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Dynamics of a Two Species Competitive System with Pure Delays

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Abstract

A class of non-autonomous two species Lotka-Volterra competitive system with pure discrete time delays is discussed. Some sufficient conditions on the boundedness, permanence, periodic solution and global attractivity of the system are established by means of the comparison method and Liapunov functional.

Keywords

Lotka-Volterra Competitive System, Discrete Time Delay, Liapunov Functional, Global Attractivity

1. Introduction

Population competition systems of Lotka-Volterra type have been investigated extensively in recent years [1]-[5]. The basic and the simplest two species nonautonomous competitive system for Lotka-Volterra type is as following form

$$\begin{aligned}\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= x_1(t)\left[r_1(t)-a_{11}(t)x_1(t)-a_{12}(t)x_2(t)\right],\\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= x_2(t)\left[r_2(t)-a_{21}(t)x_1(t)-a_{22}(t)x_2(t)\right].\end{aligned} \tag{1}$$

There is an extensive literature concerned with the properties of system (1) that has been discussed by many authors[1]-[4].

However, in the real world, the growth rate of a natural species will not often respond immediately to changes in its own population or that of an interacting species, but will rather do so after a time lag [6]. Recently, many

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people are doing research on the dynamics of population with time delays, which is useful for the control of the population of mankind, animals and the environment. Therefore, it is essential for us to investigate population systems with time delays. In this paper, we investigate the following two species Lotka-Volterra type competitive systems with pure discrete time delays

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[r_1(t) - a_{11}(t)x_1(t-\tau_1) - a_{12}(t)x_2(t-\tau_1)\right], \\ \dot{x}_2(t) &= x_2(t)\left[r_2(t) - a_{21}(t)x_1(t-\tau_2) - a_{12}(t)x_2(t-\tau_2)\right]. \end{aligned} \tag{2}$$

By using the technique of comparison method and Liapunov function method, we will establish some sufficient conditions on the boundedness, permanence, existence of positive periodic solution and global attractivity of the system.

The organization of this paper is as follows. In the next Section, we will present some basic assumptions and main definition and lemmas. In Section 3, conditions for the positivity and boundedness are considered. In the final Section, we considered the conditions for the permanence, existence of positive periodic solution and global attractivity of the system.

2. Preliminaries

In system (2), we have that $x_i(t)(i=1,2)$ represent the density of two competitive species $x_i(t)(i=1,2)$ at time *t*, respectively; $r_i(t)(i=1,2)$ represent the intrinsic growth rate of species $x_i(t)(i=1,2)$ at time *t*, respectively; $a_{11}(t)$ and $a_{22}(t)$ represent the intra patch restriction density of species $x_i(t)(i=1,2)$ at time *t*, respectively; $a_{12}(t)$ and $a_{21}(t)$ represent the competitive coefficients between two species $x_i(t)(i=1,2)$ at time *t*, respectively. $\tau_i(t)(i=1,2)$ represent the time delay in the model. In this paper, we always assume that

$(\mathrm{H}_1)\tau_i > 0(i=1,2)$ are positive constants, $r_i(t)(i=1,2)$, $a_{ij}(t)(i,j=1,2)$ are continuous positive functions.

$(\mathrm{H}_1')\tau_i > 0(i=1,2)$ are positive constants, $r_i(t)(i=1,2)$, $a_{ij}(t)(i,j=1,2)$ are continuous positive ω-periodic functions.

From the viewpoint of mathematical biology, in this paper for system (2) we only consider the solution with the following initial conditions

$$x_i(t) = \phi_i(t), \text{ for all } t \backslash \in [-\tau, 0], i=1,2, \tag{3}$$

where $\phi_i(t)(i=1,2)$ are nonnegative continuous functions defined on $[-\tau,0]$ satisfying $\phi_i(0)>0(i=1,2)$ with $\tau = \max\{\tau_1,\tau_2\}$.

In this paper, for any continuous function $f(t)$ we denote

$$f^L = \min_{t\in[0,+\infty)} f(t), \quad f^M = \max_{t\in[0,+\infty)} f(t).$$

Now, we present some useful definitions.

