

Well-Posedness of an N-Unit Series System with Finite Number of Vacations

Abdugeni Osman, Abdukerim Haji

College of Mathematics and Systems Science, Xinjiang University, Urumqi, China
 Email: abdugeniho@163.com, abdukerimhaji@sina.com.cn

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Abstract

We investigate the solution of an N-unit series system with finite number of vacations. By using C_0 -semigroup theory of linear operators, we prove well-posedness and the existence of the unique positive dynamic solution of the system.

Keywords

N-Unit Series System, C_0 -Semigroup, Dynamic Solution, Well-Posedness

1. Introduction

The study of repairable systems is an important topic in reliability. The series system is one of the classical repairable systems. Since the strong practical background of the series systems, many researchers have studied them extensively under varying assumptions on the failures and repairs, see [1]-[4]. In [4], the authors studied an N-unit series system with finite number of vacations and obtained some reliability expressions such as the Laplace transform of the reliability, the mean time to the first failure, the availability and the failure frequency of the system. In [4], the authors used the dynamic solution in calculating the availability and the reliability. But they did not discuss the existence of the positive dynamic solution. Motivated by this, we study in this paper the well-posedness and the existence of a unique positive dynamic solution of the system, by using C_0 -semigroup theory of linear operators. For background reading on semigroup theory we refer to [5] or [6]. The N-unit series system with finite number of vacations can be described by the following equations (see [4]).

$$(R) \left\{ \begin{array}{l} \left(\frac{d}{dt} + \Lambda \right) p_0(t) = \int_0^{+\infty} r(w) p_{0M}(t, w) dw, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu_i(y) \right) p_i(t, y) = 0, i = 1, 2, \dots, n, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial w} + r(w) \right) p_{ij}(t, w) = \lambda_i p_{0j}(t, w), i = 1, 2, \dots, n, j = 1, 2, \dots, M, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial w} + \Lambda + r(w) \right) p_{0j}(t, w) = 0, j = 1, 2, \dots, M, \end{array} \right.$$

with the boundary conditions

$$(BC) \left\{ \begin{aligned} p_i(t, 0) &= \lambda_i p_0(t) + \sum_{j=1}^M \int_0^\infty r(\omega) p_{ij}(t, \omega) d\omega, i = 1, 2, \dots, n \\ p_{ij}(t, 0) &= 0, i = 1, 2, \dots, n, j = 1, 2, \dots, M, \\ p_{01}(t, 0) &= \sum_{i=1}^n \int_0^\infty \mu_i(y) p_i(t, y) dy, \\ p_{0j}(t, 0) &= \int_0^\infty r(\omega) p_{0,j-1}(t, \omega) d\omega, j = 2, \dots, M, \end{aligned} \right.$$

and the initial conditions

$$(IC) \left\{ \begin{aligned} p_0(0, \omega) &= 1, \\ p_i(0, y) &= 0, i = 1, 2, \dots, n, \\ p_{ij}(0, \omega) &= 0, i = 1, 2, \dots, n, j = 1, 2, \dots, M, \\ p_{0j}(0, \omega) &= 0, j = 1, 2, \dots, M, \end{aligned} \right.$$

where $\Lambda = \lambda_1 + \lambda_2 + \dots + \lambda_n$

Here $(t, y) \in [0, \infty) \times [0, \infty), (t, \omega) \in [0, \infty) \times [0, \infty)$; $p_0(t)dx$ gives the probability that at time t N units are in working state and the repairman is idle; Analogously, $p_i(t, y)dy$ represents the probability that at time t the repairman is repairing the failed unit i ($i = 1, 2, \dots, n$), and the elapsed repair time lies in $[y, y + dy)$; $p_{0j}(t, \omega)dx$ represents the probability that at time t N units are in working state, the repairman is in j^{th} vacation ($j = 1, 2, \dots, M$) and the elapsed vacation time lies in $[\omega, \omega + d\omega)$; $p_{ij}(t, \omega)d\omega$ represents the probability that at time t unit i ($i = 1, 2, \dots, n$) is waiting for repair, the repairman is in j^{th} vacation ($j = 1, 2, \dots, M$) and the elapsed vacation time lies in $[\omega, \omega + d\omega)$; λ_i is positive constant; $r(\omega)$ is the vacation rate function of the repairman; $\mu_i(y)$ is the repair rate function of unit i ($i = 1, 2, \dots, n$).

