r)=\left(\frac{2(N-s)(\bar\mu-\mu)}{\sqrt{\bar\mu}}\right)^{\frac{\sqrt{\bar\mu}}{2-s}}\Bigg/\left(r^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}\left(1+r^{\frac{(2-s)\sqrt{\bar\mu-\mu}}{\sqrt{\bar\mu}}}\right)^{\frac{N-2}{2-s}}\right),$$

then

$$z_\varepsilon(r)=\varepsilon^{-\frac{N-2}{2}}U\left(\frac{r}{\varepsilon}\right).$$

I_0 has a (non-compact) 1-dimensional critical manifold given by

$$Z=\{z=z_\varepsilon(r):\varepsilon>0\}.$$

The unperturbed problem is invariant under the transformation that transforms the function $u(r)$ in the function $\varepsilon^{-\frac{N-2}{2}}u\left(\frac{r}{\varepsilon}\right)$. The purpose of this section is to show the following lemmas.

Lemma 2.1. For all $\varepsilon>0$, $T_{z_\varepsilon}Z=\mathrm{Ker}\left[I_0''(z_\varepsilon)\right]$.

Proof. We will prove the lemma by taking $\varepsilon=1$, hence $z_\varepsilon=U$. The case of a general $\varepsilon>0$ will follow immediately. It is always true that $T_U Z\subseteq \mathrm{Ker}\left[I_0''(U)\right]$. We will show the converse, *i.e.*, that if $v\in\mathrm{Ker}\left[I_0''(U)\right]$, namely v is a solution of

$$\begin{cases}-\Delta u-\mu\dfrac{u}{|x|^2}=\left(2^*(s)-1\right)\dfrac{U^{2^*(s)-2}(x)}{|x|^s}u, & x\in\mathbb{R}^N\\ u\in D_r^{1,2}\left(\mathbb{R}^N\right),\ u>0, & x\in\mathbb{R}^N\end{cases}\tag{2.2}$$

then $v\in T_U Z$, namely $\exists a\in\mathbb{R}$ such that $v=aD_\varepsilon z_\varepsilon\big|_{\varepsilon=1}$, where D_ε denotes the derivatives with respect to the parameter ε. We look for solutions $u\in D_r^{1,2}\left(\mathbb{R}^N\right)$ of problem (2.2). One has

$$-\Psi''-\frac{n-1}{r}\Psi'-\mu\frac{\Psi}{r^2}=\left(2^*(s)-1\right)\frac{U^{2^*(s)-2}}{|r|^s}\Psi,$$

and then a first solution is given by

$$w=D_\varepsilon z_\varepsilon\big|_{\varepsilon=1}=C\left(r^{\frac{(2-s)\sqrt{\bar{\mu}-\mu}}{\sqrt{\bar{\mu}}}}-1\right)\Bigg/\left(r^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}\left(r^{\frac{(2-s)\sqrt{\bar{\mu}-\mu}}{\sqrt{\bar{\mu}}}}+1\right)^{\frac{N-s}{2-s}}\right)$$

which belongs to $D_r^{1,2}\left(\mathbb{R}^N\right)$, where $C=\sqrt{\bar{\mu}-\mu}\left(\dfrac{4(N-s)(\bar{\mu}-\mu)}{N-2}\right)^{\frac{N-2}{2(2-s)}}$. If we look for a second independent solution of the form $u(r)=c(r)w(r)$, we will check that u is not in $D_r^{1,2}\left(\mathbb{R}^N\right)$. A direct computation gives

$$-\left(c''w+2c'w'+cw''\right)-\frac{N-1}{r}\left(c'w+cw'\right)-\mu\frac{cw}{r^2}=\left(2^*(s)-1\right)\frac{U^{2^*(s)-2}(r)}{|r|^s}cw,$$

