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Solutions of the Exponential Equation $y^{\frac{x}{y}} = x$ or $\frac{\ln x}{x} = \frac{\ln y}{y}$ and Fine Structure Constant

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Abstract

In this paper, we study the equation of the form of $y^{\frac{x}{y}} = x$ which can also be written as $\frac{\ln x}{x} = \frac{\ln y}{y}$. Apart from the trivial solution $x = y$, a non-trivial solution can be expressed in terms of Lambert *W* function as $y = \frac{W\left[-\frac{\ln(x)}{x}\right]}{\left[-\frac{\ln(x)}{x}\right]}$. For $y > \mathrm{e}$, the solutions of x are in-between 1 and e. For integer y values between 4 and 12, the solutions of x written in base y are in-between 1.333 and 1.389. The non-trivial solutions of the equations $y^{x/y^2} = x/y$ and $y^{x/y^3} = x/y^2$ written in base y are exactly one and two orders higher respectively than the solutions of the equation $y^{\frac{x}{y}} = x$. If $y = 10$, the rounded nontrivial solutions for the three equations are 1.3713, 13.713 and 137.13, *i.e.* $10^{0.13713}$ = 1.3713. Further, ln(1.3713)/1.3713 = 0.2302 and *W*(−0.2302) = −2.302. The value 137.13 is very close to the fine structure constant value of 137.04 within 0.1%.

Keywords

Exponential Equation, Lambert W Function, Fine Structure Constant

1. Introduction

Lambert *W* function is a transcendental function [1] [2] which has applications in many areas of science which include QCD renormalisation, Planck's spectral distribution law, water movement in soil and population growth [3]-[8].

Considering the equation

$$y^{\frac{x}{y}} = x \tag{1}$$

The Equation (1) can be written as

$$\log_y x = \frac{x}{y} \tag{2}$$

Converting the Equation (2) in terms of natural log gives

$$\frac{\ln x}{x} = \frac{\ln y}{y} \tag{3}$$

Equations ((1)-(3)) have a trivial solution $x = y$, but they also have a non-trivial solution.

Figure 1 shows the plot of the function $\frac{\ln x}{x}$. The plot indicates that, for any value of the function $\frac{\ln x}{x}$ in the range of 1 to infinity, it has two different solutions of x. *i.e.* for any value of y between 1 and infinity, a non-trivial solution of x can be found. The plot also indicates that, at $y = \mathrm{e}$, there is only one solution $x = \mathrm{e}$ and $\frac{\ln x}{x} = 1/\mathrm{e} = 0.3679$ (rounded). For any value of y between e and infinity, a solution for x can be found in-between 1 and e.

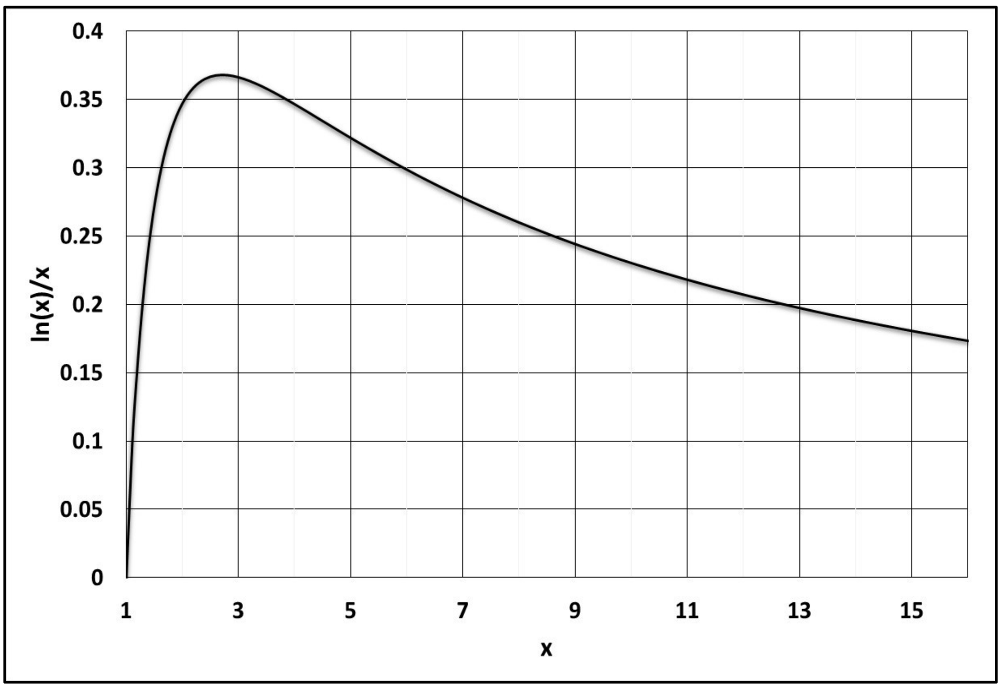

Figure 1. The plot x vs $\ln(x)/x$.