Throughout the paper we require the following assumption for the vacation rate function $r(\omega)$ and the repair rate functions $\mu_i(y)$ ($i = 1, 2, \dots, n$).

General Assumption 1.1: The functions $r(\omega)$ and $\mu_i(y): [0, +\infty) \rightarrow [0, +\infty)$ ($i = 1, 2, \dots, n$) are measurable and bounded such that

$$r = \lim_{\omega \rightarrow +\infty} r(\omega), \mu_i = \lim_{y \rightarrow +\infty} \mu_i(y) (i = 1, 2, \dots, n),$$

$$\mu_\infty = \min(r, \mu_1, \mu_2, \dots, \mu_n).$$

2. Problem as an Abstract Cauchy Problem

In order to apply semigroup theory we transform in this section the system (RS), (BC) and (IC) into an abstract Cauchy problem [5, Def. II. 6.1] on the Banach space $(X, \|\cdot\|)$, where

$$X = \mathbb{C} \times (L^1_y[0, +\infty))^n \times (L^1_\omega[0, +\infty))^{(n+1) \times M}$$

and

$$\|p\| = |p_0| + \sum_{i=1}^n \|p_i\|_{L^1_y[0, +\infty)} + \sum_{i=1}^n \sum_{j=1}^M \|p_{ij}\|_{L^1_\omega[0, +\infty)} + \sum_{j=1}^M \|p_{0j}\|_{L^1_\omega[0, +\infty)}$$

$$p = (p_0, p_1(y), p_2(y), \dots, p_n(y), p_{11}(\omega), p_{12}(\omega), \dots, p_{1M}(\omega), p_{21}(\omega), p_{22}(\omega), \dots, p_{2M}(\omega), \dots, p_{n1}(\omega), p_{n2}(\omega), \dots, p_{nM}(\omega), p_{01}(\omega), p_{02}(\omega), \dots, p_{0M}(\omega))^T \in X$$

To define the system operator $(A, D(A))$ we introduce a “maximal operator” $(A_m, D(A_m))$ on X given as

$$A_m = \begin{pmatrix} -\Lambda & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \varphi_M \\ 0 & D_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & D_2 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots \\ 0 & 0 & 0 & \dots & D_n & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & D_{11} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \lambda_1 \\ 0 & 0 & 0 & \dots & 0 & 0 & D_{12} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \lambda_1 \\ \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & D_{1M} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \lambda_1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & D_{21} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \lambda_2 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & D_{22} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \lambda_2 \\ \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & D_{2M} & \dots & 0 & 0 & 0 & 0 & 0 & \dots & \lambda_2 \\ \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & D_{n1} & 0 & 0 & 0 & 0 & \dots & \lambda_n \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & D_{n2} & \dots & 0 & 0 & \dots & \lambda_n \\ \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & D_{nM} & 0 & 0 & 0 & \dots & \lambda_n \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & D_{01} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & D_{02} & 0 & 0 & \dots & 0 \\ \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & D_{0M} \end{pmatrix}$$

where

$$\varphi_M: f \mapsto \varphi_M(f) = \int_0^\infty r(\omega)f(\omega)d\omega, \quad D_i = -\frac{d}{dy} - \mu_i(y), i = 1, 2, \dots, n,$$

$$D_{ij} = -\frac{d}{d\omega} - r(\omega), i = 1, 2, \dots, n, j = 1, 2, \dots, M, \quad D_{0j} = -\left[\frac{d}{d\omega} + \Lambda + r(\omega)\right], \quad j = 1, 2, \dots, M$$

Applying an abstract approach as in [7] we can model the boundary conditions (BC) with operators. For this purpose we consider the “boundary space” $\partial X := C^{n+(n+1)M}$ and then define “boundary operators” L and Φ as follows.