and because w is a solution, we have

$$-c''w-c'\left(2w'+\frac{N-1}{r}w\right)=0.$$

Setting $v=c'$, we obtain

$$-\frac{v'}{v}=2\frac{w'}{w}+\frac{N-1}{r},$$

namely

$$v(r)=\frac{1}{r^{N-1}w^2(r)}\approx Cr^{1-N+2\sqrt{\bar{\mu}}-2\sqrt{\bar{\mu}-\mu}}\quad\left(r\to0^+\right),$$

where C is a constant. This implies $c(r)\approx Cr^{2-N+2\sqrt{\bar{\mu}}-2\sqrt{\bar{\mu}-\mu}}$ as well as

$$u(r)\approx Cr^{\frac{2-N}{2}-\sqrt{\bar{\mu}-\mu}},$$

as $r\to0^+$. Since $\dfrac{2-N}{2}-\sqrt{\bar{\mu}-\mu}<0$, we have $u\notin D_r^{1,2}\left(\mathbb{R}^N\right)$. This implies a contradiction to assumption which had been made. So $T_U Z = \mathrm{Ker}\left[I_0''(U)\right]$. This completes the proof of Lemma. □

Lemma 2.2. For all $\varepsilon > 0$, $I_0''(z_\varepsilon)$ is a Fredholm operator with index zero.

Proof. Indeed, $D_r^{1,2}(\mathbb{R}^N)$ is a Hilbert space, this implies

$D_r^{1,2}(\mathbb{R}^N) \cong \left[D_r^{1,2}(\mathbb{R}^N)\right]^*$ and $T_U Z = \mathrm{Ker}\left[I_0''(U)\right]$, we have

$$I_0''(U): D_r^{1,2}(\mathbb{R}^N) \to \left[D_r^{1,2}(\mathbb{R}^N)\right]^* = D_r^{1,2}(\mathbb{R}^N);$$

$$I_0''(U)(v+w) = I_0''(U)(w), \quad \text{where } v \in T_U Z, w \in \left[T_U Z\right]^\perp;$$

$$I_0''(U)(w) = -\Delta w - \mu \frac{w}{|x|^2} - \left(2^*(s)-1\right)\frac{U^{2^*(s)-2}(x)}{|x|^s} w.$$

It is obviously that $I_0''(U)$ is a self-adjoint operator on $D_r^{1,2}(\mathbb{R}^N)$, we have $\left(\mathrm{Im}\left(I_0''(U)\right)\right)^\perp = T_U Z$, hence

$$\mathrm{codim}\left(\mathrm{Im}\left(I_0''(U)\right)\right) = \dim\left(D_r^{1,2}(R^N)\big/\left[T_U Z\right]^\perp\right) = \dim T_U Z = 1.$$

Moreover, fox fixed $u \in D_r^{1,2}(\mathbb{R}^N)$, the map

$$v \mapsto \int_{\mathbb{R}^N} a(x) u v \mathrm{d}x$$

is a bounded linear functional in $D_r^{1,2}(\mathbb{R}^N)$, where

$a(x) = \left(2^*(s)-1\right)\frac{U^{2^*(s)-2}(x)}{|x|^s}$. So by the Riesz representation theorem, there is an element in $D_r^{1,2}(\mathbb{R}^N)$, denote it by Tu, such that

$$\langle Tu, v\rangle = \int_{\mathbb{R}^N} a(x) u v \mathrm{d}x. \tag{2.3}$$

Clearly $T: D_r^{1,2}(\mathbb{R}^N) \to D_r^{1,2}(\mathbb{R}^N)$ is linear symmetric and bounded. Moreover T is compact; indeed, let $\{u_n\}$ be a bounded sequence in $D_r^{1,2}(\mathbb{R}^N)$. Passing to a subsequence we may assume that $u_n \rightharpoonup u$ in $D_r^{1,2}(\mathbb{R}^N)$, $u_n \to u$ in $L^{2^*(s)}(\mathbb{R}^N)$. Use u replaced by $u_n - u$ and v by $Tu_n - Tu$ in (2.3), and apply Hölder's inequality with $\frac{1}{2^*(s)} + \frac{1}{2^*(s)} + \frac{1}{p} = 1\left(p = \frac{N-s}{2-s}\right)$ to get

