Scientific Research Publishing

Numerical Solution of Two Dimensional Fredholm Integral Equations of the Second Kind by the Barycentric Lagrange Function*

Hongyan Liu, Jin Huang, Yubin Pan

School of Mathematical sciences, University of Electronic Science and Technology of China, Chengdu, China
Email: Hongyan_Liu1528@163.com, huangjin12345@163.com, yubinpan2014@163.com

Abstract

This paper solves the two dimensional linear Fredholm integral equations of the second kind by combining the meshless barycentric Lagrange interpolation functions and the Gauss-Legendre quadrature formula. Inspired by this thought, we convert the equations into the associated algebraic equations. The results of the numerical examples are given to illustrate that the approximated method is feasible and efficient.

Keywords

Two Dimensional Fredholm Integral Equations, Barycentric Lagrange
Interpolation Functions, Gauss-Legendre Quadrature Formula

1. Introduction

Many of the mathematical physics problems and the engineering problems can be transformed into solving Fredholm integral equations [1] [2] [3]. In this paper, we pay attention to the two dimensional linear Fredholm integral equations (FIEs) of the second kind

$$a(x,y)u(x,y) = f(x,y) + b(x,y)\int_a^b\int_c^d k(x,y;s,t)u(s,t)\mathrm{d}s\mathrm{d}t, (x,y)\in\Omega, \quad (1)$$

where $a(x,y), b(x,y)$ and $f(x,y)$ are non-zero continuous functions on the bounded region $\Omega = [a,b]\times[c,d]$, and the kernel function $k(x,y;s,t)\in C(\Omega\times\Omega)$, $u(x,y)$ is the undetermined function. Usually, it's difficult to obtain the analytic solutions of integral equations, so the numerical solutions of the integral equations are necessarily needed.

There are some researches for obtaining the numerical solutions of two di-

mensional FIEs, such as the radial basis functions method [4], Haar wavelets [5] and integral mean value method [6] [7]. Berrut specializes the barycentric Lagrangian interpolation formula [8], and some other authors present the corresponding numerical stability analyses in the literatures [9] [10] [11]. The authors take advantage of the equally spaced barycentric Lagrange polynomial to solve one dimensional linear Volterra-Fredholm integro-differential equations in [12]. There exists an intrinsic problem that polynomial interpolation is ill-posed at the equispaced nodes. This paper presents a modified Lagrange interpolation method with Chebyshev nodes to solve two dimensional linear Fredholm integral equations of the second kind.

This paper is constructed as follows: Section 2, we display the barycentric interpolation function. Section 3, we transform the two dimensional FIEs into the algebraic equations by utilizing the barycentric function and the composite Gauss-Legendre quadrature formula. Section 4, the numerical examples illustrate the efficiency and applicability of our method via comparing with the Lagrange method.

2. Barycentric Interpolation Function

Barycentric Lagrange interpolation function (BLIF) is a variant of Lagrange interpolation function (LIF).

2.1. One Dimensional Barycentric Interpolation Function

The one dimensional barycentric Lagrange formula about continuous function $u(x), x \in [a,b]$ at the nodes $a = x_0 \leq x_1 \leq \ldots \leq x_n = b$ is introduced as

$$u_n(x) = \sum_{j=0}^{n} L_j(x) u_j, \tag{2}$$

where $u_j = u(x_j), j = 0,1,\ldots,n$ are the function values of $u(x)$, and $L_j(x)$ are the barycetric Lagrange interpolation basis functions

$$L_j(x) = \left(\frac{w_j}{x - x_j} \right) \Bigg/ \left(\sum_{k=0}^{n} \frac{w_k}{x - x_k} \right), \quad j = 0,1,\cdots,n, \tag{3}$$

which satisfy the property $L_j(x_i) = \delta_{ij}$, where δ_{ij} is the Kronecker-delta functions. And w_j are the barycentric interpolation weight functions

$$w_j = 1 \Bigg/ \left[\prod_{j=0, j\neq k}^{n} (x_j - x_k) \right], \tag{4}$$

and w_j only depend on the distribution of nodes.

The barycentric Lagrange interpolation formula is stable forward when we choose the Chebyshev points as interpolating points [9]. At the same time, the weight functions are simplified as $w_j = (-1)^j \xi_j$, where $\xi_j = \frac{1}{2}$ for $j = 0, m$ and $\xi_j = 1$ for $j \neq 0, m$.

