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Boundedness for Commutators of Calderón-Zygmund Operator on Herz-Type Hardy Space with Variable Exponent

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Abstract

Our aim in this paper is to prove the boundedness of commutators of Calderón-Zygmund operator with the Lipschitz function or BOM function on Herz-type Hardy space with variable exponent.

Keywords

Commutator, Variable Exponent, Herz-Taype Hardy Spaces, BMO, Calderón-Zygmund Operator

1. Introduction

In 2012, Hongbin Wang and Zongguang Liu [1] discussed boundedness Calderón-Zygmund operator on Herz-type Hardy space with variable exponent. M. Luzki [2] introduced the Herz space with variable exponent and proved the boundedness of some sublinear operator on these spaces. Li'na Ma, Shuhai Li and Huo Tang [3] proved the boundedness of commutators of a class of generalized Calderón-Zygmund operators on Labesgue space with variable exponent by Lipschitz function. Mitsuo Izuki [4] proved the boundedness of commutators on Herz spaces with variable exponent. Lijuan Wang and S. P. Tao [5] proved the boundedness of Littlewood-Paley operators and their commutators on Herz-Morrey space with variable exponent. In this paper we prove the boundedness of commutators of singular integrals with Lipschitz function or BMO function on Herz-type Hardy space with variable exponent.

In this section, we will recall some definitions.

Definition 1.1. Let T be a singular integral operator which is initially defined on the Schwartz space $S(\mathbb{R}^n)$.

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Its values are taken in the space of tempered distributions $S'(\mathbb{R}^n)$ such that for x not in the support of f,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,\mathrm{d}y, \tag{1.1}$$

where f is in $L_c^\infty(\mathbb{R}^n)$, the space of compactly bounded function.

Let $0 < \delta, D < \infty$. Here the kernel k is function in $(\mathbb{R}^n)$ away from the diagonal $x = y$ and satisfies the standard estimate

$$|K(x,y)| \le \frac{D}{|x-y|^n}, x \ne y \tag{1.2}$$

and

$$|k(x,y) - k(x',y)| \le \frac{D|x-x'|^\sigma}{|x-y| + |x'-y|^{n+\sigma}}, \tag{1.3}$$

provided that $|x-x'| \le \frac{1}{2}\max\{|x-y|, |x'-y|\}$

$$|k(x,y) - k(x',y)| \le \frac{D|x-x'|^\sigma}{|x-y| + |x'-y|^{n+\sigma}}, \tag{1.4}$$

provided that $|y-y'| \le \frac{1}{2}\max\{|x-y|, |x-y'|\}$ such that is called standard kernel and the class of all kernels that satisfy (1.2), (1.3), (1.4) is denoted by $SK(\sigma, D)$. Let T be as in (1.1) with kernel $SK(\sigma, D)$. If T is bounded from L^p to L^p with $1 < p < \infty$, then we say that T is Calderón-Zygmund operator.

Let Ω be a measurable set in $\mathbb{R}^n$ with $|\Omega| > 0$. We first defined Lebesgue spaces with variable exponent.

Definition 1.2. [4] Let $p(\cdot): \Omega \to [1, \infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ f \text{ is measurable} : \int_\Omega \left(\frac{|f(x)|}{\eta} \right)^{p(x)} \mathrm{d}x < \infty \text{ for some constant } \eta > 0 \right\}. \tag{1.5}$$

The space $L_{Loc}^{p(\cdot)}(\Omega)$ is defined by

$$L_{Loc}^{p(\cdot)}(\Omega) = \left\{ f \text{ is measurable} : f \in L^{p(\cdot)}(K) \text{ for all compact } K \subset \Omega \right\}.$$

The Lebesgue space $L^{p(\cdot)}(\Omega)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \eta > 0 : \int_\Omega \left(\frac{|f(x)|}{\eta} \right)^{p(x)} \mathrm{d}x \le 1 \right\}. \tag{1.6}$$

We denote

$$p_- = \operatorname{essinf}\{p(x) : x \in \Omega\}, \quad p_+ = \operatorname{esssup}\{p(x) : x \in \Omega\}.$$

Then $\mathcal{P}(\Omega)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Let M be the Hardy-Littlewood maximal operator. We denote $\mathfrak{B}(\Omega)$ to be the set of all function $p(\cdot) \in \mathcal{P}(\Omega)$ satisfying that M is bounded on $L^{p(\cdot)}(\Omega)$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}, C_k = B_k \setminus B_{k-1}, \chi_k = \chi_{B_k}, k \in \mathbb{Z}$.