$$\|Tu_n - Tu\|^2 \le \|a\|_{L^p} \|u_n - u\|_{L^{2^*(s)}} \|Tu_n - Tu\|_{L^{2^*(s)}}$$
$$\Rightarrow \|Tu_n - Tu\| \le c\|u_n - u\|_{L^{2^*(s)}},$$

which implies that $Tu_n \to Tu$ in $D_r^{1,2}(\mathbb{R}^N)$. This shows that T is compact. We have

$$\langle I_0''(U)u, v\rangle = \langle u, v\rangle - \langle Tu, v\rangle = \langle u - Tu, v\rangle = \langle (I-T)u, v\rangle.$$

So $I_0''(U) = I - T$, where I is an identical operator. By the fact that $\lambda I - T$ is a Fredholm operator with index zero, where $\lambda \ne 0$ and T is compact, we obtain that $I_0''(U) = I - T$ is a Fredholm operator with index zero. This completes the proof of Lemma. □

Now, we give the abstract perturbation method, which is crucial in our proof of the main results of this paper.

Lemma 2.3. [13] **(Abstract Perturbation Method)** Let E be a Hilbert space and let $f_0, G \in C^2(E, \mathbb{R})$ be given. Consider the perturbed functional $f_\varepsilon(u) = f_0(u) - \varepsilon G(u)$.

Suppose that f_0 satisfies:

1) f_0 has a finite dimensional manifold of critical points Z, let $b = f_0(z)$, for all $z \in Z$;

2) for all $z \in Z$, $D^2 f_0(z)$ is a Fredholm operator with index zero;

3) for all $z \in Z$, $T_z Z = \mathrm{Ker} D^2 f_0(z)$.

Hereafter we denote by Γ the functional $G|_Z$.

Let f_0 satisfy (1)-(3) above and suppose that there exists a critical point $\bar{z} \in Z$ of Γ such that one of the following conditions hold:

1) $\bar{z}$ is nondegenerated;

2) $\bar{z}$ is a proper local minimum or maximum;

3) $\bar{z}$ is isolated and the local topological degree of Γ' at $\bar{z}$, $\deg_{loc}(\Gamma', 0)$ is different from zero. Then for $|\varepsilon|$ small enough, the functional f_ε has a critical point u_ε such that $u_\varepsilon \to \bar{z}$, as $\varepsilon \to 0$.

Remark 2.4. [13] If $Z_0 := \{z \in Z : \Gamma(z) = \min_Z \Gamma\}$ is compact, then one can still prove that f_ε has a critical point near Z_0. The set Z_0 could also consist of local minima and the same for maxima.

3. Proof of the Theorems

We will now solve the bifurcation equation. In order to do this, let us define the reduced functional, see [14],

$$\Phi_\delta : Z \to \mathbb{R}$$

$$\Phi_\delta(z_\varepsilon) = I_\delta(z_\varepsilon + \omega_\delta(z_\varepsilon))$$
$$= c_0 - \frac{\delta}{2^*(s)} \int_{\mathbb{R}^N} h(x) \frac{z_\varepsilon^{2^*(s)}(x)}{|x|^s} \mathrm{d}x + o(\delta),\ c_0 = I_0(U),$$

where $\omega_\delta(z_\varepsilon) \perp T_{z_\varepsilon} Z$ and verifies $\|\omega_\delta(z_\varepsilon)\delta^{-1}\| \le C$ as $\delta \to 0$. Hence we are led to study the finite-dimensional functional

$$\Gamma(\varepsilon) := \int_{\mathbb{R}^N} h(x) \frac{z_\varepsilon^{2^*(s)}(x)}{|x|^s} \mathrm{d}x = \int_{\mathbb{R}^N} h(\varepsilon x) \frac{U^{2^*(s)}(x)}{|x|^s} \mathrm{d}x,\ (\varepsilon > 0).$$

