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Nonlinear Super Integrable Couplings of Super Yang Hierarchy and Its Super Hamiltonian Structures

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Abstract

Nonlinear super integrable couplings of the super Yang hierarchy based upon an enlarged matrix Lie super algebra were constructed. Then its super Hamiltonian structures were established by using super trace identity. As its reduction, nonlinear integrable couplings of Yang hierarchy were obtained.

Keywords

Lie Super Algebra, Nonlinear Super Integrable Couplings, Super Yang Hierarchy, Super Hamiltonian Structures

1. Introduction

With the development of soliton theory, super integrable systems associated with Lie superalgebra have aroused growing attentions by many mathematicians and physicists. It was known that super integrable systems contained the odd variables, which would provide more prolific fields for mathematical researchers and physical ones. Several super integrable systems including super AKNS hierarchy, super KdV hierarchy, super KP hierarchy etc., have been studied in [1] [2] [3] [4]. There are some interesting results on the super integrable systems, such as Darboux transformation [5], super Hamiltonian structures in [6] [7], binary nonlinearization [8] and reciprocal transformation [9] and so on.

There search of integrable couplings of the well known integrable hierarchy has received considerable attention [10] [11] [12]. A few approaches to construct linear integrable couplings of the classical soliton equation are presented by permutation, enlarging spectral problem, using matrix Lie algebra [12] constructing new loop Lie algebra and creating semi-direct sums of Lie algebra. Zhang [13] once employed two kinds of explicit Lie algebra F and G to obtain the nonlinear integrable couplings of the GJ hierarchy and Yang hierarchy, respectively. Recently, You [14] presented a scheme for constructing nonlinear super integrable couplings for the super integrable hierarchy.

Inspired by Zhang [13] and You [14], we hope to construct nonlinear super integrable couplings of the super Yang hierarchy through enlarging matrix Lie super algebra. We take the Lie algebra $sl(2|1)$ as an example to illustrate the approach for extending Lie super algebras. Based on the enlarged Lie super algebra $sl(4|1)$, we work out nonlinear super integrable Hamiltonian couplings of the super Yang hierarchy. Finally, we will reduce the nonlinear super Yang integrable Hamiltonian couplings to some special cases.

2. Enlargement of Lie Super Algebra

Consider the Lie super algebra $sl(2|1)$. Its basis is

$$E_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},E_2=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},E_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},E_4=\begin{pmatrix}0&0&1\\0&0&0\\0&-1&0\end{pmatrix},E_5=\begin{pmatrix}0&0&0\\0&0&1\\1&0&0\end{pmatrix}. \quad (1)$$

where E_1,E_2,E_3 are even element and E_4,E_5 are odd elements. Their non-zero (anti) commutation relations are

$$\begin{aligned}&[E_1,E_2]=-2E_3,[E_1,E_3]=-2E_2,[E_1,E_4]=E_5,[E_1,E_5]=E_4,[E_2,E_3]=-2E_1,\\&[E_2,E_4]=-E_5,[E_2,E_5]=E_4,[E_3,E_4]=E_4,[E_3,E_5]=-E_5,[E_4,E_4]=-(E_1+E_2),\\&[E_4,E_5]=[E_5,E_4]=E_3,[E_5,E_5]=E_1-E_2.\end{aligned} \quad (2)$$

Let us enlarge the Lie super algebra $sl(2|1)$ to the Lie super algebra $sl(4|1)$ with a basis

$$\begin{aligned}&e_1=\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&0\end{pmatrix},e_2=\begin{pmatrix}0&1&0&0&0\\-1&0&0&0&0\\0&0&0&1&0\\0&0&-1&0&0\\0&0&0&0&0\end{pmatrix},e_3=\begin{pmatrix}1&0&0&0&0\\0&-1&0&0&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&0\end{pmatrix},e_4=\begin{pmatrix}0&0&0&1&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&0\end{pmatrix},\\&e_5=\begin{pmatrix}0&0&0&1&0\\0&0&-1&0&0\\0&0&0&1&0\\0&0&-1&0&0\\0&0&0&0&0\end{pmatrix},e_6=\begin{pmatrix}0&0&1&0&0\\0&0&0&-1&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&0\end{pmatrix},e_7=\begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&-1&0&1&0\end{pmatrix},e_8=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\1&0&-1&0&0\end{pmatrix}.\end{aligned} \quad (3)$$

where e_1,e_2,e_3,e_4,e_5,e_6 are even, and e_7,e_8 are odd.