$$L: D(A_m) \rightarrow \partial X, \quad \begin{pmatrix} p_0 \\ p_1(y) \\ p_2(y) \\ \vdots \\ p_n(y) \\ p_{11}(\omega) \\ p_{12}(\omega) \\ \vdots \\ p_{1M}(\omega) \\ p_{21}(\omega) \\ p_{22}(\omega) \\ \vdots \\ p_{2M}(\omega) \\ \vdots \\ p_{n1}(\omega) \\ p_{n2}(\omega) \\ \vdots \\ p_{nM}(\omega) \\ p_{01}(\omega) \\ p_{02}(\omega) \\ \vdots \\ p_{0M}(\omega) \end{pmatrix} \rightarrow L \begin{pmatrix} p_0 \\ p_1(y) \\ p_2(y) \\ \vdots \\ p_n(y) \\ p_{11}(\omega) \\ p_{12}(\omega) \\ \vdots \\ p_{1M}(\omega) \\ p_{21}(\omega) \\ p_{22}(\omega) \\ \vdots \\ p_{2M}(\omega) \\ \vdots \\ p_{n1}(\omega) \\ p_{n2}(\omega) \\ \vdots \\ p_{nM}(\omega) \\ p_{01}(\omega) \\ p_{02}(\omega) \\ \vdots \\ p_{0M}(\omega) \end{pmatrix} = \begin{pmatrix} p_1(0) \\ p_2(0) \\ \vdots \\ p_n(0) \\ p_{11}(0) \\ p_{12}(0) \\ \vdots \\ p_{1M}(0) \\ p_{21}(0) \\ p_{22}(0) \\ \vdots \\ p_{2M}(0) \\ \vdots \\ p_{n1}(0) \\ p_{n2}(0) \\ \vdots \\ p_{nM}(0) \\ p_{01}(0) \\ p_{02}(0) \\ \vdots \\ p_{0M}(0) \end{pmatrix}$$

and

$$\Phi \begin{pmatrix} p_1(0) \\ p_2(0) \\ \vdots \\ p_n(0) \\ p_{11}(0) \\ p_{12}(0) \\ \vdots \\ p_{1M}(0) \\ p_{21}(0) \\ p_{22}(0) \\ \vdots \\ p_{2M}(0) \\ \vdots \\ p_{n1}(0) \\ p_{n2}(0) \\ \vdots \\ p_{nM}(0) \\ p_{01}(0) \\ p_{02}(0) \\ \vdots \\ p_{0M}(0) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 & \varphi_{11} & \varphi_{12} & \dots & \varphi_{1M} & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \lambda_2 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \varphi_{21} & \varphi_{22} & \dots & \varphi_{2M} & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots \\ \lambda_n & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & \varphi_{n1} & \varphi_{n2} & \dots & \varphi_{nM} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \varphi_1 & \varphi_2 & \dots & \varphi_n & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \varphi_{01} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1(y) \\ p_2(y) \\ \vdots \\ p_n(y) \\ p_{11}(w) \\ p_{12}(w) \\ \vdots \\ p_{1M}(w) \\ p_{21}(w) \\ p_{22}(w) \\ \vdots \\ p_{2M}(w) \\ \vdots \\ p_{n1}(w) \\ p_{n2}(w) \\ \vdots \\ p_{nM}(w) \\ p_{01}(w) \\ p_{02}(w) \\ \vdots \\ p_{0M}(w) \end{pmatrix},$$

where: $\varphi_{ij}: f \mapsto \varphi_{ij} = \int_0^\infty r(w) f_{ij}(w) dw, i = 1, 2, \dots, n, j = 1, 2, \dots, M$

$$\varphi_i: f \mapsto \varphi_i(f) = \int_0^\infty \mu_i(y) f_i(y) dy, i = 1, 2, \dots, n,$$

$$\varphi_{0j}: f \mapsto \varphi_{0j} = \int_0^\infty r(w) f_{0j-1}(w) dw, j = 2, \dots, n,$$

The system operator $(A, D(A))$ on X is then defined as

$$Ap = A_m p, D(A) = \{p \in D(A_m) \mid Lp = \Phi p\}.$$

With these definitions the above equations (R), (BC) and (IC) are equivalent to the abstract Cauchy problem

$$\begin{cases} \frac{dp(t)}{dt} = Ap(t), & t \in [0, \infty), \\ p(0) = (1, 0, \dots, 0)^T \in X. \end{cases} \quad (\text{ACP})$$

