

Gauss-Legendre Iterative Methods and Their Applications on Nonlinear Systems and BVP-ODEs

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Abstract

In this paper, a group of Gauss-Legendre iterative methods with cubic convergence for solving nonlinear systems are proposed. We construct the iterative schemes based on Gauss-Legendre quadrature formula. The cubic convergence and error equation are proved theoretically, and demonstrated numerically. Several numerical examples for solving the system of nonlinear equations and boundary-value problems of nonlinear ordinary differential equations (ODEs) are provided to illustrate the efficiency and performance of the suggested iterative methods.

Keywords

Iterative Method, Gauss-Legendre Quadrature Formula, Nonlinear Systems, Third-Order Convergence, Nonlinear ODEs

1. Introduction

We consider the general form of a system of nonlinear equations as follows:

$$F(x) = (f_1(x), f_2(x), \dots, f_n(x))^T = 0, \quad (1)$$

where $F: D \subseteq R^n \rightarrow R^n$ is a given nonlinear vector function, and each function $f_i(x)$ can be thought of as mapping of a vector $x = (x_1, x_2, \dots, x_n)^T$, which is the n -dimensional space R^n into the real number R .

Numerical solutions for systems of nonlinear equations have always appealed greatly to people in scientific computation and engineering fields. Some boundary-value problems of nonlinear ordinary differential equations (ODEs) can be transformed to nonlinear systems like (1) by the finite difference method. Constructing an efficiently iterative method to approximate the root $\xi \in D$ (such that $F(\xi) = 0$) of Equation (1) is a

typical and important issue in nonlinear numerical computation. The Newton's method (see [1] [2]) is one of the widely used methods for solving nonlinear equations by iteration as follows:

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k), \quad k = 0, 1, 2, \dots, \quad (2)$$

where $F'(x)^{-1}$ is the inverse of first Fréchet derivative $F'(x)$ which is a Jacobian matrix of the nonlinear vector function $F(x)$. It converges quadratically when an initial guess value x_0 is close to the root ξ of Equation (1). In recent years, several iterative methods have been used to solve nonlinear equations and systems of nonlinear equations. In order to improve the order of convergence, a few two-step variants of Newton's methods with cubic convergence have been proposed in some literature [3]-[11] and references therein for solving systems of nonlinear equations. S. Weera-akoon and T. Fernando [5] using the Newton theorem $f(x) = f(x)_n + \int_{x_n}^x f'(t) dt$, proposed the Newton-type method with third-order convergence for nonlinear equation and systems. M. Darvishi and A. Barati [6] received a third-order convergence iterative method based on Adomian decomposition method to the systems of nonlinear equations. M. Frontini and E. Sormani [7] presented third-order midpoint-methods using numerical quadrature formula. A. Cordero and J. R. Torregrosa [8] developed third-order convergence Newton-Simpson's method and Open Newton's method using the simple Simpson's rule and an open quadrature formula of high order respectively. These are all classic two-step Newton-type methods to approximate the root of a system of nonlinear equations.

In Section 2 of this paper, we propose a group of two-step iterative methods with third-order convergence by Gauss-Legendre quadrature formula [12]:

$$\int_a^b f(x) dx = \frac{b-a}{2} \sum_{i=1}^n w_i f\left(\frac{a+b}{2} + \frac{b-a}{2} t_i\right), \quad (3)$$

where the necessary symmetrical conditions is

$$\sum w_i = 2, \quad \sum t_i = 0, \quad \sum w_i t_i = 0. \quad (4)$$

Several numerical examples are provided in Section 3 using Gauss-Legendre iterative method for solving systems of nonlinear equations and boundary-value problems of nonlinear ODEs, and we finally make conclusions in Section 4.

2. The Iterative Methods and Cubic Convergence

Assume $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Fréchet differentiable function in a convex $D \subset \mathbb{R}^n$. We have the mean-value theorem of multivariable vectors function $F(x)$ in [1]:

$$F(x) - F(x_k) = \int_0^1 F'(x_k + t(x - x_k))(x - x_k) dt, \quad (5)$$

Using the left rectangular integral rule:

$$\int_0^1 F'(x_k + t(x - x_k))(x - x_k) dt \approx F'(x_k)(x - x_k), \quad (6)$$

And by $F(x) = 0$, we can get Newton's Method (2).