The solution of Equations ((1)-(3)) can be written in terms of Lambert W function [9],

$$y=\frac{W\left[-\frac{\ln(x)}{x}\right]}{\left[-\frac{\ln(x)}{x}\right]} \tag{4}$$

If $x=\mathrm{e}$, $y=\frac{W\left[-\frac{1}{\mathrm{e}}\right]}{\left[-\frac{1}{\mathrm{e}}\right]}$ and according to Dence [2], $W\left[-\frac{1}{\mathrm{e}}\right]=-1$, hence $y=\mathrm{e}$, which is the result obtained graphically and numerically.

Some variations of Equation (1) are:

$$y^{x/y^2}=x/y \tag{5}$$

$$y^{x/y^3}=x/y^2 \tag{6}$$

Equation (5) can be written as

$$\frac{\ln x}{\ln y}=\frac{x}{y^2}+1 \tag{7}$$

Equation (6) can be written as

$$\frac{\ln x}{\ln y}=\frac{x}{y^3}+2 \tag{8}$$

Equation (5) and Equation (6) have trivial solutions of $x=y^2$ and $x=y^3$ respectively.

2. Non-Trivial Solutions

If $y=10$ then Equation (1) becomes $10^{(x/10)}=x$ and $x=1.3713$ (rounded) is the nontrivial solution, *i.e.* $10^{0.13713}=1.3713$ and

$$\frac{\ln x}{x}=\frac{\ln y}{y}=0.2302$$

If $y=10$ then Equation (5) and Equation (6) become $10^{(x/100)}=x/10$ and $10^{(x/1000)}=x/100$ respectively and their solutions are 13.713 (rounded) and 137.13 (rounded) respectively. These solutions are exactly one and two orders larger than the solution of Equation (1).

Also if $x=1.3713$ and $y=10$, Equation (4) gives

$$W\left[-\frac{\ln(1.3713)}{1.3713}\right]=10\left[-\frac{\ln(1.3713)}{1.3713}\right]$$

Hence $W(-0.2302)=-2.302$

For the range of integer y values of 4 to 12, the non-trivial solutions for x of Equations ((1), (5) and (6)) were obtained using iterative method. The solutions of x are written in base 10 and in base y (**Table 1**). Plots of y vs x with x in base 10 and in base y are shown in **Figures 2-4** respectively.

3. Conclusions

The non-trivial solutions of Equations ((1), (5) and (6)) written in base *y*, differ exactly by one order. For *y* values in the range of 4 to 12, the solutions of Equation (6) written in base *y* are in the range of 133.33 to 138.99.

When $y = 10$, the rounded nontrivial solutions for Equation (1), Equation (5) and Equation (6) are 1.3713, 13.713 and 137.13, *i.e.* $10^{0.13713} = 1.3713$, $\ln(1.3713)/1.3713 = 0.2302$ and $W(-0.2302) = -2.302$, *i.e.* for the argument values of 1.3713 and −0.2302, the function values are exactly one order higher. To our knowledge, these results were not reported before.

Table 1. Rounded non-trivial solutions for *x* of Equations ((1), (5) and (6)) for *y* values from 4 to 12 are written in base 10 and base *y*.

y	Solutions of Equation (1)		Solutions of Equation (5)		Solutions of Equation (6)	
	In base 10	In base *y*	In base 10	In base *y*	In base 10	In base *y*
12	1.3122	1.389	15.75	13.89	189.0	138.9
11	1.3389	1.380	14.73	13.80	162.0	138.0
10	**1.3713**	**1.371**	**13.71**	**13.71**	**137.1**	**137.1**
9	1.4114	1.363	12.70	13.63	114.3	136.3
8	1.4625	1.355	11.70	13.55	93.6	135.5
7	1.5301	1.350	10.71	13.50	75.0	135.0
6	1.6242	1.343	9.75	13.42	58.5	134.3
5	1.7649	1.340	8.82	13.40	44.1	1340
4	2.0000	1.333..	8.00	13.33..	32.0	133.3