The functional $\Gamma(\varepsilon)$ can be extended by continuity to $\varepsilon = 0$ by setting

$$\Gamma(0) = h(0) \int_{\mathbb{R}^N} \frac{U^{2^*(s)}(x)}{|x|^s} \mathrm{d}x.$$

Here we will prove the existence result by showing that problem (1.1) has a positive radial solution provided that h satisfies some integrability conditions. Before giving the proof of the main results, we need the following lemma.

Lemma 3.1. If (H) holds, then $\Gamma_r(\varepsilon)\to 0$ as $\varepsilon\to+\infty$.

Proof. From the definition of $\Gamma(\varepsilon)$ and *U*, we have

$$\Gamma_r(\varepsilon)=\int_0^{+\infty}h(\varepsilon r)U^{2^*(s)}(r)r^{N-1-s}\mathrm{d}r$$

$$=\int_0^{+\infty}h(\varepsilon r)\frac{\left((N-s)(N-2)\right)^{\frac{N-s}{2-s}}}{\left(1+|r|^{2-s}\right)^{\frac{2(N-s)}{2-s}}}r^{N-1-s}\mathrm{d}r$$

$$=\int_0^{+\infty}h(r)\frac{\left((N-s)(N-2)\right)^{\frac{N-s}{2-s}}}{\left(1+\left|\frac{r}{\varepsilon}\right|^{2-s}\right)^{\frac{2(N-s)}{2-s}}}\frac{r^{N-1-s}}{\varepsilon^{N-s}}\mathrm{d}r$$

$$=\left((N-s)(N-2)\right)^{\frac{N-s}{2-s}}\int_0^{+\infty}h(r)\frac{\varepsilon^{N-s}\cdot r^{N-1-s}}{\left(\varepsilon^{2-s}+r^{2-s}\right)^{\frac{2(N-s)}{2-s}}}\mathrm{d}r$$

$$\le C\varepsilon^{-(N-s)}\int_0^1 h(r)\cdot r^{N-1-s}\mathrm{d}r+C_1\varepsilon^{\alpha-(N-s)}\int_1^{+\infty}\frac{h(r)}{r^\alpha}\cdot r^{N-1-s}\mathrm{d}r,$$

where $\alpha<N-s$. It is easy to get the first integral in the right hand side; next we show the second integral: In fact,

$$\left(1+\left(\frac{r}{\varepsilon}\right)^{2-s}\right)^{\frac{2(N-s)}{2-s}}\cdot\frac{\varepsilon^\alpha}{r^\alpha}\ge 1\quad(\alpha<N-s),$$

so we have

$$\int_1^{+\infty}h(r)\frac{\varepsilon^{N-s}\cdot r^{N-1-s}}{\left(\varepsilon^{2-s}+r^{2-s}\right)^{\frac{2(N-s)}{2-s}}}\mathrm{d}r\le\varepsilon^{\alpha-(N-s)}\int_1^{+\infty}\frac{h(r)}{r^\alpha}\cdot r^{N-1-s}\mathrm{d}r\ (\alpha<N-s).$$

we deduce that $\Gamma_r(\varepsilon)\to 0$ as $\varepsilon\to+\infty$.