The generator of Lie super algebra $sl(4|1)$, $e_i(1\le i\le 8)$ satisfy the following (anti) commutation relations:

$$\begin{aligned}&[e_1,e_2]=-2e_3,[e_1,e_3]=-2e_2,[e_1,e_5]=-2e_6,[e_1,e_6]=-2e_5,[e_1,e_7]=e_8,[e_1,e_8]=e_7,[e_2,e_3]=-2e_1,[e_2,e_4]=2e_6,\\&[e_2,e_6]=-2e_4,[e_2,e_7]=-e_8,[e_2,e_8]=e_7,[e_3,e_4]=2e_5,[e_3,e_5]=2e_4,[e_3,e_7]=e_7,[e_3,e_8]=-e_8,[e_4,e_5]=-2e_6,\\&[e_4,e_6]=-2e_5,[e_5,e_6]=-2e_4,[e_7,e_7]=-e_1-e_2+e_4+e_5,[e_7,e_8]=[e_8,e_7]=e_3-e_6,[e_8,e_8]=e_1-e_2-e_4+e_5,\\&[e_1,e_4]=[e_2,e_5]=[e_3,e_6]=[e_4,e_7]=[e_4,e_8]=[e_5,e_7]=[e_5,e_8]=[e_6,e_7]=[e_6,e_8]=0.\end{aligned} \quad (4)$$

Define a loop super algebra corresponding to the Lie super algebra *gl*(6, 2), denote by

$$\begin{aligned}\widetilde{sl}(4|1) &= sl(4|1)\otimes\mathbb{C}\left[\lambda,\lambda^{-1}\right]\\ &=\left\{e_i\lambda^m, e_i\in sl(4|1), m=0,\pm1,\cdots\right\}.\end{aligned} \tag{5}$$

The corresponding (anti) commutative relations are given as

$$\left[e_i\lambda^m, e_j\lambda^n\right]=\left[e_i,e_j\right]\lambda^{m+n}, \forall e_i,e_j\in sl(4|1). \tag{6}$$

3. Nonlinear Super Integrable Couplings of Super Yang Hierarchy

If Let us start from an enlarged spectral problem associated with $sl(4|1)$,

$$\begin{aligned}\phi_x &= U(u,\lambda)\phi, U\\ &= e_2(1)+qe_2(0)+re_1(0)+se_3(0)+u_1e_5(0)+u_2e_4(0)\\ &\quad +u_3e_6(0)+\alpha e_7(0)+\beta e_8(0).\end{aligned} \tag{7}$$

where q,r,s,u_1,u_2,u_3 are even potentials, but α,β are odd ones.

In order to obtain super integrable couplings of super Yang hierarchy, we solve the adjoint representation of (7),

$$V_x=[U,V], \tag{8}$$

with

$$\begin{aligned}V &= Ae_1(0)+Be_2(0)+Ce_3(0)+Ee_4(0)+Fe_5(0)\\ &\quad +Ge_6(0)+\rho e_7(0)+\delta e_8(0).\end{aligned} \tag{9}$$

where A,B,C,E,F and G are commuting fields, and ρ,δ are anti-commuting fields.

Substituting

$$\begin{aligned}&A=\sum_{m\ge0}A_m\lambda^{-m}, B=\sum_{m\ge0}B_m\lambda^{-m}, C=\sum_{m\ge0}C_m\lambda^{-m}, E=\sum_{m\ge0}E_m\lambda^{-m},\\ &F=\sum_{m\ge0}F_m\lambda^{-m}, G=\sum_{m\ge0}G_m\lambda^{-m}, \rho=\sum_{m\ge0}\rho_m\lambda^{-m}, \delta=\sum_{m\ge0}\delta_m\lambda^{-m}.\end{aligned} \tag{10}$$