3. Characteristic Equation

In this section we characterize $\sigma(A)$ by the spectrum of a scalar (2×2) -matrix, i.e., or we obtain a characteristic equation which relates $\sigma(A)$ to the spectrum of an operator on the boundary space ∂X . For this purpose, we apply techniques and results from [7]. We start from the operator $(A_0, D(A_0))$ defined by

$$D(A_0) = \{p \in D(A_m) \mid Lp = 0\}, A_0 p = A_m p.$$

The elements in $\ker(\gamma - A_m)$ can be expressed as follows.

Lemma 3.1: For $\gamma \in \rho(A_0)$, we have

$$p \in \ker(\gamma - A_m) \tag{1}$$

$$\Leftrightarrow p = (p_0, p_1(y), p_2(y), \dots, p_n(y), p_{11}(w), p_{12}(w), \dots, p_{1M}(w), p_{21}(w), p_{22}(w), \dots, p_{2M}(w), \dots, p_{n1}(w), p_{n2}(w), \dots, p_{nM}(w), p_{01}(w), p_{02}(w), \dots, p_{0M}(w))^T \in X$$

with

$$\begin{aligned} p_0 &= \frac{c_{0M}}{\gamma + \Lambda} \int_0^\infty r(w) e^{-\gamma w - \Lambda w - \int_0^w r(s) ds} dw, p_1(y) = c_1 e^{-\gamma y - \int_0^y \mu_1(s) ds}, \\ p_2(y) &= c_2 e^{-\gamma y - \int_0^y \mu_2(s) ds}, \dots, p_n(y) = c_n e^{-\gamma y - \int_0^y \mu_n(s) ds}, \\ p_{11}(w) &= c_{11} e^{-\gamma w - \int_0^w r(s) ds} + \frac{\lambda_1 c_{01}}{\Lambda} e^{-\gamma w - \int_0^w r(s) ds} (1 - e^{-\Lambda w}), \\ p_{12}(w) &= c_{12} e^{-\gamma w - \int_0^w r(s) ds} + \frac{\lambda_1 c_{02}}{\Lambda} e^{-\gamma w - \int_0^w r(s) ds} (1 - e^{-\Lambda w}), \dots \\ p_{1M}(w) &= c_{1M} e^{-\gamma w - \int_0^w r(s) ds} + \frac{\lambda_1 c_{0M}}{\Lambda} e^{-\gamma w - \int_0^w r(s) ds} (1 - e^{-\Lambda w}), \\ p_{21}(w) &= c_{21} e^{-\gamma w - \int_0^w r(s) ds} + \frac{\lambda_2 c_{01}}{\Lambda} e^{-\gamma w - \int_0^w r(s) ds} (1 - e^{-\Lambda w}) \end{aligned}$$

$$\begin{aligned}
 p_{22}(\omega) &= c_{22}e^{-\gamma\omega - \int_0^\omega r(s)ds} + \frac{\lambda_2 c_{02}}{\Lambda} e^{-\gamma\omega - \int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}), \dots, \\
 p_{2M}(\omega) &= c_{2M}e^{-\gamma\omega - \int_0^\omega r(s)ds} + \frac{\lambda_2 c_{0M}}{\Lambda} e^{-\gamma\omega - \int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}), \dots \\
 p_{n1}(\omega) &= c_{n1}e^{-\gamma\omega - \int_0^\omega r(s)ds} + \frac{\lambda_n c_{01}}{\Lambda} e^{-\gamma\omega - \int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}), \\
 p_{n2}(\omega) &= c_{n2}e^{-\gamma\omega - \int_0^\omega r(s)ds} + \frac{\lambda_n c_{02}}{\Lambda} e^{-\gamma\omega - \int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}), \dots \\
 p_{nM} &= c_{nM}e^{-\gamma\omega - \int_0^\omega r(s)ds} + \frac{\lambda_n c_{0M}}{\Lambda} e^{-\gamma\omega - \int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}), \\
 p_{01}(\omega) &= c_{01}e^{-\gamma\omega - \Lambda\omega - \int_0^\omega r(s)ds}, \quad p_{02}(\omega) = c_{02}e^{-\gamma\omega - \Lambda\omega - \int_0^\omega r(s)ds}, \dots \\
 p_{0M}(\omega) &= c_{0M}e^{-\gamma\omega - \Lambda\omega - \int_0^\omega r(s)ds}
 \end{aligned}$$

