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Numerical Solution of the Diffusion Equation with Restrictive Pade Approximation

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Abstract

The problem of solving the linear diffusion equation by a method related to the Restrictive Pade Approximation (RPA) is considered. The advantage is that it has the exact value at certain *r*. This method will exhibit several advantages for example highly accurate, fast and with good results, etc. The absolutely error is still very small. The obtained results are compared with the exact solution and the other methods. The numerical results are in agreement with the exact solution.

Keywords

Restrictive Pade Approximation (RPA), Diffusion Equation, Finite Difference

1. Introduction

In this paper, we apply a new implicit method of high accuracy and the number of linear systems which to be solved are smaller than that for many famous known implicit methods of small step length. Therefore, our required machine time is less than that for the other implicit methods.

Restrictive Pade Approximation (RPA) for parabolic Partial Differential Equation (PDE) and Partial Difference Equations is a new technique done by İsmail and Elbarbary [1]. In addition, they studied numerical solution of the Convection Diffusion Equation [2]. RPA for hyperbolic PDE is done by İsmail and Younes [3] [4] [5]. Restrictive Taylor approximation solution for the parabolic PDE is studied by İsmail and Elbarbary [6]. Schrodinger and Singularly perturbed parabolic PDE studied by İsmail and Elbeetar [7] [8]. G. Gurarslan [9] studied numerical modelling of linear and nonlinear diffusion equations by compact finite difference method.

In this work, we consider the following one dimensional diffusion equation;

$$u_t = u_{xx} + f(u), \quad 0 < x < L \tag{1}$$

$$u_t = \left(D(u)u_x\right)_x, \; 0 < x < L \tag{2}$$

Subject to the initial condition

$$u(x,0) = f(x), \; 0 < x < L \tag{3}$$

and boundary conditions

$$u(0,t) = g_0(x), \; t > 0 \tag{4}$$

$$u(1,t) = g_1(x), \; t > 0 \tag{5}$$

The functions $f(u)$ are linear source functions. The function "$D(u)$" is the diffusion term that plays a crucial role in a wide range of applications in diffusion process. [10] [11] [12] The diffusion term $D(u)$ appears in several forms. Some of the well known diffusion proses are the fast and the slow diffusion process where the diffusion term is of the form $D(u) = u^n$ where $n < 0$ and $n > 0$ respectively.

2. Method

2.1. Restrictive Pade Approximation (RPA)

The Restrictive Pade Approximation (RPA) of function $f(x)$ is a particular type of rational functions, it can be written in the form [13] [14];

$$RPA[M+\alpha/N]_{f(x)}(x) = \frac{\sum_{i=0}^{M} \alpha_i x^i + \sum_{i=1}^{\alpha} \varepsilon_i x^{M+i}}{1 + \sum_{i=1}^{N} b_i x^i} \tag{6}$$

where the positive integer α doesn't exceed the degree of the numerator N, $\alpha = 0(1)N$,

$$f(x) - RPA[M+\alpha/N]_{f(x)}(x) = 0\left(x^{M+N+1}\right). \tag{7}$$

Let $f(x)$ have a Maclaurin Series;

$$f(x) = \sum_{i=0}^{\infty} c_i x^i \tag{8}$$

From Equations (6)-(8) we get, [1]

$$\left(\sum_{i=0}^{\infty} c_i x^i\right)\left(1 + \sum_{i=1}^{N} b_i x^i\right) - \sum_{i=0}^{N} a_i x^i - \sum_{i=1}^{\alpha} \varepsilon_i x^{i+M} = 0\left(x^{M+N+1}\right) \tag{9}$$

The varishing of the first $(M+N+1)$ power of x on the left hand side of (9) implies a system of $(M+N+1)$ equations.

$$\begin{gathered} a_r = c_r + \sum_{i=1}^{r} c_{r-i} b_i, \; r = 0(1)M \quad (b_i = 0 \text{ if } i > M) \\ c_{M+N-s} + \sum_{i=1}^{N} c_{M+N-i-s} b_i = \varepsilon_{N-s} \quad (c_i = 0 \text{ if } i > 0) \end{gathered} \tag{10}$$

Hence we can determine the coefficient, a_i and b_i as a function of $\varepsilon_i, i = 0(1)\alpha$, where the parameters ε_i are to be determined, such that [15];

$$f(x_i) = RPA[M+\alpha/N]_{f(x)}(x_i), \; i = 1(1)\alpha \tag{11}$$

Note: $\varepsilon_i = 0, i = 1(1)\alpha$ gives the classical Pade Approximation (RPA) of the form;

$$PA[M/N]_{f(x)}(x) = \frac{\sum_{i=0}^{M} \beta_i x^i}{1+\sum_{i=1}^{N} \gamma_i x^i}. \tag{12}$$

The local truncation error form the RPA can be summarized by the following theorem.