Definition 1. (see [7]) System (2) is said to be permanent if there exists a compact region $D \subset IntR_+^2$ such that every solution $z(t)=(x_1(t),x_2(t))$ of system (2) with initial conditions (3) eventually enters and remains in the region *D*.

Definition 2. (see [8]) System (2) is said to be global attractive, if for any two positive solutions $(x_1(t),x_2(t))$ and $(y_1(t),y_2(t))$ of system (2), one has

$$\lim_{t\to\infty}(x_i(t)-y_i(t))=0, \quad i=1,2.$$

The following two lemmas will be used in the proof of the main results of system (2).

Lemma 1. (see [9]) Consider the following equation:

$$\dot{u}(t) = u(t)(d_1 - d_2 u(t)),$$

where, $d_2 > 0$, we have

1) If $d_1 > 0$, then $\lim_{t\to+\infty} u(t) = d_1/d_2$.

2) If $d_1 < 0$, then $\lim_{t\to+\infty} u(t) = 0$.

Lemma 2. (see [10]) Let $f(t)$ be a nonnegative function defined on $[0,\infty)$, such that $f(t)$ is integrable on $[0,\infty)$ and uniformly continuous on $[0,\infty)$. Then, $\lim_{t\to\infty} f(t) = 0$.

3. Positivity and Boundedness

In this section, we will obtain positivity and boundedness of system (2). The following Lemma is about the positivity of system (2).

Lemma 1. Set $R_+^2 = \{(x_1, x_2) : x_i > 0, i = 1,2\}$ is positively invariant for system (2).

The proof of Lemma1 is simple, and here we omit it.

The following theorem is about the boundedness of system (2).

Theorem 1. Suppose that assumption (H_1) holds, then there exist positive constants $M_i\,(i = 1,2)$ such that $x_i(t) \le M_i$, for any positive solution $x_i(t)$ of system (2).

Proof: Let $(x_1(t), x_2(t))$ be a solution of system (2). Firstly, it follows from the first equation of system (2) that for $t > \tau$, we have

$$\begin{aligned}\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= x_1(t)\left[r_1(t) - a_{11}(t)x_1(t-\tau_1) - a_{12}(t)x_2(t-\tau_1)\right] \\ &\le x_1(t)\left[r_1^M - a_{11}^L \mathrm{e}^{-r_1^M\tau_1} x_1(t)\right] \text{ for } t > \tau\end{aligned}$$

We consider the following auxiliary equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = u(t)\left[r_1^M - a_{11}^L \mathrm{e}^{-r_1^M\tau_1} u(t)\right].$$

By Lemma 2, we derive

$$\lim_{t\to+\infty} u(t) = \frac{r_1^M \mathrm{e}^{r_1^M\tau_1}}{a_{11}^L} =: M_1.$$

By comparison, there exists a $T_1 > \tau$ such that $x_1(t) \le M_1$ for $t \ge T_1$.

Next, by using an argument similar in the above, there exist a $T_2 > \tau$ such that $x_2(t) \le M_2$, where

$$M_2 = \frac{r_2^M \mathrm{e}^{r_2^M\tau_2}}{a_{22}^L}.$$

This completes the proof.

The following theorem is about the global attractivity of system (2). Firstly, for convenience we denote the following functions

$$\lambda_1(t) = \gamma_{12}(t),\ \lambda_2(t) = \gamma_{21}(t)$$

where,

$$\begin{aligned}\gamma_{ij}(t) = \mu_i a_{ii}(t) - \mu_i \int_{t-\tau_i}^{t} a_{ii}(u+\tau_i)\,\mathrm{d}u\left[r_i(t) + \left(a_{ii}(t) + a_{ij}(t)\right)M\right] \\ - M\sum_{l=1}^{2}\mu_l\tau_l a_{ll}^M a_{lj}(t+\tau_l) - \mu_j a_{ji}(t+\tau_j) \quad i \ne j, i, j = 1,2,\end{aligned} \tag{4}$$

where, $M = \max\{M_1, M_2\}$ and $\mu_i > 0\,(i = 1,2)$ are constants.

4. Permanence, Existence of Positive Periodic Solution and Global Attractivity

In this section, we will obtain the permanence, existence of positive periodic solution and global attractivity of system (2). First we obtain the global attractivity of system (2).