Using [8, Lemma 1.2], the domain $D(A_m)$ of the maximal operator A_m decomposes as

$$D(A_m) = D(A_0) \oplus \ker(\Phi_\gamma - A_m).$$

Moreover, since L is surjective, $L|_{\ker(\Phi_\gamma - A_m)}: (\gamma - A_m) \rightarrow \partial X$ is invertible for each $\gamma \in \rho(A_0)$, see [8, Lemma 1.2]. We denote its inverse by

$$D_\gamma := (L|_{\ker(\Phi_\gamma - A_m)})^{-1}: \partial X \rightarrow \ker(\gamma - A_m)$$

and call it ‘‘Dirichlet operator’’.

We can give the explicit form of D_γ as follows.

Lemma 3.2: For each $\gamma \in \rho(A_0)$, the operator D_γ has the form

$$D_\gamma = \begin{pmatrix}
 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & d_0 \\
 d_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
 0 & d_2 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 0 & \dots & d_n & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
 0 & 0 & \dots & 0 & d_{11} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & h_{01} \\
 0 & 0 & \dots & 0 & 0 & d_{12} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & h_{02} \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 0 & \dots & 0 & 0 & 0 & \dots & d_{1M} & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & h_{0M} \\
 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & d_{21} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & h_{01} \\
 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & d_{22} & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & h_{02} \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & d_{2M} & \dots & 0 & 0 & 0 & 0 & \dots & h_{0M} \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & d_{n1} & 0 & 0 & 0 & \dots & h_{01} \\
 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & d_{n2} & \dots & 0 & 0 & h_{02} \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & d_{nM} & 0 & 0 & \dots & h_{0M} \\
 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_1 & 0 & \dots & 0 \\
 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & d_1 & \dots & 0 \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & d_1
 \end{pmatrix},$$

where

$$\begin{aligned}
 d_0 &= \frac{1}{\gamma + \Lambda} \int_0^\infty r(\omega) e^{-\gamma\omega - \Lambda\omega - \int_0^\omega r(s)ds} d\omega, \\
 d_i &= e^{-\gamma\gamma - \int_0^\gamma \mu_i(s)ds}, \quad i = 1, 2, \dots, n, \\
 d_{ij} &= e^{-\gamma\omega - \int_0^\omega r(s)ds}, \quad i = 1, 2, \dots, n, j = 1, 2, \dots, M, \\
 h_{0j} &= \frac{\lambda_n}{\Lambda} e^{-\gamma\omega - \int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}), \quad j = 1, 2, \dots, M \\
 d_{0j} &= e^{-\gamma\omega - \Lambda\omega - \int_0^\omega r(s)ds}, \quad j = 1, 2, \dots, M.
 \end{aligned}$$

For $\gamma \in \rho(A_0)$, the operator ΦD_γ can be represented by the $(n + n \times M + M) \times (n + n \times M + M)$ -matrix

$$\phi D_Y = \left(\begin{array}{cccccccccccccccc} 0 & 0 & \dots & 0 & a_{1,n+1} & a_{1,n+2} & \dots & a_{1,n+M} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & a_{2,n+M+1} & a_{2,n+M+2} & \dots & a_{2,n+M+M} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n+nM+1,1} & a_{n+nM+1,2} & \dots & a_{n+nM+1,n} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots \end{array} \right),$$

$$= \left(\begin{array}{cccccccccccccccc} 0 & 0 & \dots & 0 & a_{1,n+nM+1} & a_{1,n+nM+2} & \dots & a_{1,n+nM+M-1} & a_{1,n+nM+M} \\ 0 & 0 & \dots & 0 & a_{2,n+nM+1} & a_{2,n+nM+2} & \dots & a_{2,n+nM+M-1} & a_{2,n+nM+M} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n,n+(n-1)M+1} & a_{n,n+(n-1)M+2} & \dots & a_{n,n+nM} & a_{n,n+nM+1} & a_{n,n+nM+2} & \dots & a_{n,n+nM+M-1} & a_{n,n+nM+M} \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & a_{n+nM+2,n+nM+1} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & a_{n+nM+M,n+nM+M} & 0 \end{array} \right)$$