Now, we apply the Gauss-Legendre quadrature formula (3) to approximate the integral on the right side of Equation (5), that is,

$$\int_0^1 F'(x_k + t(x - x_k))(x - x_k) dt \approx \frac{1}{2} \sum_{i=1}^n w_i F' \left(\frac{x_k + x}{2} + \frac{x - x_k}{2} t_i \right) (x - x_k), \tag{7}$$

and using $F(x) = 0$ we have

$$x - x_k \approx -2 \left[\sum_{i=1}^n w_i F' \left(\frac{x_k + x}{2} + \frac{x - x_k}{2} t_i \right) \right]^{-1} F(x_k). \tag{8}$$

Herein, taking x as the next iterative step of x_k , we get the following implicit structure:

$$x_{k+1} = x_k - 2 \left[\sum_{i=1}^n w_i F' \left(\frac{x_k + x_{k+1}}{2} + \frac{x_{k+1} - x_k}{2} t_i \right) \right]^{-1} F(x_k). \tag{9}$$

We use the k -th iteration y_k of the Newton's method to replace x_{k+1} on the right side of the expression (9), and we construct the following group of two-step iterative schemes:

$$\begin{cases} y_k = x_k - F'(x_k)^{-1} F(x_k) \\ x_{k+1} = x_k - 2 \left[\sum_{i=1}^n w_i F' \left(\frac{x_k + y_k}{2} + \frac{y_k - x_k}{2} t_i \right) \right]^{-1} F(x_k) \end{cases}. \tag{10}$$

We state and prove the convergence theorem for the schemes (10) as follows:

Theorem *Let $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a k -time Fréchet differentiable function in a convex set D with a root $\xi \in D$ and x_0 be close to ξ , then the group of iterative methods (10) is cubically convergent, and the error equation is*

$$e_{k+1} = \left(C_2^2 - C_3 + \frac{3}{8} \left(2 + \sum_{i=1}^n w_i t_i^2 \right) C_3 \right) e_k^3 + O(e_k^4). \tag{11}$$

Proof. As $F(\xi) = 0$, and noting that $x_k - \xi = e_k$, $C_k = F'(\xi)^{-1} \frac{F^{(k)}(\xi)}{k!}$.

By Taylor's expansion, we have

$$\begin{aligned} F(x_k) &= F'(\xi)(x_k - \xi) + \frac{1}{2!} F''(\xi)(x_k - \xi)^2 + \frac{1}{3!} F'''(\xi)(x_k - \xi)^3 + O(x_k - \xi)^4 \\ &= F'(\xi) [e_k + C_2 e_k^2 + C_3 e_k^3] + O(e_k^4), \end{aligned} \tag{12}$$

and

$$F'(x_k) = F'(\xi) [I + 2C_2 e_k + 3C_3 e_k^2 + O(e_k^3)]. \tag{13}$$

Then, suppose that

$$F'(x_k)^{-1} = [D(e_k)^{-1}] F'(\xi)^{-1}, \tag{14}$$

where

$$D(e_k) = I + 2C_2 e_k + 3C_3 e_k^2 + O(e_k^3). \tag{15}$$

The inverse of $D(e_k)$ is given by

$$D(e_k)^{-1} = I + K_1 e_k + K_2 e_k^2,$$

where K_1 and K_2 will be satisfied the definition of the following inversion.

$$D(e_k)D(e_k)^{-1} = D(e_k)^{-1}D(e_k) = I. \quad (16)$$

That is

$$(I + 2C_2 e_n + 3C_3 e_n^2)(I + K_1 e_n + K_2 e_n^2) = I. \quad (17)$$

From (17), we have

$$K_1 = -2C_2, K_2 = 4C_2^2 - 3C_3. \quad (18)$$

Therefore,

$$F'(x_k)^{-1} = [I - 2C_2 e_k + (4C_2^2 - 3C_3) e_k^2 + \dots] F'(\xi)^{-1}. \quad (19)$$

And

$$\begin{aligned} F'(x_k)^{-1} F(x_k) &= [I - 2C_2 e_k + (4C_2^2 - 3C_3) e_k^2 + \dots] \times (e_k + C_2 e_k^2 + C_3 e_k^3 + \dots) \\ &= e_k - C_2 e_k^2 + (2C_2^2 - 2C_3) e_k^3 + O(e_k^4). \end{aligned} \quad (20)$$

By the first step of (10), we have

$$y_k = \xi + C_2 e_k^2 + 2(C_3 - C_2^2) e_k^3 + O(e_k^4) = \xi + \Delta, \quad (21)$$

Here, suppose $\Delta = C_2 e_k^2 + 2(C_3 - C_2^2) e_k^3 + O(e_k^4)$.