Theorem 2. Suppose that (H_1) and there exists a constant $\mu_i > 0\,(i = 1,2)$ such that

$$\liminf_{t\to\infty} \lambda_i(t) > 0, i = 1,2,$$

Then system (2) has a positive solution which is globally attractive.

Proof: Let $(x_1(t), x_2(t))$ and $(y_1(t), y_2(t))$ are any two positive solutions of system (2). From Theorem 1, choose positive constants $M_i > 0$ such that

$$x_i(t) \le M,\ i = 1,2, \tag{5}$$

for all $t \ge T = \max\{T_1 < T_2\}$. Let

$$L_1(t) = \sum_{i=1}^{2} \mu_i \left|\ln x_i(t) - \ln y_i(t)\right|.$$

Calculating the upper right derivation of $L_1(t)$ along system (2) for all $t \ge T$, we have

$$\begin{aligned}
D^+ L_1(t) &= \sum_{i=1}^{2}\sum_{j\ne i}^{2} \mu_i \operatorname{sign}\left(x_i(t) - y_i(t)\right)\Big[-a_{ii}(t)\left(x_i(t-\tau_i) - y_i(t-\tau_i)\right) \\
&\quad - a_{ij}(t)\left(x_j(t-\tau_i) - y_j(t-\tau_i)\right)\Big] \\
&= \sum_{i=1}^{2}\sum_{j\ne i}^{2} \mu_i \operatorname{sign}\left(x_i(t) - y_i(t)\right)\Big[-a_{ii}(t)\left(x_i(t) - y_i(t)\right) \\
&\quad - a_{ij}(t)\left(x_j(t-\tau_i) - y_j(t-\tau_i)\right) + a_{ii}(t)\int_{t-\tau_i}^{t}\left(\dot{x}_i(u) - \dot{y}_i(u)\right)\mathrm{d}u\Big] \\
&= \sum_{i=1}^{2}\sum_{j\ne i}^{2} \mu_i \operatorname{sign}\left(x_i(t) - y(t)\right)\Big[-a_{ii}(t)\left(x_i(t) - y_i(t)\right) - a_{ij}(t)\left(x_j(t-\tau_i) - y_j(t-\tau_i)\right) \\
&\quad + a_{ii}(t)\int_{t-\tau_i}^{t}\Big(\left(x_i(u) - y_i(u)\right)\left[r_i(u) - a_{ii}(u)y_i(u-\tau_i) - a_{ij}(u)y_j(u-\tau_i)\right] \\
&\quad + x_i(u)\Big[-a_{ii}(u)\left(x_i(u-\tau_i) - y_i(u-\tau_i)\right) - a_{ij}(u)\left(x_j(u-\tau_i) - y_j(u-\tau_i)\right)\Big]\Big)\mathrm{d}u\Big] \\
&\le \sum_{i=1}^{2}\sum_{j\ne i}^{2}\Big(-\mu_i a_{ii}(t)\left|x_i(t) - y_i(t)\right| + \mu_i a_{ij}(t)\left|x_j(t-\tau_i) - y_j(t-\tau_i)\right| \\
&\quad + \mu_i a_{ii}(t)\int_{t-\tau_i}^{t}\Big(\left|x_i(u) - y_i(u)\right|\left[r_i(u) + a_{ii}(u)y_i(u-\tau_i) + a_{ij}(u)y_j(u-\tau_i)\right] \\
&\quad + x_i(u)\Big[a_{ii}(u)\left|x_i(u-\tau_i) - y_i(u-\tau_i)\right| + a_{ij}(u)\left|x_j(u-\tau_j) - y_j(u-\tau_i)\right|\Big]\Big).
\end{aligned} \tag{6}$$

Define

$$L_2(t) = \mu_1 V_{12}(t) + \mu_2 V_{21}(t),$$

where

$$\begin{aligned}
V_{ij}(t) &= \int_{t-\tau_i}^{t}\int_{u}^{t} a_{ii}(u+\tau_i)\Big(\left[r_i(s) + a_{ii}(s)y_i(s-\tau_i) + a_{ij}(s)y_j(s-\tau_i)\right]\left|x_i(s) - y_i(s)\right| \\
&\quad + x_i(s)a_{ii}(s)\left|x_i(s-\tau_i) - y_i(s-\tau_i)\right| + a_{ij}(s)\left|x_j(s-\tau_i) - y_j(s-\tau_i)\right|\Big)\mathrm{d}s\mathrm{d}u,\quad i \ne j, i, j = 1, 2.
\end{aligned}$$