where

$$\begin{aligned}
 a_{n+nM+1,1} &= \int_0^\infty \mu_1(y) e^{-\gamma y - \int_0^y \mu_1(t) dt} dy, \quad a_{n+nM+1,2} = \int_0^\infty \mu_2(y) e^{-\gamma y - \int_0^y \mu_2(t) dt} dy, \dots, \\
 a_{n+nM+1,n} &= \int_0^\infty \mu_n(y) e^{-\gamma y - \int_0^y \mu_n(t) dt} dy, \quad a_{1,n+1} = \int_0^\infty r(w) e^{-\gamma w - \int_0^w r(s) ds} dw, \\
 a_{1,n+2} &= \int_0^\infty r(w) e^{-\gamma w - \int_0^w r(s) ds} dw, \dots, \quad a_{1,n+M} = \int_0^\infty r(w) e^{-\gamma w - \int_0^w r(s) ds} dw, \\
 a_{2,n+M+1} &= \int_0^\infty r(w) e^{-\gamma w - \int_0^w r(s) ds} dw, \quad a_{2,n+M+2} = \int_0^\infty r(w) e^{-\gamma w - \int_0^w r(s) ds} dw, \dots, \\
 a_{2,n+M+M} &= \int_0^\infty r(w) e^{-\gamma w - \int_0^w r(s) ds} dw, \dots, \quad a_{n,n+(n-1)M+1} = \int_0^\infty r(w) e^{-\gamma w - \int_0^w r(s) ds} dw, \\
 a_{n,n+(n-1)M+2} &= \int_0^\infty r(w) e^{-\gamma w - \int_0^w r(s) ds} dw, \dots, \quad a_{n,n+nM} = \int_0^\infty r(w) e^{-\gamma w - \int_0^w r(s) ds} dw, \\
 a_{1,n+nM+1} &= \frac{\lambda_1}{\Lambda} \int_0^\infty r(w) e^{-\gamma w - \int_0^w r(s) ds} (1 - e^{-\Lambda w}) dw, \\
 a_{2,n+nM+1} &= \frac{\lambda_2}{\Lambda} \int_0^\infty r(w) e^{-\gamma w - \int_0^w r(s) ds} (1 - e^{-\Lambda w}) dw, \dots, \\
 a_{n,n+nM+1} &= \frac{\lambda_n}{\Lambda} \int_0^\infty r(w) e^{-\gamma w - \int_0^w r(s) ds} (1 - e^{-\Lambda w}) dw,
 \end{aligned}$$