Proof of Theorem 1. Firstly, we claim that $\Gamma_r(\varepsilon)$ is not identically equal to 0. To prove this claim we will use Fourier analysis. We introduce some notation that will be used in the following discussion. If $g\in L^1\left([0,\infty),\frac{\mathrm{d}r}{r}\right)$, we define

$$M[g](s)=\int_0^\infty r^{-is}g(r)\frac{\mathrm{d}r}{r},$$

M is nothing but the Mellin transform. The associated convolution is defined by

$$(g\times k)(s)=\int_0^\infty g(r)k\left(\frac{s}{r}\right)\frac{\mathrm{d}r}{r}.$$

From the definition, it follows that $M[g\times k]=M[g]\cdot M[k]$. Indeed,

$$M[g(x)\times k(x)](s)=F\left[g(e^x)\times k(e^x)\right](s)$$

$$=\int_{-\infty}^{+\infty}\left[g(e^x)\times k(e^x)\right]e^{-ixs}\mathrm{d}x=\int_{-\infty}^{+\infty}\left[\int_0^{+\infty}g(z)k\left(\frac{e^x}{z}\right)\frac{\mathrm{d}z}{z}\right]e^{-ixs}\mathrm{d}x\ \left(z=e^t\right)$$

$$=\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}g(e^t)k(e^{x-t})\mathrm{d}t\right]e^{-ixs}\mathrm{d}x$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} g\left(e^t\right)e^{-its}k\left(e^{x-t}\right)e^{-i(x-t)s}\mathrm{d}t\mathrm{d}x$$
$$= \int_{-\infty}^{+\infty} g\left(e^t\right)e^{-its}\mathrm{d}t\int_{-\infty}^{\infty} k\left(e^{x-t}\right)e^{-i(x-t)s}\mathrm{d}x$$
$$= M\left[g(x)\right](s)\cdot M\left[k(x)\right](s).$$

With this notation we can write our Γ_r in the form

$$\Gamma_r(\varepsilon) = \int_0^{\infty} h(r)U^{2^*(s)}\left(\frac{r}{\varepsilon}\right)\left(\frac{r}{\varepsilon}\right)^{N-s}\frac{\mathrm{d}r}{r}.$$

We set $m = N - s - \alpha$ and

$$g(r) = h(r)r^m,\ k(r) = U^{2^*(s)}\left(\frac{1}{r}\right)\left(\frac{1}{r}\right)^{N-s-m}.$$

Note that $g,k \in L^1\left([0,\infty),\frac{\mathrm{d}r}{r}\right)$. We have $\Gamma_r(\varepsilon) = \varepsilon^{-m}(g\times k)(\varepsilon)$ and hence if, by contradiction, $\Gamma \equiv 0$ then $g\times k \equiv 0$ and one has

$$M[g\times k] = M[g]\cdot M[k] \equiv 0.$$

On the other hand, $M[k]$ is real analytic and so has a discrete number of zeros. By continuity it follows that $M[g]\equiv 0$. Then g and hence h are identically equal to 0. We arrive at a contradiction. This proves the claim. Since $\Gamma_r(0) = 0$, $\Gamma_r(\varepsilon)\to 0$ as $\varepsilon\to+\infty$, and $\Gamma_r \not\equiv 0$, it follows that Γ_r has a maximum or a minimum at some $\bar{\varepsilon} > 0$. By a straight application of **Lemma 2.3** jointly with **Remark 2.4**, the result follows. □

Proof of Theorem 2. Using **Lemma 3.1**, we have $\Gamma_r(\varepsilon)\to 0$ as $\varepsilon\to+\infty$. and Γ_r can be extended to $\varepsilon = 0$ by continuity setting $\Gamma_r(0) = a_0 h(0)$, where $a_0 = \int_0^{+\infty} U^{2^*(s)}(r)r^{N-1-s}\mathrm{d}r > 0.$ From the assumption, we have

$$\Gamma_r'(0) = 0,\ \Gamma_r''(0) = a_1 h''(0),\ a_1 = \int_0^{+\infty} U^{2^*(s)}(r)r^{N+1-s}\mathrm{d}r > 0$$

and the condition $h(0)h''(0) > 0$ implies that Γ_r has a (global) maximum (if $h(0) > 0$) or a (global) minimum (if $h(0) < 0$), at some $\bar{\varepsilon} > 0$. This allows us to use the abstract results, yielding a radial solution of problem (1.1), for $|\delta|$ small. □