So,

$$\frac{x_k + y_k}{2} = \xi + \frac{1}{2} x_k + \frac{1}{2} \Delta = \xi + \frac{1}{2} e_k + \frac{1}{2} \Delta, \quad (22)$$

$$\frac{y_k - x_k}{2} = \frac{\xi}{2} + \frac{1}{2} \Delta - \frac{1}{2} (\xi + e_k) = -\frac{1}{2} e_k + \frac{1}{2} \Delta. \quad (23)$$

Furthermore,

$$\frac{x_k + y_k}{2} + \frac{y_k - x_k}{2} t_i = \xi + \frac{1}{2} (1 - t_i) e_k + \frac{1}{2} (1 + t_i) \Delta = \xi + d_k, \quad (24)$$

where,

$$d_k = \frac{1}{2} (1 - t_i) e_k + \frac{1}{2} (1 + t_i) \Delta. \quad (25)$$

Then,

$$\begin{aligned} &F'\left(\frac{x_k + y_k}{2} + \frac{y_k - x_k}{2} t_i\right) \\ &= F'(\xi) (I + 2C_2 d_k + 3C_3 d_k^2 + \dots) \\ &= F'(\xi) \left[I + C_2 (1 - t_i) e_k + C_2 (1 + t_i) \Delta + \frac{3}{4} C_3 (1 - t_i)^2 e_k^2 + O(e_k^3) \right] \\ &= F'(\xi) \left[I + C_2 (1 - t_i) e_k + C_2^2 (1 + t_i) e_k^2 + \frac{3}{4} C_3 (1 - t_i)^2 e_k^2 + O(e_k^3) \right]. \end{aligned} \quad (26)$$

Let

$$\begin{aligned}
 B &= \sum_{i=1}^n w_i F' \left(\frac{x_k + y_k}{2} + \frac{y_k - x_k}{2} t_i \right) \\
 &= \sum_{i=1}^n w_i F'(\xi) \left[I + C_2(1-t_i)e_k + C_2^2(1+t_i)e_k^2 + \frac{3}{4}C_3(1-t_i)^2 e_k^3 + O(e_k^3) \right] \\
 &= F'(\xi) \left[\sum_{i=1}^n w_i I + C_2 e_k \sum_{i=1}^n w_i (1-t_i) + C_2^2 e_k^2 \sum_{i=1}^n w_i (1+t_i) + \frac{3}{4} C_3 e_k^3 \sum_{i=1}^n w_i (1-2t_i+t_i^2) + O(e_k^3) \right].
 \end{aligned}
 \tag{27}$$

By the conditions (4), we have

$$B = F'(\xi) \left[2I + 2C_2 e_k + 2C_2^2 e_k^2 + \frac{3}{4} C_3 e_k^3 \left(2 + \sum_{i=1}^n w_i t_i^2 \right) + O(e_k^3) \right].
 \tag{28}$$

So the iterative schemes (10) can be written as $x_{k+1} = x_k - 2B^{-1}F(x_k)$. Subtracting ξ from both sides of this equation, we can obtain

$$\frac{1}{2} B e_{k+1} = \frac{1}{2} B e_k - F(x_k),
 \tag{29}$$

that is

$$\begin{aligned}
 (I + C_2 e_k + O(e_k^2)) e_{k+1} &= e_k + C_2 e_k^2 + C_2^2 e_k^3 + \frac{3}{8} C_3 e_k^3 \left(2 + \sum_{i=1}^n w_i t_i^2 \right) + O(e_k^4) - F(x_k) \\
 &= (C_2^2 - C_3) e_k^3 + \frac{3}{8} C_3 \left(2 + \sum_{i=1}^n w_i t_i^2 \right) e_k^3 + O(e_k^4)
 \end{aligned}
 \tag{30}$$

Therefore, the error equation is

$$e_{k+1} = \left(C_2^2 - C_3 + \frac{3}{8} \left(2 + \sum_{i=1}^n w_i t_i^2 \right) C_3 \right) e_k^3 + O(e_k^4).
 \tag{31}$$

This shows that the group of iterative methods (10) is third-order convergent.