2.2. Two Dimensional Barycentric Interpolation Function

Basing on the one dimensional barycentric Lagrange function, we define the two

dimensional barycentric Lagrange interpolation function about a continuous function $u(x,y), (x,y) \in [a,b] \times [c,d]$ as follows

$$u_{mn}(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} L_{ij}(x,y) u_{ij}, \tag{5}$$

where $u_{ij} = u(x_i, y_j)$, and the tensor product points (x_i, y_j) are obtained by subdividing $[a,b]$ and $[c,d]$ into the calculation nodes $a = x_0 < x_1 < \cdots < x_m = b$ and $c = x_0 < x_1 < \cdots < x_n = d$, respectively. The two dimensional barycentric interpolation basis functions $L_{ij}(x,y)$ satisfy

$$L_{ij}(x,y) = L_i(x) L_j(y), \tag{6}$$

where $L_i(x)$ and $L_j(y)$ are equal to (4). The above equation has the following property

$$L_{ij}(x_k, y_l) = \begin{cases} 1, & i = k \text{ and } j = l, \\ 0, & \text{others.} \end{cases} \tag{7}$$

In the practical calculation, we take the second Chebyshev nodes (x_i, y_j), $x_i = -\cos\left(\dfrac{i}{m}\pi\right)$, $i = 0,1,\cdots,m$, $y_j = -\cos\left(\dfrac{j}{n}\pi\right)$, $j = 0,1,\cdots,n$ as the interpolation nodes.

3. The Barycentric Method of Two Dimensional Fredholm Integral Equation

We enter (5) into (1) to approximate two dimensional Fredholm integral equation

$$a(x,y)\sum_{i=0}^{m}\sum_{j=0}^{n} L_{ij}(x,y) u_{ij}$$
$$= f(x,y) + b(x,y)\int_a^b\int_c^d k(x,y;s,t)\sum_{i=0}^{m}\sum_{j=0}^{n} L_{ij}(s,t) u_{ij}\,\mathrm{d}s\mathrm{d}t, \tag{8}$$

then taking the collocation points (x_k, y_l) into (8), and exchanging the integral and summation sign

$$a(x_k,y_l)\sum_{i=0}^{m}\sum_{j=0}^{n} L_{ij}(x_k,y_l) u_{ij}$$
$$= f(x_k,y_l) + b(x_k,y_l)\sum_{i=0}^{m}\sum_{j=0}^{n}\left[\int_a^b\int_c^d k(x_k,y_l;s,t) L_{ij}(s,t)\,\mathrm{d}s\mathrm{d}t\right] u_{ij}. \tag{9}$$

Now, we deal with the integral part and let $s = \dfrac{b+a}{2} + \dfrac{b-a}{2}\sigma$ and $t = \dfrac{c+d}{2} + \dfrac{d-c}{2}\tau$, then we can easily have

$$\int_a^b\int_c^d k(x_k,y_l;s,t) L_{ij}(s,t)\,\mathrm{d}s\mathrm{d}t = \int_{-1}^{1}\mathrm{d}\sigma\int_{-1}^{1}k_1(x_k,y_l;\sigma,\tau) L_{1,ij}(\sigma,\tau)\,\mathrm{d}\tau, \tag{10}$$

where $k_1(x_k,y_l;\sigma,\tau) = \dfrac{b-a}{2}\dfrac{d-c}{2}k\left(x_k,y_l;\dfrac{a+b}{2} + \dfrac{b-a}{2}\sigma, \dfrac{c+d}{2} + \dfrac{d-c}{2}\tau\right)$,

$L_{1,ij} = L_{ij}(\sigma,\tau)\left(\dfrac{a+b}{2} + \dfrac{b-a}{2}\sigma, \dfrac{c+d}{2} + \dfrac{d-c}{2}\tau\right)$. Then, we apply the composite

Gauss-Legendre quadrature formula to approximate integral (10). Then the Equation (9) becomes