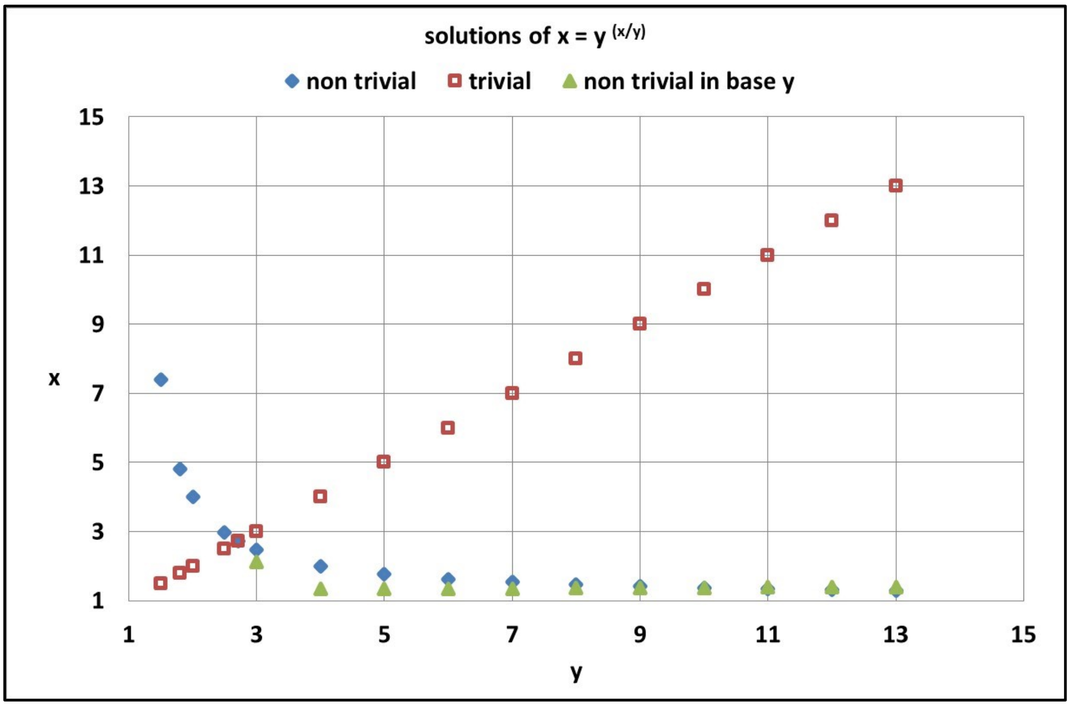

Figure 2. Solutions of *x* in base 10 and in base *y* for Equation (1) for *y* values of 1 to 13.