Proof of Theorem 3. It suffices to remark that

$$\Gamma_r(1) = \left[(N-s)(N-2)\right]^{\frac{N-s}{2-s}}\int_0^{\infty} h(r)\left(1+r^{2-s}\right)^{-\frac{2(N-s)}{2-s}} r^{N-s-1}\mathrm{d}r \neq 0.$$

If

$$\int_0^{\infty} h(r)\left(1+r^{2-s}\right)^{-\frac{2(N-s)}{2-s}} r^{N-s-1}\mathrm{d}r > 0$$
$$\left(\text{resp. } \int_0^{\infty} h(r)\left(1+r^{2-s}\right)^{-\frac{2(N-s)}{2-s}} r^{N-s-1}\mathrm{d}r < 0\right)$$

then $h(0)\le 0$ (resp. $h(0)\ge 0$) and, once more Γ_r has a (global) maximum (resp. a (global) minimum) at some $\bar{\varepsilon} > 0$. □

In the rest of the section we will give the proof of **Theorem 4** and **Theorem 5**.

First we give the following **Lemma**. Hypothesis (H') allows us to use the following Riemann-Lebesgue convergence result.

Lemma 3.2 [13] Let $Q=[0,T]^N$ be a cube in $\mathbb{R}^N$, and $f\in L^q(Q)$ be a T-periodic function. Consider $f_\varepsilon(x)=f(\varepsilon x)$, then

$$f_\varepsilon \rightharpoonup \overline{f}=\frac{1}{|Q|}\int_Q f\,\mathrm{d}x,\ \text{weakly in } L^q_{loc}(\mathbb{R}^N),\ \text{as } \varepsilon\to\infty.$$

Lemma 3.3 If (H') holds, then

$$\Gamma_r(\varepsilon)\to 0,\ \varepsilon\to+\infty.$$

Proof. Given $\varepsilon>0$, there exists $R>0$ large enough such that

$$\left|\int_R^\infty h(r)z_\varepsilon^{2^*(s)}(r)r^{N-s-1}\mathrm{d}r\right|$$
$$\le\|h(r)\|_\infty\int_R^\infty z_\varepsilon^{2^*(s)}(r)r^{N-s-1}\mathrm{d}r<\varepsilon.$$

On the other hand, the remainder integral over the interval $0\le r<R$ tends to 0 as $\varepsilon\to\infty$ because of hypothesis (H') and the Riemann-Lebesgue lemma. □

Proof of Theorem 4. Using **Lemma 3.3**, we have $\Gamma_r(\varepsilon)\to 0$ as $\varepsilon\to+\infty$. and Γ_r can be extended to $\varepsilon=0$ by continuity setting $\Gamma_r(0)=a_0h(0)$, where $a_0=\int_0^{+\infty}U^{2^*(s)}(r)r^{N-1-s}\mathrm{d}r>0$. From the assumption, we have

$$\Gamma_r'(0)=0,\ \Gamma_r''(0)=a_1h''(0),\ a_1=\int_0^{+\infty}U^{2^*(s)}(r)r^{N+1-s}\mathrm{d}r>0.$$

and the condition $h(0)h''(0)>0$ implies that Γ_r has a (global) maximum (if $h(0)>0$) or a (global) minimum (if $h(0)<0$), at some $\overline{\varepsilon}>0$. This allows us to use the abstract results, yielding a radial solution of problem (1.1), for $|\delta|$ small. □

Proof of Theorem 5. It suffices to repeat the arguments used to prove **Theorem 1** using **Lemma 3.1** instead of **Lemma 3.3**.

4. Conclusion

We study a class of semilinear elliptic problems involving critical Hardy-Sobolev exponent and Hardy terms, and obtain positive radial solutions for these problems via an abstract perturbation method in critical point theory. Extensions of nonradial solutions for these problems are being investigated by the author. Results will be submitted for publication in the near future.

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