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\left[a\left(x_k,y_l\right)L_{ij}\left(x_k,y_l\right)-b\left(x_k,y_l\right)\sum_{p=1}^{M}\sum_{q=1}^{N}l_pl_qk_1\left(x_k,y_l;\sigma_p,\tau_q\right)L_{1,ij}\left(\sigma_p,\tau_q\right)\right]u_{ij} \quad (11)$$
$$= f\left(x_k,y_l\right),$$

where l_p, l_q; σ_p, τ_q are the quadrature coefficients and points of the composite Gauss-Legendre quadrature formula, respectively. Finally, we transform the Equation (1) into algebraic equations. The solution of the Equation (11) is close to (1). The whole process is called discrete collocation method whose nature is the Nyström iterative approach. We can analyze the existence, uniqueness and convergence of the approximate solution under the theoretical framework of the Nyström from [13]. Once we obtain u_{ij}, we can get the values of the discrete collocation solution $u(x,y)$ at any interior points.

The specific Algorithm is as follows

Step 1: Construct two dimensional Chebyshev nodes $\left(x_i, y_j\right)$,

$$x_i = -\cos\left(\frac{i}{m}\pi\right), \quad y_j = -\cos\left(\frac{j}{n}\pi\right),$$

Step 2: Approximate integral operator K in (10) by using composite Gauss-Legendre quadrature formula,

Step 3: Solve the algebraic Equations (11) using the above steps and Gaussian elimination method.

4. Numerical Example

In this section, we present two numerical examples to verify the effectiveness and accuracy of the barycentric method. Defining the absolute error (AE) and the relative error (RE) as

$$e(x,y) = \left\| u(x,y) - u_{mn}(x,y) \right\|, \quad (12)$$

$$re(x,y) = \frac{\left\| u(x,y) - u_{mn}(x,y) \right\|}{\left\| u(x,y) \right\|}, \quad (13)$$

and $u(x,y)$, $u_{mn}(x,y)$ represent the exact and approximate solution of Equation (1), respectively.

Example 1. For the following two dimensional linear Fredholm integral equation of the second kind

$$u(x,y) = \sin(x+y)\int_0^1\int_0^1 u(s,t)\,\mathrm{d}t\mathrm{d}s + f(x,y), (x,y)\in[0,1]\times[0,1], \quad (14)$$

where $f(x,y) = \left[1+\sin(2)-2\sin(1)\right]\sin(x+y)$, the accurate solution is $u(x,y) = \sin(x+y)$.

Table 1 and **Table 2** are the absolute and relative errors of the barycentric Lagrange interpolation function method and the Lagrangian interpolation function method at the equidistant nodes and Chebyshev nodes, respectively, where *m, n* represent the interpolation node numbers and *M, N* are the quadrature node numbers. Usually, the accuracy of the BLIF method is higher than the LIF

method, and the numerical results at the Chebyshev nodes are superior to equidistant nodes. From **Table 1**, we can see that the errors increase with m varying from 8 to 16. In **Table 2**, the numerical results of the BLIF are stable with Chebyshev nodes and those of the LIF method are oscillatory. From **Table 3**, we find that the errors first reduce and remain stable finally at the Chebyshev nodes. In reverse, the errors of the BLIF method increase at the equal nodes with the increasing of m, especially the method is invalid when $m = 32$. In **Table 4**, we list

Table 1. The absolute and relative errors of equal nodes, $M = 8$, $N = 7$, example 1.

Node numbers	The BLIF method		The LIF method	
	AE	RE	AE	RE
$m = n = 8$	2.9984e−11	4.1823e−12	1.1735e−10	1.6369e−11
$m = n = 16$	3.9568e−08	2.9075e−09	2.4235e−03	1.7808e−04

Table 2. The absolute and relative errors of Chebyshev nodes, $M = 8$, $N = 7$, example 1.

Node numbers	The BLIF method		The LIF method	
	AE	RE	AE	RE
$m = n = 8$	4.6649e−13	6.6148e−14	7.0170e−11	9.9500e−12
$m = n = 16$	3.7665e−14	2.8191e−15	6.2815e−05	4.7015e−06

Table 3. The absolute errors of the barycentric Lagrange method, $M = 8$, $N = 7$, example 1.