into previous equation gives the following recursive formulas

$$\begin{cases}A_{m,x}=-2rB_m+2C_{m+1}+2qC_m+\beta\rho_m+\alpha\delta_m,\\ B_{m,x}=-2rA_m+2sC_m-\alpha\rho_m-\beta\delta_m,\\ C_{m,x}=-2qA_m-2A_{m+1}+2sB_m-\alpha\rho_m+\beta\delta_m,\\ E_{m,x}=-2u_2B_m+2u_1C_m-2rF_m-2u_2F_m+2G_{m+1}+2qG_m+2u_1G_m-\beta\rho_m-\alpha\delta_m,\\ F_{m,x}=-2u_2A_m+2u_3C_m-2rE_m-2u_2E_m+2sG_m+2u_3G_m+\alpha\rho_m+\beta\delta_m,\\ G_{m,x}=-2u_1A_m+2u_3B_m-2E_{m+1}-2qE_m-2u_1E_m+2sF_m+2u_3F_m+\alpha\rho_m-\beta\delta_m,\\ \rho_{m,x}=-\alpha A_m-\beta B_m-\beta C_m+s\rho_m+\delta_{m+1}+q\delta_m+r\delta_m,\\ \delta_{m,x}=\beta A_m+\alpha B_m-\alpha C_m-\rho_{m+1}-q\rho_m+r\rho_m-s\delta_m.\end{cases} \tag{11}$$

From previous equations, we can successively deduce

$$\begin{aligned}
&B_0 = 1, A_0 = C_0 = E_0 = G_0 = \rho_0 = \delta_0 = 0, F_0 = \varepsilon = \text{const.},\\
&A_1 = s, B_1 = 0, C_1 = r, E_1 = u_3 + \varepsilon s + +\varepsilon u_3, F_1 = 0,\\
&G_1 = \varepsilon r + u_2 + \varepsilon u_2, \rho_1 = \alpha, \delta_1 = \beta, A_2 = -\frac{1}{2} r_x - qs,\\
&B_2 = \frac{1}{2} r^2 + \frac{1}{2} s^2 + \alpha\beta, C_2 = \frac{1}{2} s_x - qr,\\
&E_2 = -\frac{1}{2}\varepsilon r_x - \frac{1}{2} u_{2x} - \frac{1}{2}\varepsilon u_{2x} - su_1 - qu_3 - \varepsilon qs,\\
&F_2 = \frac{1}{2}\varepsilon r^2 + \frac{1}{2}\varepsilon s^2 + (1+\varepsilon)\left(ru_2 + su_3 + \frac{1}{2}u_2^2 + \frac{1}{2}u_3^2\right) - \alpha\beta,\\
&G_2 = \frac{1}{2}\varepsilon s_x - \varepsilon qr + (1+\varepsilon)\left(\frac{1}{2}u_{3x} - ru_1 - qu_2 - u_1 u_2\right),\\
&\rho_2 = -\beta_x - q\alpha, \delta_2 = \alpha_x - q\beta.
\end{aligned}$$

Equation (11) can be written as

$$\begin{aligned}
&\left(-2B_{n+1} - F_{n+1}, 2C_{n+1} + G_{n+1}, 2A_{n+1} + E_{n+1}, -B_{n+1} - F_{n+1}, C_{n+1} + G_{n+1}, A_{n+1} + E_{n+1}, \delta_{n+1}, -\rho_{n+1}\right)^T\\
&= L\left(-2B_n - F_n, 2C_n + G_n, 2A_n + E_n, -B_n - F_n, C_n + G_n, A_n + E_n, \delta_n, -\rho_n\right)^T,
\end{aligned} \tag{12}$$

where

$$L = \begin{pmatrix}
0 & -\partial^{-1}r\partial + 2\partial^{-1}qs & -\partial^{-1}s\partial - 2\partial^{-1}qr & 0 & -\partial^{-1}u_2\partial - 2\partial^{-1}(su_1 + qu_3 + u_1u_3) & -\partial^{-1}u_3\partial - 2\partial^{-1}(ru_1 + qu_2 + u_1u_2) & -\partial^{-1}\alpha\partial - \partial^{-1}q\beta & -\partial^{-1}\beta\partial + \partial^{-1}q\alpha\\
-r & -q & \frac{1}{2}\partial & -u_2 & -u_1 & 0 & -\frac{1}{2}\alpha & \frac{1}{2}\beta\\
-s & -\frac{1}{2}\partial & -q & -u_3 & 0 & -u_1 & \frac{1}{2}\partial & \frac{1}{2}\alpha\\
0 & 0 & 0 & 0 & -\partial^{-1}r\partial - \partial^{-1}u_2\partial + 2\partial^{-1}(qs + su_1 + qu_3 + u_1u_3) & -\partial^{-1}s\partial - \partial^{-1}u_3\partial - 2\partial^{-1}(qr + ru_1 + qu_2 + u_1u_2) & 0 & 0\\
0 & 0 & 0 & -r - u_2 & -q - u_1 & \frac{1}{2}\partial & 0 & 0\\
0 & 0 & 0 & -s - u_3 & -\frac{1}{2}\partial & -q - u_1 & 0 & 0\\
-\beta & \beta & \alpha & \beta & -\beta & -\alpha & -q - r & s - \partial\\
\alpha & \alpha & -\beta & -\alpha & -\alpha & \beta & s + \partial & -q + r
\end{pmatrix}.$$

