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A Maximum Principle Result for a General Fourth Order Semilinear Elliptic Equation

A. Mareno

Penn State University, Middletown, USA
Email: aum24@psu.edu

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Abstract

We obtain maximum principles for solutions of some general fourth order elliptic equations by modifying an auxiliary function introduced by L.E. Payne. We give a brief application of these maximum principles by deducing apriori bounds on a certain quantity of interest.

Keywords

Nonlinear, Fourth Order, Partial Differential Equation, Semilinear

1. Introduction

In [1], Payne obtains maximum principle results for the semilinear fourth order elliptic equation

$$\Delta^2 u = f(u) \tag{1}$$

by proving that certain functionals defined on the solution of (1) are subharmonic. In this work, functionals containing the terms $\left|\nabla^2 u\right| - u_{,i}\Delta u_{,i}$ are utilized and apriori bounds on the integral of the square of the second gradient and on the square of the gradient of the solution are deduced. Since then, many authors [2]-[11] and references therein have used this technique to obtain maximum principle results for other fourth order elliptic differential equations whose principal part is the biharmonic operator.

Other works deal with the more general fourth order elliptic operator L^2u, where $Lu := a_{ij}u_{,ij}$ and $a_{ij} = a_{ji}$. In [12], Dunninger mentions that functionals containing the term $(Lu)^2$ can be used to obtain maximum principle results for such linear equations as

$$L^2 u + aLu + bu = 0.$$

A similar approach is taken in [13] for a class of nonlinear fourth order equations.

In this paper, we modify the results in [1] and a matrix result from [14] to deduce maximum principles defined on the solutions to semilinear fourth order elliptic equations of the form:

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$$L^2u = f(u). \tag{2}$$

Then we briefly indicate how these maximum principles can be used to obtain apriori bounds on a certain quantity of interest.

2. Results

Throughout this paper, the summation convention on repeated indices is used; commas denote partial differentiation. Let $a_{ij}(x)$ be a symmetric matrix. Moreover let $Lu := a_{ij}u_{,ij}$, be a uniformly elliptic operator, *i.e*, the symmetric matrix $a_{ij}(x)$ is positive definite and satisfies the uniform ellipticity condition:
$a_{ij}(x)v_i v_j \geq |v|^2, x \in \Omega, v \in R^n$, where Ω is a bounded domain in R^n and $n \geq 2$.

Let u be a C^5 solution to the equation

$$L^2u = f(u) \text{ in } \Omega. \tag{3}$$

where f is say, a C^1 function. Now we define the functional

$$P = c_1(Lu)^2 - (a_{mn}Lu)_{,n}u_{,m} + c_2|\nabla u|^2 + 2(1-c_1)\int_0^u f(s)\mathrm{d}s + \beta(x).$$

We show that $L(P)$ is subharmonic and note that the constants c_1 and c_2 and any constraints on f are yet to be determined.

By a straight-forward calculation, we have

$$\begin{aligned} P_{,i} &= 2c_1 LuLu_{,i} - (a_{nm}Lu)_{,ni}u_{,m} - (a_{mn}Lu)_{,n}u_{,mi} \\ &\quad + 2c_2u_{,m}u_{,mi} + 2(1-c_1)f(u)u_{,i} + \beta_{,i}. \end{aligned}$$

Now we write

$$\begin{aligned} L(P) &= a_{ij}P_{,ij} \\ &= 2c_1a_{ij}LuLu_{,ij} + 2c_1a_{ij}Lu_{,i}Lu_{,j} - a_{ij}(a_{mn}Lu)_{,nij}u_{,m} \\ &\quad - a_{ij}(a_{mn}Lu)_{,ni}u_{,mj} - a_{ij}(a_{nm}Lu)_{,nj}u_{,mi} - a_{ij}(a_{mn}Lu)_{,n}u_{,mij} \\ &\quad + 2c_2a_{ij}u_{,mi}u_{,mj} + 2c_2a_{ij}u_{,m}u_{,mij} + 2(1-c_1)f(u)a_{ij}u_{,ij} \\ &\quad + 2(1-c_1)f'(u)a_{ij}u_{,i}u_{,j} + L(\beta). \end{aligned} \tag{4}$$