Nodes (x, y)	The Chebyshev nodes			The equidistant nodes	
	$m = 8$	$m = 16$	$m = 32$	$m = 8$	$m = 16$
(0.1, 0.1)	3.8528e−11	7.4940e−16	2.4980e−16	3.6538e−11	9.3405e−07
(0.3, 0.3)	8.1330e−12	1.6653e−15	1.6653e−15	1.3789e−11	2.6547e−06
(0.5, 0.5)	5.5622e−14	2.5535e−15	2.4425e−15	8.3724e−12	3.9562e−06
(0.7, 0.7)	4.0140e−12	2.4425e−15	1.2212e−15	5.8561e−12	4.6331e−06
(0.9, 0.9)	5.8634e−12	2.9976e−15	2.9976e−15	1.5618e−11	4.5786e−06

Table 4. The numerical results with Chebyshev nodes, $M = 8$, $N = 7$, example 1.

Nodes (x, y)	The BLIF method		The LIF method	
	$m = 8$	$m = 16$	$m = 8$	$m = 16$
$(2^{-1}, 2^{-1})$	1.1935e−13	1.8203e−10	8.3724e−12	3.9562e−06
$(2^{-2}, 2^{-2})$	4.6897e−11	2.0025e−11	5.1702e−11	2.2540e−06
$(2^{-3}, 2^{-3})$	2.2128e−11	9.7228e−14	1.9650e−11	1.1632e−06
$(2^{-4}, 2^{-4})$	2.7555e−11	7.9936e−15	2.6306e−11	5.8616e−07
$(2^{-5}, 2^{-5})$	7.5557e−12	1.1727e−15	8.1812e−12	2.9365e−07
$(2^{-6}, 2^{-6})$	1.6780e−11	1.0755e−16	1.7093e−11	1.4690e−07

the numerical results of the BLIF method and the LIF method. We can conclude that the barycentric Lagrange method is more efficient and stable than the Lagrange method.

Example 2. Consider two dimensional Fredholm integral equation of the following form

$$(e^{x+y}u(x,y)$$
$$= x^2 y \int_0^{\frac{\pi}{2}} \int_0^1 (x+y+s+t) \cdot u(s,t) \mathrm{d}t \mathrm{d}s + f(x,y), (x,y) \in \left[0, \frac{\pi}{2}\right] \times [0,1], \quad (15)$$

where

$$f(x,y) = (e-1)\left(1 - e^{\frac{\pi}{2}}\right)\left(x^3 y + x^2 y^2\right)$$
$$+ \left[(e-1)\left(e^{\frac{\pi}{2}}\left(1 - \frac{1}{2}\pi\right) - 1\right) + \left(1 - e^{\frac{\pi}{2}}\right)\right] x^2 y + e^{2x+2y},$$

the accurate solution is $u(x,y) = e^{x+y}$.

Table 5 and **Table 6** reveal that the absolute and relative errors of the barycentric and Lagrangian methods with $m = n = 8,16$ at the interior points. Generally, the approximate solutions of barycentric method are more accurate than Lagrange method, especially for large node numbers. In the tables, the absolute errors are decreasing by barycentric method and increasing by the Lagrange method with the increasing of Chebyshev node numbers. In a word, the BLIF

Table 5. The absolute errors with Chebyshev nodes, $M = N = 11$, example 2.

Nodes (x, y)	The BLIF method		The LIF method	
	$m = 8$	$m = 16$	$m = 8$	$m = 16$
(0.1, 0.1)	2.2598e−09	6.6613e−16	2.2597e−09	2.4946e−08
(0.3, 0.3)	4.5070e−09	6.6613e−16	4.5078e−09	5.3816e−07
(0.5, 0.5)	1.3426e−09	1.3323e−15	1.3395e−09	1.9373e−06
(0.7, 0.7)	8.2825e−09	0.0000e−00	8.2724e−09	4.4205e−06
(0.9, 0.9)	1.2460e−08	1.7764e−15	1.2495e−08	1.7531e−05

Table 6. The absolute errors with Chebyshev nodes, $M = N = 11$, example 2.