Then, let us consider the spectral problem (7) with the following auxiliary problem

$$\phi_{t_n} = V^{(n)}\phi \tag{13}$$

with

$$\begin{aligned}
V^{(n)} = &\sum_{j=1}^{n}\Big(A_j e_1(n-j) + B_j e_2(n-j) + C_j e_3(n-j) + E_j e_4(n-j) + F_j e_5(n-j)\\
&+ G_j e_6(n-j) + \rho_j e_7(n-j) + \delta_j e_8(n-j)\Big) + B_{n+1} e_2(0) + F_{n+1} e_5(0),
\end{aligned} \tag{14}$$

From the compatible condition $\phi_{x,t_n} = \phi_{t_n,x}$, according to (7) and (13), we get the zero curvature equation

$$U_{t_n} - V_x^{(n)} + \left[U, V^{(n)}\right] = 0. \tag{15}$$

which gives a nonlinear Lax super integrable hierarchy

$$u_{t_n} = \begin{pmatrix} q\\ r\\ s\\ u_1\\ u_2\\ u_3\\ \alpha\\ \beta \end{pmatrix}_{t_n} = \begin{pmatrix} B_{n+1,x}\\ -2A_{n+1} - 2sB_{n+1}\\ 2rB_{n+1} + 2B_{n+1}\\ F_{n+1,x}\\ -2u_3 A_{n+1} - 2sF_{n+1} - 2u_3 F_{n+1}\\ 2u_2 B_{n+1} + 2rF_{n+1} + 2u_2 F_{n+1} + 2G_{n+1}\\ \delta_{n+1} + \beta\\ -\rho_{n+1} - \alpha B_{n+1} \end{pmatrix}. \tag{16}$$

The super integrable hierarchy (16) is a nonlinear super integrable couplings for the Yang hierarchy in [15]

$$\begin{aligned}\tilde{u}_{t_n} &= \left(q,r,s,\alpha,\beta\right)^T_{t_n} \\ &= \left(B_{n+1,x}, -2A_{n+1}-2sB_{n+1}, 2rB_{n+1}+2B_{n+1}, \ \delta_{n+1}+\beta B_{n+1}, -\rho_{n+1}-\alpha B_{n+1}\right)^T.\end{aligned} \tag{17}$$

4. Super Hamiltonian Structures

A direct calculation reads

$$\begin{aligned}&Str\left(U_\lambda,V\right)=-4B-2F, Str\left(U_q,V\right)=-4B-2F,\\ &Str\left(U_r,V\right)=4C+2G, Str\left(U_s,V\right)=4A+2E,\\ &Str\left(U_{u_1},V\right)=-2B-2F, Str\left(U_{u_2},V\right)=2C+2G,\\ &Str\left(U_{u_3},V\right)=2A+2E, Str\left(U_\alpha,V\right)=2\delta, Str\left(U_\beta,V\right)=-2\rho.\end{aligned} \tag{18}$$

Substituting above results into the super trace identity [7]

$$\frac{\delta}{\delta u}\int Str\left(\frac{\delta U}{\delta\lambda}V\right)\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}Str\left(\frac{\delta U}{\delta u}V\right). \tag{19}$$

Comparing the coefficients of λ^{-n-1} on both side of (19). From the initial values in (11), we obtain $\gamma=0$. Thus we have

$$\begin{cases}\dfrac{\delta H_n}{\delta u}=\left(-2B_n-F_n, 2C_n+G_n, 2A_n+E_n, -B_n-F_n, C_n+G_n, A_n+E_n, \delta_n, -\rho_n\right)^T,\\ H_n=\displaystyle\int\frac{2B_{n+1}+F_{n+1}}{n+1}\mathrm{d}x, n\geq 0.\end{cases} \tag{20}$$