By expanding out the derivative terms in parentheses, we see that $L(P)$ is

$$\begin{aligned} &= 2c_1f(u)Lu + 2c_1a_{ij}Lu_{,i}Lu_{,j} - a_{ij}u_{,m}\big[a_{mn,nij}Lu + a_{mn,ni}Lu_{,j} + a_{mn,nj}Lu_{,i} \\ &\quad + a_{mn,n}Lu_{,ij} + a_{mn,ij}Lu_{,n} + a_{mn,i}Lu_{,nj} + a_{mn,j}Lu_{,ni} + a_{mn}Lu_{,nij}\big] \\ &\quad - 2a_{ij}u_{,mj}\left(a_{mn,i}Lu_{,n} + a_{mn}Lu_{,ni} + a_{mn,ni}Lu + a_{mn,n}Lu_{,i}\right) \\ &\quad - a_{ij}u_{,mij}\left(a_{mn,n}Lu + a_{mn}Lu_{,n}\right) + 2c_2a_{ij}u_{,mi}u_{,mj} + 2c_2a_{ij}u_{,m}u_{,mij} \\ &\quad + 2(1-c_1)f(u)Lu + 2(1-c_1)f'(u)a_{ij}u_{,i}u_{,j} + L(\beta). \end{aligned} \tag{5}$$

The terms in lines 2 and 3 above containing two or more derivatives of Lu can be rewritten using (3) in the form $A^{ij}f(u) = Lu_{,ij}$, where A^{ij} denotes the matrix which is the inverse of the positive definite matrix (a_{ij}). Furthermore, we use the identity $a_{ij}u_{,mij} = Lu_{,m} - a_{ij,m}u_{,ij}$ to rewrite the last two terms in line 4. Hence,

$$\begin{aligned} L(P) &= 2f(u)Lu + 2c_1a_{ij}Lu_{,i}Lu_{,j} - f(u)u_{,m}a_{mn,n} - a_{ij}Lu\left(u_{,m}a_{mn,nij} + 2a_{mn,ni}u_{,mj}\right) \\ &\quad - a_{mn,n}LuLu_{,m} + a_{mn,n}Lua_{ij,m}u_{,ij} - a_{mn}Lu_{,n}Lu_{,m} + a_{mn}a_{ij,m}u_{,ij}Lu_{,n} \\ &\quad - a_{ij}a_{mn}u_{,m}\left(A^{ni}_{,j}f(u) + A^{ni}f'(u)u_{,j}\right) - 2a_{ij}a_{mn}u_{,mj}A^{ni}f(u) \\ &\quad - 2a_{ij}a_{mn,i}u_{,m}A^{nj}f(u) - 2a_{ij}u_{,m}a_{mn,ni}Lu_{,j} - 2a_{mn,n}a_{ij}u_{,mj}Lu_{,i} \\ &\quad - a_{ij}Lu_{,n}u_{,m}a_{mn,ij} - 2a_{ij}Lu_{,n}u_{,mj}a_{mn,i} + 2c_2a_{ij}u_{,mi}u_{,mj} \\ &\quad + 2c_2a_{ij}u_{,m}u_{,mij} + 2(1-c_1)f'(u)a_{ij}u_{,i}u_{,j} + L(\beta). \end{aligned} \tag{6}$$