$$\begin{aligned}
 a_{n+nM+2,n+nM+1} &= \int_0^\infty r(\omega) e^{-\gamma\omega-\Lambda\omega-\int_0^\omega r(s)ds} d\omega, \\
 a_{1,n+nM+2} &= \frac{\lambda_1}{\Lambda} \int_0^\infty r(\omega) e^{-\gamma\omega-\int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}) d\omega, \\
 a_{2,n+nM+2} &= \frac{\lambda_2}{\Lambda} \int_0^\infty r(\omega) e^{-\gamma\omega-\int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}) d\omega, \dots, \\
 a_{n,n+nM+2} &= \frac{\lambda_n}{\Lambda} \int_0^\infty r(\omega) e^{-\gamma\omega-\int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}) d\omega, \\
 a_{n+nM+3,n+nM+2} &= \int_0^\infty r(\omega) e^{-\gamma\omega-\Lambda\omega-\int_0^\omega r(s)ds} d\omega, \dots, \\
 a_{1,n+nM+M-1} &= \frac{\lambda_1}{\Lambda} \int_0^\infty r(\omega) e^{-\gamma\omega-\int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}) d\omega, \\
 a_{2,n+nM+M-1} &= \frac{\lambda_2}{\Lambda} \int_0^\infty r(\omega) e^{-\gamma\omega-\int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}) d\omega, \dots, \\
 a_{n,n+nM+M-1} &= \frac{\lambda_n}{\Lambda} \int_0^\infty r(\omega) e^{-\gamma\omega-\int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}) d\omega, \\
 a_{n+nM+M,n+nM+M+1} &= \int_0^\infty r(\omega) e^{-\gamma\omega-\Lambda\omega-\int_0^\omega r(s)ds} d\omega, \\
 a_{1,n+nM+M} &= \frac{\lambda_1}{\gamma + \Lambda} \int_0^\infty r(\omega) e^{-\gamma\omega-\Lambda\omega-\int_0^\omega r(s)ds} d\omega + \frac{\lambda_1}{\Lambda} \int_0^\infty r(\omega) e^{-\gamma\omega-\int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}) d\omega, \\
 a_{2,n+nM+M} &= \frac{\lambda_2}{\gamma + \Lambda} \int_0^\infty r(\omega) e^{-\gamma\omega-\Lambda\omega-\int_0^\omega r(s)ds} d\omega + \frac{\lambda_2}{\Lambda} \int_0^\infty r(\omega) e^{-\gamma\omega-\int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}) d\omega, \dots \\
 a_{n,n+nM+M} &= \frac{\lambda_n}{\gamma + \Lambda} \int_0^\infty r(\omega) e^{-\gamma\omega-\Lambda\omega-\int_0^\omega r(s)ds} d\omega + \frac{\lambda_n}{\Lambda} \int_0^\infty r(\omega) e^{-\gamma\omega-\int_0^\omega r(s)ds} (1 - e^{-\Lambda\omega}) d\omega,
 \end{aligned}$$

The Following result, which can be found in [9], plays important role for us to prove the well-posedness of the system.

Lemma 3.3 (The characteristic equation): If $\gamma \in \rho(A_0)$ and there exists $\gamma_0 \in \mathcal{C}$ such that $1 \notin \sigma(\Phi_{D_{\gamma_0}})$, then

$$\gamma \in \sigma(A) \iff 1 \in \sigma(\Phi_{D_\gamma}).$$

4. Well-Posedness of the System

Our main goal in this section is to prove the well-posedness of the system. We first prove that the operator A generates a positive contraction C_0 -semigroup $(T(t))_{t \geq 0}$. For this purpose we will check that operator A fulfills all the conditions in the Phillips' theorem, see [6, Thm. C-II 1.2]. The following lemma shows the surjectivity of $\gamma - A$ for $\gamma > 0$.

Lemma 4.1: If $\gamma \in R, \gamma > 0$, then $\gamma \in \rho(A)$.

Proof: Let $\gamma \in R, \gamma > 0$. Then all the entries of Φ_{D_γ} are positive and using only elementary calculations one can show that both column sums are strictly less than 1. Hence, $\|\Phi_{D_\gamma}\| < 1$ and thus $1 \in \sigma(\Phi_{D_\gamma})$. Using Lemma 3.3 we conclude that $\gamma \in \rho(A)$.

Lemma 4.2: $A: D(A) \rightarrow R(A) \subset X$ is a closed linear operator and $D(A)$ is dense in X .

If X' denotes the dual space of X , then $X' = \mathbb{C} \times (L^\infty_\gamma[0, \infty))^n \times (L^\infty_\omega[0, \infty))^{nM+M+1}$

It is obvious that X' is a Banach space endowed with the norm

$$\|q\| = |q_0| + \sum_{i=1}^n \|q_i\|_{L^\infty_\gamma[0, +\infty)} + \sum_{i=1}^n \sum_{j=1}^M \|q_{ij}\|_{L^\infty_\omega[0, +\infty)} + \sum_{j=1}^M \|q_{0j}\|_{L^\infty_\omega[0, +\infty)}$$

where $q = (q_0, p_1(y), q_2(y), \dots, q_n(y), q_{11}(\omega), q_{12}(\omega), \dots, qp_{1M}(\omega), q_{21}(\omega), q_{22}(\omega), \dots, q_{2M}(\omega), \dots, q_{n1}(\omega), q_{n2}(\omega), \dots, q_{nM}(\omega), q_{01}(\omega), q_{02}(\omega), \dots, q_{0M}(\omega))^T \in X'$

Lemma 4.3: The operator $(A, D(A))$ is dispersive.