It then follows that the nonlinear super integrable couplings (16) possess the following super Hamiltonian form

$$u_{t_n}=K_n\left(u\right)=J\frac{\delta H_n}{\delta u}. \tag{21}$$

where

$$J=\begin{pmatrix}-\partial & -s & r & \partial & s & -r & \frac{1}{2}\beta & -\frac{1}{2}\alpha\\ s & 0 & -1 & -s & 0 & 1 & 0 & 0\\ -r & 1 & 0 & r & -1 & 0 & 0 & 0\\ \partial & s & -r & -2\partial & -2s-u_3 & 2r-u_2 & -\frac{1}{2}\beta & \frac{1}{2}\alpha\\ -s & 0 & 1 & 2s+u_3 & 0 & -2 & 0 & 0\\ r & -1 & 0 & -2r+u_2 & 2 & 0 & 0 & 0\\ -\frac{1}{2}\beta & 0 & 0 & \frac{1}{2}\beta & 0 & 0 & \frac{1}{2} & 0\\ \frac{1}{2}\alpha & 0 & 0 & -\frac{1}{2}\alpha & 0 & 0 & 0 & \frac{1}{2}\end{pmatrix}. \tag{22}$$

is a super Hamiltonian operator and $H_n(n\geq 0)$ are Hamiltonian functions.

5. Reductions

Taking $\alpha=\beta=0,$ (16) reduces to a nonlinear integrable couplings of the Yang hierarchy in [13].

When $n=2$ in (16), we obtain the nonlinear super integrable couplings of the second order super Yang equations