Using the identity above for $a_{ij}u_{,mij}$ and the additional identity, $a_{ij}A^{ni}_{,j} = -a_{ij,j}A^{ni}$, which can be obtained by computing $\left(a_{ij}A^{ni}\right)_{,j}$, for the terms at the ends of lines 6 and 3 respectively, we obtain

$$\begin{aligned} L(P) = {} & L(\beta) + (1-2c_1)a_{ij}f'(u)u_{,i}u_{,j} - 2a_{mn,n}u_{,m}f(u) + 2c_2u_{,k}Lu_{,k} \\ & + 2c_2a_{ij}u_{,ik}u_{,jk} - 2c_2a_{ij,k}u_{,ij}u_{,k} - a_{mn,n}LuLu_{,m} + a_{mn,n}Lua_{ij,m}u_{,ij} \\ & - a_{rs}a_{ij}u_{,m}a_{mn,nrs}u_{,ij} - 2a_{mn,n}a_{ij}Lu_{,i}u_{,mj} - 2a_{ij}Lua_{mn,ni}u_{,mj} \\ & - 2a_{ij}a_{mn,i}Lu_{,n}u_{,mj} + a_{mn}a_{ij,m}Lu_{,n}u_{,ij} + (2c_1-1)a_{ij}Lu_{,i}Lu_{,j} \\ & - a_{ij}a_{mn,ij}u_{,m}Lu_{,n} - 2a_{ij}a_{mn,ni}u_{,m}Lu_{,j}. \end{aligned} \tag{7}$$

To show that $L(P)$ is nonnegative, we establish a series of inequalities based on the following one from [14]: Let $\left(s_{pk}\right)$ be any $n\times n$ matrix. From the inequality

$$a_{ij}\left(u_{ik} + \frac{1}{2}A^{ip}s_{pk}\right)\left(u_{,jk} + \frac{1}{2}A^{jq}s_{qk}\right) \geq 0, \tag{8}$$

One can deduce

$$a_{ij}u_{,kj}u_{,ki} + s_{ki}u_{,ki} \geq -\frac{1}{4}A^{pq}s_{pk}s_{qk}. \tag{9}$$

Repeated use of (9) on terms in lines 2, 3, 4, 5 in (7) yields the following:

$$a_{ij}u_{,ik}u_{,jk} + a_{mn}a_{ij,m}Lu_{,n}u_{,ij} \geq -\frac{1}{4}A^{pq}\left(a_{mn}a_{pi,m}Lu_{,n}a_{rs}a_{qi,r}Lu_{,s}\right) \tag{10}$$

$$a_{ij}u_{,ik}u_{,jk} - 2a_{ij}a_{mn,i}Lu_{,n}u_{,mj} \geq -A^{pq}\left(a_{pm,r}a_{ri}Lu_{,n}a_{qn,s}a_{si}Lu_{,m}\right) \tag{11}$$

$$a_{ij}u_{,ik}u_{,jk} - 2a_{ij}a_{mn,ni}Luu_{,mj} \geq -A^{pq}\left(Lua_{ps,sr}a_{ri}Lua_{ql,lw}a_{wi}\right) \tag{12}$$

$$a_{ij}u_{,ik}u_{,jk} + a_{mn,n}a_{ij,m}Lu_{,n}u_{,ij} \geq -\frac{1}{4}A^{pq}\left(a_{mn,n}a_{pi,m}Lua_{rs,s}a_{qi,r}Lu\right) \tag{13}$$

$$a_{ij}u_{,ik}u_{,jk} - a_{ij}a_{rs}a_{mn,nrs}u_{,ij}u_{,m} \geq -\frac{1}{4}A^{pq}\left(a_{rs}a_{pi}u_{,m}a_{mn,nrs}a_{lw}a_{qi}u_{,z}a_{zt,tlw}\right) \tag{14}$$

$$a_{ij}Lu_{,i}Lu_{,j} - 2a_{mn,ni}a_{ij}u_{,m}Lu_{,j} \geq -A^{pq}\left(a_{sk}u_{,r}a_{rn,ns}a_{mq}u_{,p}a_{kw,wm}\right) \tag{15}$$

$$a_{ij}Lu_{,i}Lu_{j} - a_{ij}a_{mn,ij}u_{,m}Lu_{,n} \geq -\frac{1}{4}A^{pq}\left(a_{in}u_{,r}a_{rk,in}a_{lm}u_{,q}a_{kp,lm}\right) \tag{16}$$

$$a_{ij}u_{,ik}u_{jk} - 2c_2a_{ij,m}u_{,m}u_{,ij} \geq -c_2^2A^{pq}\left(a_{pi,m}u_{,m}a_{qi,l}u_{,l}\right) \tag{17}$$

$$a_{ij}u_{,ik}u_{jk} - 2a_{ij}a_{mn,n}Lu_{,i}u_{,mj} \geq -A^{pq}\left(a_{pr,r}a_{si}Lu_{,s}a_{ql,l}a_{mi}Lu_{,m}\right) \tag{18}$$