THEOREM: If the function $f(x)$ has an $(n+1)^{+h}$ derivative, then for every argument x there exist a number η in the smallest interval I containing the set of points $\{x_0, x_1, x_2, \cdots, x_\alpha, x\}$, such that;

$$R_{M,N,\alpha}(x) = f(x) - RPA[M+\alpha/N]_{f(x)} = \frac{\pi_{\alpha+1}(x)}{(\alpha+1)!}\left(R_{M,N,\alpha}(\eta)\right)^{(\alpha+1)} \tag{13}$$

where $\prod_{\alpha+1} = x(x-x_1)(x-x_2)\cdots(x-x_\alpha)$ and $R_{M,N,\alpha}$ is the local truncation error for the RPA [1].

2.2. Restrictive Pade Approximation of the Exponential Matrix

The exponential matrix exp(rA) can be formally defined by the convergent power series,

$$\exp(rA) = I + rA + \frac{r^2}{2!}A^2 + \frac{r^3}{3!}A^3 + \cdots = \sum_{n=0}^{\infty} \frac{r^n}{n!}A^r, \quad A^0 = I \tag{14}$$

where A is $(N-1)\times(N-1)$ matrix.

In the case of Restrictive Pade Approximation of single function the term ε_i in Equation (4) can be reduced to the square matrix ε_i in the case of Restrictive Pade approximation of the exponential matrix, where

$$\varepsilon_i = \begin{pmatrix} \varepsilon_{1,i} & & & 0 \\ & \varepsilon_{2,i} & \cdots & \\ & \vdots & \ddots & \vdots \\ 0 & & \cdots & \varepsilon_{N-1,i} \end{pmatrix}$$

for example,

$$RPA[1/1]_{\exp(rA)}(r) = \left(I + \left(\varepsilon_1 - \frac{1}{2}A\right)r\right)^{-1}\left(I + \left(\varepsilon_1 + \frac{1}{2}A\right)r\right). \tag{15}$$

3. Method of Solution

We consider the diffusion for $f(u) = -u$ [16]. Thus, we use the equation;

$$U_t = U_{xx} - U \tag{16}$$

Subject to the initial condition;

$$U(x,0) = \sin x, \quad 0 < x < 1 \tag{17}$$

and boundary conditions;

$$\begin{aligned} U(0,t) &= 0, \quad t > 0 \\ U(1,t) &= 0, \quad t > 0 \end{aligned} \tag{18}$$

The exact solution of the Equation (16) is given by:

$$U\left(x,t\right)=\mathrm{e}^{-2t}\sin x \tag{19}$$

Condiser the diffusion Equation (16) with the initial and boundary condition. The open rectangular domain is covered by a rectangular grid with spacing h and k in the x and t directions respectively. The grid point $\left(x,t\right)$ denoted by $\left(ih, jk\right)$ and $U\left(ih, jk\right)=U_{i,j}$ where $i=0\left(1\right)N$ j is non-negaive integer.

The exact solution of grid representation of (6) is given by [3];

$$U_{i,j+1}=\exp\left(k\left(D_x^2-1\right)\right)U_{i,j} \tag{20}$$

The approximation of the partical derivatives D_x^2 at the grid point $\left(ih, jk\right)$ will take the usual form:

$$D_x^2=\frac{1}{h^2}\left(U_{i+1,j}-2U_{i,j}+U_{i-1,j}\right) \tag{21}$$

and according to central finite difference for mulation

$$U=U_{i+1}-2U_i+U_{i-1} \tag{22}$$

The result of making this approximation is to replace (20) by the following equation

$$U^{j+1}=\exp\left(rA\right)U^j,\ r=\frac{k}{h^2} \tag{23}$$

where $U^j=\left(U_{1,j},U_{2,j},\cdots,U_{N-1,j}\right)^{\mathrm{T}}, Nh=1$ and

$$A=\begin{bmatrix} -2+2h^2 & 1-h^2 & & & & & 0 \\ 1-h^2 & -2+2h^2 & 1-h^2 & & & & \\ & 1-h^2 & -2+2h^2 & 1-h^2 & & & \\ & & & \ddots & & & \\ & & & & 1-h^2 & -2+2h^2 & 1-h^2 \\ 0 & & & & & 1-h^2 & -2+2h^2 \end{bmatrix}_{(N-1)\times(N-1)} \tag{24}$$