$$
\left\{
\begin{aligned}
q_{t_2} &= -\frac{1}{2}r_{xx}s + \frac{1}{2}rs_{xx} - 2qqr_x - 2qss_x - q_x r^2 - q_x s^2 + \alpha\alpha_{xx} + \beta\beta_{xx} - 2\left(q\alpha\beta\right)_x, \\
r_{t_2} &= \frac{1}{2}s_{xx} - q_x r - 2qr_x + r_x s^2 - rss_x - 2q^2 s - r^2 s - s^3 + 2qs^3 + 2qr^2 s - 2s\alpha\alpha_x \\
&\quad - 2s\beta\beta_x + 4qs\alpha\beta - 2s\alpha\beta - \alpha\beta_x + \alpha_x\beta, \\
s_{t_2} &= -\frac{1}{2}r_{xx} - q_x s - 2qs_x - rr_x s + r^2 s_x + 2q^2 r - 2qrs^2 - 2qr^3 + r^3 + rs^2 + 2r\alpha\beta \\
&\quad + 2r\alpha\alpha_x + 2r\beta\beta_x - 4qr\alpha\beta - \alpha\alpha_x + \beta\beta_x, \\
u_{1,t_2} &= -\frac{1}{2}\varepsilon r_{xx}s + \frac{1}{2}\varepsilon rs_{xx} - 2\varepsilon qrr_x - 2\varepsilon qss_x - \varepsilon q_x r^2 - \varepsilon q_x s^2 + \left(\varepsilon+1\right) \\
&\quad \Big(-\frac{1}{2}u_{2xx}u_3 + \frac{1}{2}ru_{3xx} + \frac{1}{2}u_2u_{3xx} - \frac{1}{2}su_{2xx} + \frac{1}{2}s_{2xx}u_2 - \frac{1}{2}r_{2xx}u_3 - 2u_1u_3u_{3x} \\
&\quad - 2rr_xu_1 - 2r_xu_1u_2 - 2qs_xu_3 - 2qru_{2x} - 2qr_xu_2 - 2qu_2u_{2x} - 2q_xru_2 - 2ru_1u_{2x} \\
&\quad - 2u_1u_2u_{2x} - 2ss_xu_1 - 2qsu_{3x} - 2su_{1x}u_3 - 2su_1u_{3x} - 2s_xu_1u_3 - 2qu_3u_{3x} \\
&\quad - 2q_xsu_3 - 2ru_{1x}u_2 - q_xu_3^2 - u_{1x}u_3^2 - r^2u_{1x} - q_xu_2^2 - u_{1x}u_2^2 - s^2u_{1x}\Big) \\
&\quad - \alpha\alpha_{xx} - \beta\beta_{xx} + 2\left(q\alpha\beta\right)_x \\
u_{2,t_2} &= \frac{1}{2}\varepsilon s_{xx} - \varepsilon q_x r - 2\varepsilon qr_x + \varepsilon r_x s^2 - \varepsilon rss_x - 2\varepsilon q^2 s - \varepsilon r^2 s - \varepsilon s^3 + 2\varepsilon qs^3 \\
&\quad + 2\varepsilon qr^2 s + \left(\varepsilon+1\right)\Big(\frac{1}{2}u_{3xx} + 4qrsu_2 + 4rsu_1u_2 + 4qru_2u_3 + 4ru_1u_2u_3 - 4qsu_1 \\
&\quad - 2rsu_2 - 4qu_1u_3 - 2ru_2u_3 - u_3^2 - r^2u_3 - 3s^2u_3 - 2q^2u_3 - 2su_1^2 - 2u_1^2u_3 - su_2^2 \\
&\quad - 3su_3^2 - u_2^2u_3 - ru_{1x} - 2r_xu_1 - q_xu_2 - 2qu_{2x} - u_{1x}u_2 - 2u_1u_{2x} + r_xu_3^2 + 2s^3u_1 \\
&\quad + s^2u_{2x} + 2u_1u_3^3 + 2qu_3^3 + u_{2x}u_3^2 + 6s^2u_1u_3 + 2qu_2^2u_3 + 2u_1u_2^2u_3 - su_2u_{3x} \\
&\quad + 6qsu_3^2 - rs_xu_3 - ru_3u_{3x} - s_xu_2u_3 + 2su_{2x}u_3 + 2r^2su_1 + 2su_1u_2^2 + 2qsu_2^2 \\
&\quad + 6qs^2u_3 + 2qr^2u_3 - rsu_{3x} + 2r_xsu_3 + 6su_1u_3^2 + 2r^2u_1u_3 - u_2u_3u_{3x} - ss_xu_2\Big) \\
&\quad + 2s\alpha\alpha_x + 2s\beta\beta_x - 4qs\alpha\beta + 2s\alpha\beta + \alpha\beta_x - \alpha_x\beta, \\
u_{3,t_2} &= -\frac{1}{2}\varepsilon r_{xx} - \varepsilon q_x s - 2\varepsilon qs_x - \varepsilon rr_x s + \varepsilon r^2 s_x + 2\varepsilon q^2 r - 2\varepsilon qrs^2 - 2\varepsilon qr^3 + \varepsilon r^3 \\
&\quad + \varepsilon rs^2 + \left(\varepsilon+1\right)\Big(-\frac{1}{2}u_{2xx} - su_{1x} - 2s_xu_1 - q_xu_3 - 2qu_{3x} - u_{1x}u_3 - 2u_1u_{3x} \\
&\quad + 2ru_1^2 + 2u_1^2u_2 + 3ru_2^2 + ru_3^2 + u_2u_3^2 + 2q^2u_2 + 3r^2u_2 + s^2u_2 + u_2^3 - 4qrsu_3 \\
&\quad - 4rsu_1u_3 - 4qsu_2u_3 - 4su_1u_2u_3 + 4qu_1u_2 + 4qru_1 + 2rsu_3 + 2su_2u_3 - 2rs^2u_1 \\
&\quad - 6qru_2^2 - 2ru_1u_3^2 - r_xu_2u_3 - rsu_{2x} - 2s^2u_1u_2 - 6qr^2u_2 - 6r^2u_1u_2 + 2ru_2u_{3x} \\
&\quad - 2qs^2u_2 - 6ru_1u_2^2 - u_2u_{2x}u_3 + 2rs_xu_2 - 2u_1u_2u_3^2 - ru_{2x}u_3 - su_2u_{2x} - 2qru_3^2 \\
&\quad - r_xsu_2 - 2qu_2u_3^2 - rr_xu_3 - 2u_1u_3^2 - 2qu_2^3 + s_xu_2^2 + u_2^2u_{3x} + r^2u_{3x} - 2r^3u_1\Big) \\
&\quad - 2r\alpha\beta - 2r\alpha\alpha_x - 2r\beta\beta_x + 4qr\alpha\beta + \alpha\alpha_x - \beta\beta_x, \\
\alpha_{t_2} &= -\beta_{xx} - q_x\alpha - 2q\alpha_x - \frac{1}{2}r_x\alpha - r\alpha_x + \frac{1}{2}r^2\beta + \frac{1}{2}s^2\beta + \frac{1}{2}s_x\beta + s\beta_x + q^2\beta \\
&\quad - \frac{1}{2}r_xs\beta + \frac{1}{2}rs_x\beta - qr^2\beta - qs^2\beta + \alpha\alpha_x\beta, \\
\beta_{t_2} &= \alpha_{xx} - q_x\beta - 2q\beta_x + \frac{1}{2}r_x\beta - \frac{1}{2}r^2\alpha - \frac{1}{2}s^2\alpha + \frac{1}{2}s_x\alpha - q^2\alpha + r\beta_x + s\alpha_x \\
&\quad + \frac{1}{2}r_xs\alpha - \frac{1}{2}rs_x\alpha + qr^2\alpha + qs^2\alpha - \alpha\beta\beta_x.
\end{aligned}
\right.
\tag{23}
$$