Furthermore, by completing the square, we obtain useful inequalities for the last two terms in line 1 and the third term in line 2 of (7):

$$2c_2u_{,k}Lu_{,k} \geq -c_2u_{,m}u_{,m} - c_2Lu_{,m}Lu_{,m} \tag{19}$$

$$-2a_{mn,n}u_{,m}f(u) \geq -a_{rp,p}u_{,r}a_{sq,q}u_{,s} - f^2 \tag{20}$$

$$-a_{mn,n}LuLu_{,m} \geq -a_{mp,p}Lua_{mq,q}Lu - Lu_{,m}Lu_{,m} \tag{21}$$

We add (10)-(21) and label the resulting inequality, for part of $L(P)$, as

$$\begin{aligned}\hat{L}(P) &= 7a_{ij}u_{ik}u_{jk} + 2a_{ij}Lu_{,i}Lu_{,j} + 2c_2u_{,k}Lu_{,k} - 2a_{mn,n}u_{,m}f(u) + a_{mn}a_{ij,m}Li_{,n}u_{,ij} \\ &\quad - 2a_{ij}a_{mn,i}Lu_{,n}u_{,mj} - 2a_{ij}a_{mn,ni}Luu_{,mj} + a_{mn,n}a_{ij,m}Lu_{,n}u_{,ij} - a_{ij}a_{rs}a_{mn,nrs}u_{,ij}u_{,m} \\ &\quad - 2a_{mn,ni}a_{ij}u_{,m}Lu_{,j} - 2c_2a_{ij,m}u_{,m}u_{,ij} - 2a_{ij}a_{mn,n}Lu_{,i}u_{,mj} + a_{mn,n}Lua_{ij,m}u_{,ij} \\ &\geq -A^{pq}\left(a_{sk}u_{,r}a_{rn,ns}a_{mq}u_{,p}a_{kw,wm}\right) - \frac{1}{4}A^{pq}\left(a_{in}u_{,r}a_{rk,in}a_{lm}u_{,q}a_{kp,lm}\right) \\ &\quad - c_2^2A^{pq}\left(a_{pi,m}u_{,m}a_{qi,l}u_{,l}\right) - A^{pq}\left(a_{pr,r}a_{si}Lu_{,s}a_{ql,l}a_{mi}Lu_{,m}\right) \\ &\quad - A^{pq}\left(a_{pm,r}a_{ri}Lu_{,n}a_{qn,s}a_{si}Lu_{,m}\right) - c_2u_{,m}u_{,m} - c_2Lu_{,m}Lu_{,m} \\ &\quad - a_{rp,p}u_{,r}a_{sq,q}u_{,s} - a_{mp,p}Lua_{mq,q}Lu - Lu_{,m}Lu_{,m} - a_{mn,n}LuLu_{,m} - f^2 \\ &\quad - \frac{1}{4}A^{pq}\left(a_{mn}a_{pi,m}Lu_{,n}a_{rs}a_{qi,r}Lu_{,s}\right) - \frac{1}{4}A^{pq}\left(a_{mn,n}a_{pi,m}Lua_{rs,s}a_{qi,r}Lu\right) \\ &\quad - \frac{1}{4}A^{pq}\left(a_{rs}a_{pi}u_{,m}a_{mn,nrs}a_{lw}a_{qi}u_{,z}a_{zt,tlw}\right) - A^{pq}\left(Lua_{ps,sr}a_{ri}Lua_{ql,lw}a_{wi}\right).\end{aligned}$$