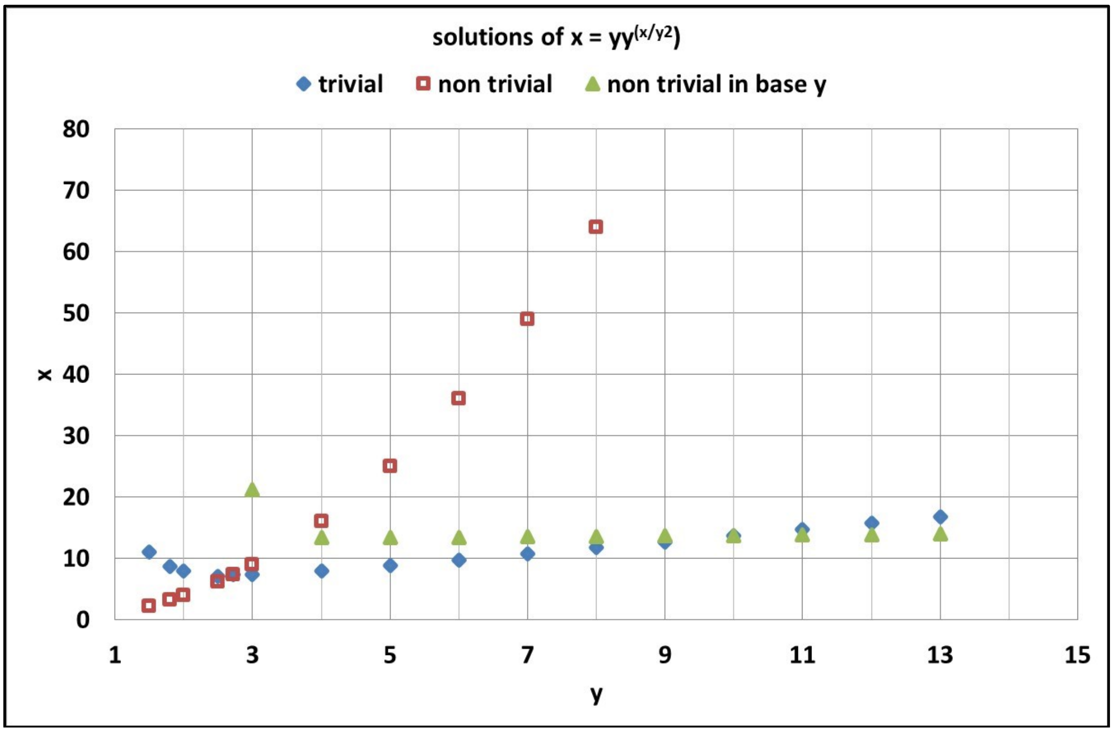

Figure 3. Solutions of x in base 10 and in base y for Equation (5) for y values of 1 to 13.

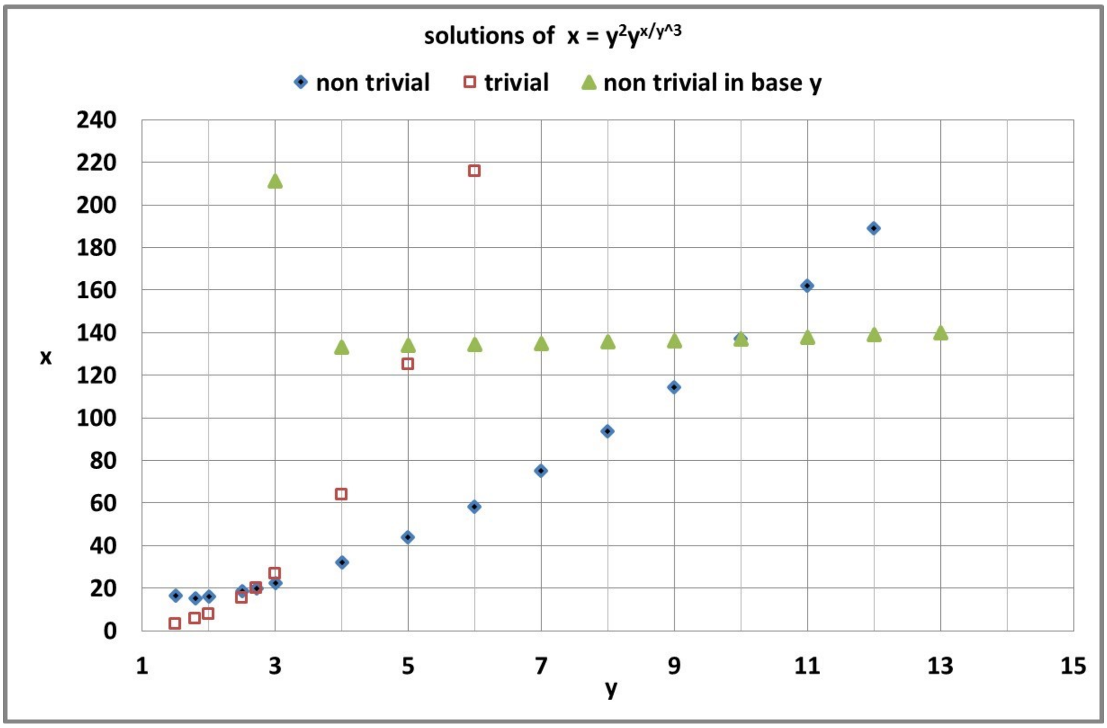

Figure 4. Solutions of x in base 10 and in base y for Equation (6) for y values of 1 to 13.

The trivial solutions of Equations ((1), (5) and (6)) can be written as 10, 100 and 1000 in base y for any y value.

The non-trivial solution for x of Equation (6), 137.128857 is within 0.1% of the reciprocal value of the atomic fine structure constant α^{-1}, 137.0359991.

4. Possible Connection to Fine Structure Constant

Allen suggested that $m_e/M_p \sim 10\alpha^2$ [10] however for the current values of m_e/M_p and α, the relationship is $m_e/M_p = 10.227\alpha^2$. Edward Teller suggested ln $T_0^{3/2} = \alpha^{-1}$, where T_o is the age of the universe [11]. There could be a connection between Equations ((1) to (8)) and α^{-1}.

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