Especially, taking $\alpha=\beta=0$ in (23), we can obtain the nonlinear integrable couplings of the second order Yang equations in [13]

$$\begin{cases} q_{t_2} = -\frac{1}{2} r_{xx} s + \frac{1}{2} r s_{xx} - 2qqr_x - 2qss_x - q_x r^2 - q_x s^2, \\ r_{t_2} = \frac{1}{2} s_{xx} - q_x r - 2qr_x + r_x s^2 - rss_x - 2q^2 s - r^2 s - s^3 + 2qs^3 + 2qr^2 s, \\ s_{t_2} = -\frac{1}{2} r_{xx} - q_x s - 2qs_x - rr_x s + r^2 s_x + 2q^2 r - 2qrs^2 - 2qr^3 + r^3 + rs^2, \\ u_{1,t_2} = -\frac{1}{2}\varepsilon r_{xx} s + \frac{1}{2}\varepsilon r s_{xx} - 2\varepsilon qrr_x - 2\varepsilon qss_x - \varepsilon q_x r^2 - \varepsilon q_x s^2 + (\varepsilon+1) \\ \quad \Big(-\frac{1}{2} u_{2xx} u_3 + \frac{1}{2} r u_{3xx} + \frac{1}{2} u_2 u_{3xx} - \frac{1}{2} s u_{2xx} + \frac{1}{2} s_{2xx} u_2 - \frac{1}{2} r_{2xx} u_3 - 2u_1 u_3 u_{3x} \\ \quad - 2rr_x u_1 - 2r_x u_1 u_2 - 2qs_x u_3 - 2qru_{2x} - 2qr_x u_2 - 2qu_2 u_{2x} - 2q_x r u_2 \\ \quad - 2ru_1 u_{2x} - 2u_1 u_2 u_{2x} - 2ss_x u_1 - 2qsu_{3x} - 2su_{1x} u_3 - 2su_1 u_{3x} - 2s_x u_1 u_3 \\ \quad - 2qu_3 u_{3x} - 2q_x s u_3 - 2ru_{1x} u_2 - q_x u_3^2 - u_{1x} u_3^2 - r^2 u_{1x} - q_x u_2^2 - u_{1x} u_2^2 - s^2 u_{1x} \Big), \\ u_{2,t_2} = \frac{1}{2}\varepsilon s_{xx} - \varepsilon q_x r - 2\varepsilon q r_x + \varepsilon r_x s^2 - \varepsilon rss_x - 2\varepsilon q^2 s - \varepsilon r^2 s - \varepsilon s^3 + 2\varepsilon qs^3 \\ \quad + 2\varepsilon qr^2 s + (\varepsilon+1)\Big(\frac{1}{2} u_{3xx} + 4qrsu_2 + 4rsu_1 u_2 + 4qru_2 u_3 + 4ru_1 u_2 u_3 - 4qsu_1 \\ \quad - 2rsu_2 - 4qu_1 u_3 - 2ru_2 u_3 - u_3^2 - r^2 u_3 - 3s^2 u_3 - 2q^2 u_3 - 2su_1^2 - 2u_1^2 u_3 - su_2^2 \\ \quad - 3su_3^2 - u_2^2 u_3 - ru_{1x} - 2r_x u_1 - q_x u_2 - 2qu_{2x} - u_{1x} u_2 - 2u_1 u_{2x} + r_x u_3^2 + 2s^3 u_1 \\ \quad + s^2 u_{2x} + 2u_1 u_3^3 + 2qu_3^3 + u_{2x} u_3^2 + 6s^2 u_1 u_3 + 2qu_2^2 u_3 + 2u_1 u_2^2 u_3 - su_2 u_{3x} \\ \quad + 6qsu_3^2 - rs_x u_3 - ru_3 u_{3x} - s_x u_2 u_3 + 2su_{2x} u_3 + 2r^2 su_1 + 2su_1 u_2^2 + 2qsu_2^2 \\ \quad + 6qs^2 u_3 + 2qr^2 u_3 - rsu_{3x} + 2r_x su_3 + 6su_1 u_3^2 + 2r^2 u_1 u_3 - u_2 u_3 u_{3x} - ss_x u_2 \Big), \\ u_{3,t_2} = -\frac{1}{2}\varepsilon r_{xx} - \varepsilon q_x s - 2\varepsilon qs_x - \varepsilon rr_x s + \varepsilon r^2 s_x + 2\varepsilon q^2 r - 2\varepsilon qrs^2 - 2\varepsilon qr^3 + \varepsilon r^3 \\ \quad + \varepsilon rs^2 + (\varepsilon+1)\Big(-\frac{1}{2} u_{2xx} - su_{1x} - 2s_x u_1 - q_x u_3 - 2qu_{3x} - u_{1x} u_3 - 2u_1 u_{3x} \\ \quad + 2ru_1^2 + 2u_1^2 u_2 + 3ru_2^2 + ru_3^2 + u_2 u_3^2 + 2q^2 u_2 + 3r^2 u_2 + s^2 u_2 + u_2^3 - 4qrsu_3 \\ \quad - 4rsu_1 u_3 - 4qsu_2 u_3 - 4su_1 u_2 u_3 + 4qu_1 u_2 + 4qru_1 + 2rsu_3 + 2su_2 u_3 - 2rs^2 u_1 \\ \quad - 6qru_2^2 - 2ru_1 u_3^2 - r_x u_2 u_3 - rsu_{2x} - 2s^2 u_1 u_2 - 6qr^2 u_2 - 6r^2 u_1 u_2 + 2ru_2 u_{3x} \\ \quad - 2qs^2 u_2 - 6ru_1 u_2^2 - u_2 u_{2x} u_3 + 2rs_x u_2 - 2u_1 u_2 u_3^2 - ru_{2x} u_3 - su_2 u_{2x} - 2qru_3^2 \\ \quad - r_x su_2 - 2qu_2 u_3^2 - rr_x u_3 - 2u_1 u_3^2 - 2qu_2^3 + s_x u_2^2 + u_2^2 u_{3x} + r^2 u_{3x} - 2r^3 u_1 \Big). \end{cases} \tag{24}$$