Calculating the upper right derivative of $L_2(t)$ and from (6), we have

$$\begin{aligned}
\sum_{i=1}^{2} D^+ L_i(t) &\le -\sum_{i=1}^{2}\Bigg(\sum_{j\ne i}^{2}\Big(\mu_i a_{ii}(t)\left|x_i(t) - y_i(t)\right| + \mu_i a_{ij}(t)\left|x_j(t-\tau_i) - y_j(t-\tau_i)\right| \\
&\quad + \mu_i\int_{t-\tau_i}^{t} a_{ii}(u+\tau_i)\mathrm{d}u\left[r_i(t) + \left(a_{ii}(t) + a_{ij}(t)\right)M\right]\left|x_i(t) - y_i(t)\right|\Big) \\
&\quad + \mu_i\tau_i a_{ii}^{M} M\sum_{l=1}^{2} a_{il}(t)\left|x_l(t-\tau_i) - y_l(t-\tau_i)\right|\Bigg).
\end{aligned} \tag{7}$$

Define

$$L_3(t) = \mu_1 W_{12}(t) + \mu_2 W_{21}(t),$$

where

$$W_{ij}(t) = \mu_i\tau_i a_{ii}^{M} M\sum_{l=1}^{2}\int_{t-\tau_i}^{t} a_{il}(u+\tau_i)\left|x_l(u) - y_l(u)\right|\mathrm{d}u + \int_{t-\tau_i}^{t} a_{ij}(u+\tau_i)\left|x_j(u) - y_j(u)\right|\mathrm{d}u,$$

Further, we define a Liapunov function as follows

$$V(t)=\sum_{i=1}^{3}L_i(t).$$

Calculating the upper right derivation of $V(t)$, from (6) and (7) we finally can obtain for all $t\ge T$

$$D^+V(t)\le-\sum_{i=1}^{2}\lambda_i(t)\left|x_i(t)-y_i(t)\right|. \tag{8}$$

From assumption (H_2), there exists a constant $\delta>0$ and $T^*\ge T$ such that for all $t\ge T^*$ we have

$$\lambda_i(t)\ge\delta>0,\ \ i=1,2. \tag{9}$$

Integrating from T^* to t on both sides of (8) and by (9) produces

$$V(t)+\alpha\int_{T^*}^{t}\left(\sum_{i=1}^{2}\left|x_i(s)-y_i(s)\right|\right)\mathrm{d}s\le V\left(T^*\right), \tag{10}$$

hence, $V(t)$ bounded on $\left[T^*,\infty\right)$ and we have

$$\int_{T^*}^{t}\left(\sum_{i=1}^{2}\left|x_i(s)-y_i(s)\right|\right)\mathrm{d}s<\infty. \tag{11}$$

From the boundedness of $x_i(t),y_i(t)(i=1,2)$ and (11), we can obtain that $\left(x_i(t)-y_i(t)\right)(i=1,2)$ and their derivatives remain bounded on $\left[T^*,\infty\right)$. Therefore $\sum_{i=1}^{2}\left|x_i(t)-y_i(t)\right|$ is uniformly continuous on $[0,+\infty)$. By Barbalat's theorem it follows that

$$\lim_{t\to+\infty}\sum_{i=1}^{2}\left|x_i(t)-y_i(t)\right|=0.$$

Therefore,

$$\lim_{t\to+\infty}\left(x_i(t)-y_i(t)\right)=0,\ \ i=1,2.$$

This completes the proof of Theorem 2.

From the global attractivity of system (2), we have the following result.

Corollary 1. Suppose that the conditions of Theorem 2 hold, then system (2) is permanent.

As a direct corollary of [11] (Theorem 2), from Corollary 1, we have the following result.

Corollary 2. Suppose that the conditions of Theorem 2 and (H_1') hold, then system (2) has a positive ω-periodic solution which is globally attractive.

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