Now,

$$\begin{aligned}L(P) &= \hat{L}(P) + L(\beta) + (1-2c_1)a_{ij}f'u_{,i}u_{,j} + (2c_2-7)a_{ij}u_{,ik}u_{,jk} + (2c_1-3)a_{ij}Lu_{,i}Lu_{,j} \\ &\geq \left((2c_2-7)a_{ij} - a_{mp,p}a_{mq,q}a_{ik}a_{jk} - \frac{1}{4}A^{pq}a_{mn,n}a_{pi,m}a_{rs,s}a_{qi,r}a_{ik}a_{jk} - A^{pq}a_{ps,sr}a_{ri}a_{ql,lw}a_{wi}a_{ik}a_{jk}\right)u_{,ik}u_{,jk} \\ &\quad + \left((2c_1-3)a_{ij} - (c_2+1)\delta_{ij} - A^{pq}a_{pj,r}a_{rl}a_{qi,s}a_{sl} - A^{pq}a_{pr,r}a_{is}a_{ql,l}a_{js} - \frac{1}{4}A^{pq}a_{mi}a_{pl,m}a_{rj}a_{ql,r}\right)Lu_{,i}Lu_{,j} \\ &\quad + \Big((1-2c_1)f'a_{ij} - c_2\delta_{ij} - A^{jq}a_{sk}a_{in,ns}a_{mq}a_{kw,wm} - c_2^2A^{pq}a_{pm,i}a_{qm,j} - \frac{1}{4}A^{pj}a_{rn}a_{ik,rn}a_{lm}a_{kp,lm} \\ &\quad - a_{ip,p}a_{jq,q} - \frac{1}{4}A^{pq}a_{rs}a_{pm}a_{in,nrs}a_{lw}a_{qm}a_{jt,tlw}\Big)u_{,i}u_{,j} + L(\beta) - f^2.\end{aligned}$$

Since $a_{ij}(x)$ is positive definite, for a sufficiently large value of c_2, where c_2 depends on the coefficients a_{ij} and their derivatives, and for a sufficiently large value of c_1, say (>1), where c_1 depends on the constants c_2, γ, a_{ij}, and various derivatives of a_{ij}, $L(P)$ can be made nonnegative as desired. Thus we have the following result.

Theorem 1. *Suppose that* $u \in C^5(\Omega) \cap C^3(\bar{\Omega})$ *is a solution of* (2) *and* $f \in C^1(R)$. *If* $f^2 \leq \gamma$, *where* $\gamma > 0$, $f'(u) \leq \alpha, \alpha < 0$, $\beta(x)$ *is a nonnegative function such that* $L(\beta) \geq \gamma$ *then there exists positive constants* c_2 *and* c_1 *sufficiently large* $(c_1 > 1)$ *such that P cannot attain its maximum value in* Ω *unless it is a constant.*

We note that the function $f(u) = -(u + u^3)$ satisfies the conditions stated in Theorem 1 for a solution that is bounded above.

3. Bounds

Here we give a brief application of Theorem 1.

Suppose that

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

By Theorem 1,

$$P \leq \max_{\partial\Omega}\left(c_1(Lu)^2 + \beta(x)\right).$$

Using integration by parts on the first two terms of P yields the identity

$$\int_\Omega a^{nm}u_{,km}a^{kp}u_{,pn} - \left(a^{mn}Lu\right)_{,n}u_{,m}\mathrm{d}x = 2\int_\Omega (Lu)^2\,\mathrm{d}x.$$

Upon integrating both sides of the previous inequality we deduce

$$2\int_{\Omega}(Lu)^2\,\mathrm{d}x + 2(1-c_1)\int_{\Omega}\left(\int_0^u f(s)\,\mathrm{d}s\right)\mathrm{d}x \tag{22}$$

$$\leq \left[\max_{\partial\Omega}\left(c_1(Lu)^2 + \beta(x)\right)\right]\mathrm{area}(\Omega). \tag{23}$$

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