We use the

$$RPA[1/1]_{\exp(rA)}\left(r\right)=\left(I+\left(\varepsilon-\frac{1}{2}A\right)r\right)^{-1}\cdot\left(I+\left(\varepsilon+\frac{1}{2}A\right)r\right) \tag{25}$$

Equation to approximate the exponential matrix in Equation (23), then the approximate solution of grid representation of Equation (16) can take the form.

$$\begin{aligned}&\left(\varepsilon_i-\frac{1}{2}\left(1-h^2\right)\right)rU_{i-1,j+1}+\left[\left(\varepsilon_i-\left(1-h^2\right)\right)r+1\right]U_{i,j+1}+\left(\varepsilon_i-\frac{1}{2}\left(1-h^2\right)\right)rU_{i+1,j+1}\\&=\left(\varepsilon_i+\frac{1}{2}\left(1-h^2\right)\right)rU_{i-1,j}+\left[\left(\varepsilon_i-\left(1-h^2\right)\right)r+1\right]U_{i,j}+\left(\varepsilon_i+\frac{1}{2}\left(1-h^2\right)\right)rU_{i+1,j}\end{aligned} \tag{26}$$

The constrat matrix ε_1 must determined by using the only one fact that the exact solution is given at first level. First choice of $\varepsilon_1=\left[\varepsilon_{i,j}\right]$ was the tridiagonal form: $\varepsilon_{i,i-1}=\varepsilon_{i,i}=\varepsilon_{i,i+1}=\varepsilon_i$ and $\varepsilon_{i,j}=0$. The second choise of ε_1 is the diagonal matrix

$\varepsilon_1 = \left[\varepsilon_{i,j}\right] : \varepsilon_{i,i} = \varepsilon_i, \varepsilon_{i,j} = 0$ otherwise.

Other implicit methods can be derived if we use the possible Restrictive Pade Approximation RPA [M/N], with non-negative integers M and N.

4. Findings

The accuracy of Restrictive Pade Approximation method are compared in tables for various values of the time t. Tables give exact value, approximate value for compact finite difference method, approximate value for Restrictive Taylor Approximation, Restrictive Pade Approximation and absolute error for $\varepsilon = 0.0032408523$. Comparison of the RPA results with RTA method for $k = 0.0001$, $N = 6$, $r = 0.0036$ given below in tables.

The absolute error (AE) is give by the following formula:

$$\text{AE} = \left|\text{Exact value} - \text{Approximate value}\right|.$$

We tabulated all AE values at Table 1, Table 2 and Table 3.

Table 1. Absolute error (AE) for RPA at $t = 0.01$.

t	x	Exact	RPA	RTA	AE [Present]
0.01	1/6	0.162611	0.162611	0.162611	1.02E−10
	2/6	0.320715	0.320715	0.320715	1.01E−10
	3/6	0.469932	0.469932	0.469932	1.01E−11
	4/6	0.606125	0.606125	0.606125	2.06E−11
	5/6	0.725520	0.725520	0.725520	1.12E−10
	1	0.824808	0.824808	0.824808	1.15E−10

Table 2. Absolute error (AE) for RPA at $t = 0.1$.

t	x	Exact	RPA	RTA	AE [Present]
0.1	1/6	0.135824	0.135824	0.135824	1.26E−9
	2/6	0.267884	0.267884	0.267884	1.16E−9
	3/6	0.392520	0.392520	0.392520	2.01E−11
	4/6	0.506278	0.506278	0.506278	1.25E−10
	5/6	0.606005	0.606005	0.606005	1.21E−9
	1	0.688938	0.688938	0.688938	1.66E−9

Table 3. Absolute error (AE) for RPA at $t = 1$.

t	x	Exact	RPA	RTA	AE [Present]
1.00	1/6	0.022451	0.022451	0.022451	2.54E−10
	2/6	0.044280	0.044280	0.044280	2.12E−10
	3/6	0.064883	0.064883	0.064883	4.25E−12
	4/6	0.083687	0.083687	0.083687	1.92E−11
	5/6	0.100172	0.100172	0.100172	2.04E−10
	1	0.113880	0.113880	0.113880	2.52E−10

5. Discussion & Conclusion

In this article, a numerical algorithm was applied in the one dimensional diffusion equation. Computed results were compared with other paper results in Table 1, Table 2 and Table 3. Especially, we compared it with Restrictive Taylor approximation method, because these methods have same properties. But as we have seen from the computational results, Restrictive Pade approximation method has more efficient results than restrictive Taylor approximation method. The proposed method results are quite satisfactory.

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