Proof: For $p = (p_0, p_1(y), p_2(y), \dots, p_n(y), p_{11}(\omega), p_{12}(\omega), \dots, p_{1M}(\omega), p_{21}(\omega), p_{22}(\omega), \dots, p_{2M}(\omega), \dots, p_{n1}(\omega), p_{n2}(\omega), \dots, p_{nM}(\omega), p_{01}(\omega), p_{02}(\omega), \dots, p_{0M}(\omega))^T \in X$,

we define

$$q = (q_0, p_1(y), q_2(y), \dots, q_n(y), q_{11}(\omega), q_{12}(\omega), \dots, qp_{1M}(\omega), q_{21}(\omega), q_{22}(\omega), \dots, q_{2M}(\omega), \dots, q_{n1}(\omega), q_{n2}(\omega), \dots, q_{nM}(\omega), q_{01}(\omega), q_{02}(\omega), \dots, q_{0M}(\omega))^T \in X'$$

where

$$\begin{aligned}
 q_0 &= \|p\| \operatorname{sgn}_+(p_0), \quad q_i(y) = \|p\| \operatorname{sgn}_+(p_i(y)), \quad i = 1, 2, \dots, n, \\
 q_{ij}(\omega) &= \|p\| \operatorname{sgn}_+(p_{ij}(\omega)), \quad i = 1, 2, \dots, n, j = 1, 2, \dots, M,
 \end{aligned}$$

$$q_{0j}(\omega) = |p| \operatorname{sgn}_+(p_{0j}(\omega)), j = 1, 2, \dots, M,$$

and

$$\operatorname{sgn}_+(p_0) = \begin{cases} 1 & \text{if } p_0 > 0, \\ 0 & \text{if } p_0 \leq 0, \end{cases} \operatorname{sgn}_+(p_i(y)) = \begin{cases} 1 & \text{if } p_i(y) > 0, \\ 0 & \text{if } p_i(y) \leq 0, \end{cases} i = 1, 2, \dots, n,$$

$$\operatorname{sgn}_+(p_{ij}(\omega)) = \begin{cases} 1 & \text{if } p_{ij}(\omega) > 0, \\ 0 & \text{if } p_{ij}(\omega) \leq 0, \end{cases} i = 1, 2, \dots, n, j = 1, 2, \dots, M,$$

$$\operatorname{sgn}_+(p_{0j}(\omega)) = \begin{cases} 1 & \text{if } p_{0j}(\omega) > 0, \\ 0 & \text{if } p_{0j}(\omega) \leq 0, \end{cases} j = 1, 2, \dots, M.$$

Noting the boundary condition, it is not difficult to see that $\langle Ap, q \rangle \leq 0$. By [6, p.49] we obtain that $(A, D(A))$ is a dispersive operator.

From Lemma 4.1- 4.3 we see that all the conditions in Phillips' theorem (see [6, Thm. C-II 1.2]) are fulfilled and thus we obtain the following result.

Theorem 4.4: The operator $(A, D(A))$ generates a positive contraction C_0 -semi-group $(T(t))_{t \geq 0}$.

From Theorem 4.4 and [5, Cor.II.6.9] we can characterize the well-posedness of The system (R) , (BC) and (IC) as follows.

Theorem 4.5: The system (R) , (BC) and (IC) is well-posed.

Combining Theorem 4.5 with [5, Prop.II.6.2] we can state our main result.

Theorem 4.6: The system (R) , (BC) and (IC) has a unique positive dynamic solution

$$p(t) = (p_0(t), p_1(t, y), p_2(t, y), \dots, p_n(t, y), p_{11}(t, \omega), p_{12}(t, \omega), \dots, \\ p_{1M}(t, \omega), p_{21}(t, \omega), p_{22}(t, \omega), \dots, p_{2M}(t, \omega), \dots, \\ p_{n1}(t, \omega), p_{n2}(t, \omega), \dots, p_{nM}(t, \omega), p_{01}(t, \omega), p_{02}(t, \omega), \dots, p_{0M}(t, \omega))^T \in X$$

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