As for the iterative methods (10), when $n=1, w_1=2, t_1=0$, then it can be transformed to the following form:

$$\begin{cases} y_k = x_k - F'(x_k)^{-1} F(x_k) \\ x_{k+1} = x_k - F' \left(\frac{x_k + y_k}{2} \right)^{-1} F(x_k) \end{cases}
 \tag{32}$$

When $n=2, w_1=w_2=1, t_1=\mu, t_2=-\mu, \mu \in R$ is a parameter, then the methods (10) can be written as follows:

$$\begin{cases} y_k = x_k - F'(x_k)^{-1} F(x_k) \\ x_{k+1} = x_k - 2 \left[\sum_{i=1}^n F' \left(\frac{x_k + y_k}{2} - \frac{y_k - x_k}{2} \mu \right) + F' \left(\frac{x_k + y_k}{2} + \frac{y_k - x_k}{2} \mu \right) \right]^{-1} F(x_k) \end{cases}
 \tag{33}$$

Especially, we take the parameter $\mu = 0.577350$ in the iterative method (33) to make applications in the following section.

3. Numerical Examples

The iterative method (33) with $\mu = 0.577350$ is demonstrated for solving systems of nonlinear equations and some two-point boundary-value problems of nonlinear ODEs.

Example 1. Consider a system of nonlinear equations with variables $n = 5$:

$$\sum_{j=1, j \neq i}^n x_j - (n-1)x_i^2 = 0, \quad 1 \leq i \leq n, \tag{34}$$

where $x_0 = (2, 2, 2, 2, 2)'$ is the guess value. We can get the exact solution $\xi = (1, 1, 1, 1, 1)'$ by the iterative method (33) for Equation (34). The numerical results are shown in **Table 1**, in which k refers to the number of iterations.

Example 2. Consider solving the following two-point boundary-value problem of nonlinear ODE:

$$\begin{cases} x''(t) + \sqrt{x(t)} = 0, & t \in (0, 1) \\ x(0) = 0, \quad x(1) = 1 \end{cases} \tag{35}$$

We discretize the nonlinear ODE (35) with the finite difference method. Taking nodes $t_i = ih$, where $h = 1/N$ and $N = 10$ herein, we can obtain the following system of nine-variables nonlinear equations:

$$\begin{cases} 2x_1 - h^2\sqrt{x_1} - x_2 = 0 \\ -x_{i-1} + 2x_i - h^2\sqrt{x_i} - x_{i+1} = 0, \quad i = 2, 3, \dots, 8, \\ -x_8 + 2x_9 - 1 - h^2\sqrt{x_9} = 0 \end{cases} \tag{36}$$

where $x_0 = (1, 1, 1, 1, 1, 1, 1, 1, 1)'$. Using the iterative method (33), we get the numerical solutions ξ of Equation (36), that is,

$$\begin{aligned} \xi = & (0.1283710267934016\dots, 0.2531591633178528\dots, 0.3729158073866021\dots, \\ & 0.4865657681707311\dots, 0.5932403023143547\dots, 0.6922126270196785\dots, \\ & 0.7828650200725204\dots, 0.8646694471440053\dots, 0.9371751138295893\dots)' \end{aligned}$$

The numerical results for the system of nonlinear Equations (36) derived from ODE (35) are shown in **Table 2**.

According to results of the above two numerical experiments, the iterative method (33) can achieve third-order convergence for systems of nonlinear equations, and their numerical solutions show also the method is feasible.

Table 1. The numerical solutions and errors of the system of Equation (34) using the method (33).

k	1	2	3	4	5	6
$\ x_k - \xi\ _2$	3.1943e-1	4.3758e-3	1.6660e-8	9.2481e-25	1.5819e-73	7.9180e-220
$\ F(x_k)\ _2$	1.4602e-0	1.7537e-2	6.6640e-8	3.6992e-24	6.3278e-73	3.1672e-219

Table 2. The numerical solutions and errors of the system of nonlinear Equation (36).

k	1	2	3	4
$\ x_k - \xi\ _2$	4.9234e-4	5.63305e-14	8.6475e-17	8.6475e-17
$\ F(x_k)\ _2$	1.3365e-4	9.35871e-15	9.7017e-45	1.5685e-134

Example 3. Solving the following two-point nonlinear boundary-value problem of ODE with exact solution:

$$\begin{cases} y''(x) = \sqrt{1 + y'^2} \\ y(0) = 1, \quad y(1) = \frac{1}{2}(e + e^{-1}) \end{cases} \tag{37}$$

The exact solution for this problem of ODE (37) is $y(x) = \frac{1}{2}(e^{-x} + e^x)$. We now find the numerical solutions using the present method and compare them with the exact solution $y(x)$.