Nodes (x, y)	The BLIF method		The LIF method	
	$m = 8$	$m = 16$	$m = 8$	$m = 16$
$(2^{-1}, 2^{-1})$	1.3426e−09	1.3323e−15	1.3395e−09	1.9373e−06
$(2^{-2}, 2^{-2})$	1.4203e−09	1.1102e−15	1.4208e−09	3.3037e−07
$(2^{-3}, 2^{-3})$	3.1271e−09	1.5543e−15	3.1270e−09	4.6671e−08
$(2^{-4}, 2^{-4})$	1.7789e−10	1.5543e−15	1.7788e−10	6.3278e−09
$(2^{-5}, 2^{-5})$	1.1286e−09	4.4409e−15	1.1287e−09	8.1572e−10
$(2^{-6}, 2^{-6})$	1.0338e−09	1.1102e−15	1.0338e−09	1.0351e−10

method is more efficient than the LIF method which is agreement with the theoretical analysis.

5. Conclusion

In this paper, we solve two dimensional linear Fredholm integral equations of the second kind by means of the barycentric Lagrange interpolation method. The modified Lagrange method with Chebyshev nodes transforms the equations into linear algebraic equations, and the corresponding numerical solutions are stable forward. The numerical results also demonstrate that the barycentric method is a simple and powerful technique. Furthermore, the barycentric method can extend to solve high dimensional FIEs.

Acknowledgements

References

[1] Yoshida, N., Imai, T., Phongphanphanee, S., Kovalenko, A. and Hirata, F. (2009) Molecular Recognition in Biomolecules Studied by Statistical-Mechanical Integral-Equation Theory of Liquids. *Journal of Physical Chemistry B*, **113**, 873-886. https://doi.org/10.1021/jp807068k

[2] Farengo, R., Lee, Y.C. and Guzdar, P.N. (1983) An Electromagnetic Integral Equation: Application to Microtearing Modes. *Physical Fluids*, **26**, 3515-3523. https://doi.org/10.1063/1.864112

[3] Kosarev, E.L. (1980) Applications of Integral Equations of the First Kind in Experiment Physics. *Computer Physics Communications*, **20**, 69-75. https://doi.org/10.1016/0010-4655(80)90110-1

[4] Alipanah, A. and Esmaeili, S. (2011) Numerical Solution of the Two Dimensional Fredholm Integral Equations Using Gaussian Radial Basis Function. *Journal of Computational and Applied Mathematics*, **235**, 5342-5347. https://doi.org/10.1016/j.cam.2009.11.053

[5] Derili, H., Sohrabi, S. and Arzhang, A. (2012) Two-Dimensional Wavelets for Numerical Solution of Integral Equations. *Mathematical Sciences*, **6**, 1-4.

[6] Heydari, M., Avazzadeh, Z., Navabpour, H. and Loghmani, G.B. (2013) Numerical Solution of Fredholm Integral Equations of the Second Kind by Using Integral Mean Value Theorem II. High Dimensional Problems. *Applied Mathematical Modelling*, **37**, 432-442. https://doi.org/10.1016/j.apm.2012.03.011

[7] Ma, Y.Y., Huang, J. and Li, H. (2015) A Novel Numerical Method of Two-Dimensional Fredholm Integral Equations of the Second Kind. *Mathematical Problems in Engineering*, **2015**, Article ID: 625013. https://doi.org/10.1155/2015/625013

[8] Berrut, J.-P. and Trefethen, L.N. (2004) Barycentric Lagrange Interpolation. *SIAM Review*, **46**, 501-517. https://doi.org/10.1137/S0036144502417715

[9] Higham, N.J. (2004) The Numerical Stability of Barycentric Lagrange Interpolation,. *IMA Journal of Numerical Analysis*, **24**, 547-556. https://doi.org/10.1093/imanum/24.4.547

[10] Mascarenhas, W.F. (2014) The Stability of Barycentric Interpolation at the Cheby-

shev Points of the Second Kind. *Numerische Mathematik*, **128**, 265-300.
https://doi.org/10.1007/s00211-014-0612-6

[11] Mascarenhas, W.F. and Camargo, A.P.D. (2014) On the Backward Stability of the Second Barycentric Formula for Interpolation. *Dolomites Research Notes on Approximation*, **7**, 1-13.

[12] Mustafa, M.M. and Muhammad, A.M. (2014) Numerical Solution of Linear Volterra-Fredholm Integro-Differential Equations Using Lagrange Polynomials. *Mathematical Theory and Modeling*, **4**, 158-167.

[13] Lv, T. and Huang, J. (2013) A High Precision Algorithm for Integral Equation. China Science Publishing. (In Chinese)