If setting $\varepsilon = -1, u_1 = -q, u_2 = -r$ in (23), we obtain the second order super Yang equations of (17) in [15]

$$\begin{cases} q_{t_2} = -\frac{1}{2} r_{xx} s + \frac{1}{2} r s_{xx} - 2qqr_x - 2qss_x - q_x r^2 - q_x s^2 + \alpha\alpha_{xx} + \beta\beta_{xx} - 2(q\alpha\beta)_x, \\ r_{t_2} = \frac{1}{2} s_{xx} - q_x r - 2qr_x + r_x s^2 - rss_x - 2q^2 s - r^2 s - s^3 + 2qs^3 + 2qr^2 s - 2s\alpha\alpha_x \\ \quad - 2s\beta\beta_x + 4qs\alpha\beta - 2s\alpha\beta - \alpha\beta_x + \alpha_x\beta, \\ s_{t_2} = -\frac{1}{2} r_{xx} - q_x s - 2qs_x - rr_x s + r^2 s_x + 2q^2 r - 2qrs^2 - 2qr^3 + r^3 + rs^2 + 2r\alpha\beta \\ \quad + 2r\alpha\alpha_x + 2r\beta\beta_x - 4qr\alpha\beta - \alpha\alpha_x + \beta\beta_x, \\ \alpha_{t_2} = -\beta_{xx} - q_x\alpha - 2q\alpha_x - \frac{1}{2} r_x\alpha - r\alpha_x + \frac{1}{2} r^2\beta + \frac{1}{2} s^2\beta + \frac{1}{2} s_x\beta + s\beta_x + q^2\beta \\ \quad - \frac{1}{2} r_x s\beta + \frac{1}{2} rs_x\beta - qr^2\beta - qs^2\beta + \alpha\alpha_x\beta, \\ \beta_{t_2} = \alpha_{xx} - q_x\beta - 2q\beta_x + \frac{1}{2} r_x\beta - \frac{1}{2} r^2\alpha - \frac{1}{2} s^2\alpha + \frac{1}{2} s_x\alpha - q^2\alpha + r\beta_x + s\alpha_x \\ \quad + \frac{1}{2} r_x s\alpha - \frac{1}{2} rs_x\alpha + qr^2\alpha + qs^2\alpha - \alpha\beta\beta_x. \end{cases} \tag{25}$$

6. Remarks

In this paper, we introduced an approach for constructing nonlinear integrable couplings of super integrable hierarchy. The method in this paper can be applied to other super integrable systems for constructing their integrable couplings. How to obtain the soliton solutions about equations deduced in this paper is worth considering for our future work.

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