By the finite difference method, partitioning the interval $[0, 1]$:

$$x_0 = 0 < x_1 < x_2 < \dots < x_{n-1} < x_n, \quad y_{i+1} = y_i + h, \quad h = 1/n.$$

Let $y_0 = y(0) = 1, y_1 = (x_1), \dots, y_{n-1} = y(x_{n-1})$, and $y_n = y(x_n) = 0.5(e^{-1} + e)$. And using the numerical differential formula for the second derivative $y''_i = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$,

and the first derivative $y'_i = \frac{y_{i+1} - y_i}{h}, (i = 1, 2, \dots, n - 1)$, we take $n = 10$ herein, and

hence obtain the following system of nonlinear equations with nine variables:

$$y_{i-1} - 2y_i + y_{i+1} - h\sqrt{h^2 + (y_{i+1} - y_i)^2} = 0, \quad i = 1, 2, \dots, 9, \tag{38}$$

where $y^{(0)} = (0, 0, 0, 0, 0, 0, 0, 0, 0)^T$ is an initial value. We obtain the approximate numerical solutions of this problem with the method (33) (iteration number $k = 5$):

$$y^* = (1.00412287612219\dots, 1.01834640006185\dots, 1.04286614560037\dots, \\ 1.07798462092560\dots, 1.12411585747856\dots, 1.18179111195371\dots, \\ 1.25166573656183\dots, 1.33452729205395\dots, 1.43130499800870\dots)'$$

Comparison of the numerical results of the boundary-value problem of ODE (37) and the exact solutions $y(x_i)$ are shown in **Table 3**, in which x_i refers to the nodes.

The error comparisons of the numerical solutions and exact solutions at different nodes x_i for solving the problem of ODE (37) are shown in **Figure 1**.

Table 3. Numerical comparison results for the problem of ODE (37).

x_i	Numerical solutions y^*	Exact solutions $y(x_i)$	The errors $\ y(x_i) - y^*\ $
0.1	1.00412287612219...	1.0050041680558035039...	8.812e-04
0.2	1.01834640006185...	1.0200667556190758933...	1.720e-03
0.3	1.04286614560037...	1.0453385141288604742...	2.472e-03
0.4	1.07798462092560...	1.0810723718384549485...	3.087e-03
0.5	1.12411585747856...	1.1276259652063806981...	3.510e-03
0.6	1.18179111195371...	1.1854652182422675821...	3.674e-03
0.7	1.25166573656183...	1.2551690056309430243...	3.503e-03
0.8	1.33452729205395...	1.3374349463048447184...	2.907e-03
0.9	1.43130499800870...	1.4330863854487745356...	1.781e-03

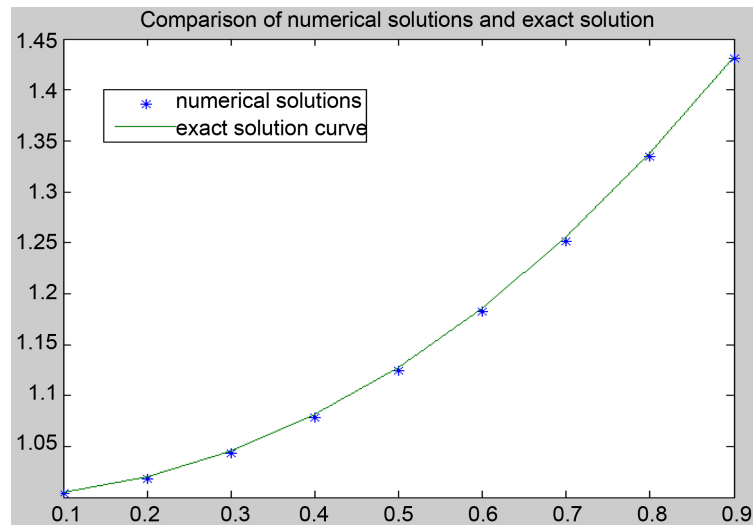


Figure 1. Error comparisons for solving the problem of ODE (37).

4. Conclusion

In this paper, we construct a group of iterative methods with cubic convergence for the systems of nonlinear equations by using the Gauss-Legendre quadrature formula. Numerical results we gave are in consistence with the theoretical analysis, and meanwhile they also demonstrate that the presented scheme is efficient and feasible to solve systems of nonlinear equations and to solve two-point boundary-value problems of nonlinear ordinary differential equations.

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