Proposition 1.1. See [1]. If $q(\cdot) \in \mathcal{P}(\Omega)$ satisfies

$$|q(x) - q(y)| \le \frac{-A}{\operatorname{Log}(|x-y|)}, |x-y| \le 1/2, \tag{1.7}$$

$$|q(x) - q(y)| \le \frac{A}{\operatorname{Log}(e+|x|)}, |y| \ge |x|, \tag{1.8}$$

then, we have $q(\cdot)\in\mathfrak{B}(\Omega)$.

Proposition 1.2. [6] Suppose that $q_1(\cdot),q_2(\cdot)\in\mathcal{P}(\mathbb{R}^n),0<\gamma<n/(q_1)_+$, if $\gamma/n=1/q_1(\cdot)-1/q_2(\cdot)$ then

$$\left\|\chi_{B_k}\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\le C2^{-k\gamma}\left\|\chi_B\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}, \tag{1.9}$$

for all balls $B_k=\{x\in\mathbb{R}^n:|x|\le 2^k\}$ with $k\in\mathbb{Z}$.

Definition 1.3. [7] Let $\alpha\in\mathbb{R}$, $0<p_1\le\infty$ and $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)=\left\{f\in L^{q(\cdot)}_{loc}(\mathbb{R}^n):\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}\mathbb{R}^n}<\infty\right\}, \tag{1.10}$$

where

$$\|f\|_{K^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)}=\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\|f\chi_k\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right\}^{\frac{1}{p_1}}. \tag{1.11}$$

The non-homogeneous Herz space with variable exponent $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)=\left\{f\in L^{q(\cdot)}_{loc}(\mathbb{R}^n):\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}<\infty\right\} \tag{1.12}$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}=\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\|f\tilde{\chi}_k\|^{p}_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{\frac{1}{p}}. \tag{1.13}$$

Definition 1.4. [1] Let $\alpha\in\mathbb{R}$, $0<p\le\infty$ and $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $N>n+1$. Suppose that $G_N f(x)$ is maximal function of *f*. Homogeneous variable exponent Herz-tybe Hardy spaces $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)=\left\{f\in S'(\mathbb{R}^n):G_N f(x)\in\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\right\}, \tag{1.14}$$

with norm

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}=\left\|G_N f(x)\right\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \tag{1.15}$$

Definition 1.5. [1] Let $n\delta_2\le\alpha<\infty,q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $(0<\delta_2<1)$, and non negative integer $s\ge[\alpha-n\delta_2]$. A function *g* on $\mathbb{R}^n$ is said to be a central $(\alpha,q(\cdot))$, if satisfies

1) $\operatorname{supp} g\subset B(0,r)=\{x\in\mathbb{R}^n:|x|<r\}$;

2) $\|g\|_{q(\cdot)(\mathbb{R}^n)}\le|B(0,r)|^{\frac{-\alpha}{n}}$;

3) $\int_{\mathbb{R}^n}g(x)x^{\ell}\mathrm{d}x=0,|\ell|\le s$.

What's more, when $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$,

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\approx inf\left\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right\}^{\frac{1}{p}}. \tag{1.16}$$

Definition 1.6. [7] $1<\gamma\le 0$ the Lipschiz space is defined by

$$Lip_\gamma(\mathbb{R}^n)=\left\{f:\|f\|_{Lip_\gamma}=\sup_{x,y\in\mathbb{R}^n;x\ne y}\frac{|f(x)-f(y)|}{|x-y|^\gamma}<\infty\right\}. \tag{1.17}$$

Definition 1.7. For $b\in L^1_{loc}(\mathbb{R}^n)$, the bounded mean oscillation space $BMO(\mathbb{R}^n)$ is defined by

$$\|b\|_{BMO(\mathbb{R}^n)}=\sup_{B:balls\in(\mathbb{R}^n)}\int_B\frac{1}{|B|}|b(x)-b_B|\mathrm{d}x.$$

2. Main Result and Proof

In order to prove result, we need recall some lemma.

Lemma 2.1. ([3]) Let $b \in lip_\gamma(0<\gamma<1)$, T be Calderón-Zygmund operator, $q_1(\cdot), q_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\frac{1}{q_1(\cdot)} - \frac{1}{q_2(\cdot)} = \frac{\beta}{n}$ Then,

$$\|[b,T]\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{lip_\gamma}\|f\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}. \tag{2.1}$$

Lemma 2.2. ([8]) Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$; if $f \in L^{q(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q'(\cdot)}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \le r_q\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \tag{2.2}$$

where $r_q = 1 + \frac{1}{q_-} - \frac{1}{q_+}$.

Lemma 2.3. ([2]) Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then for all ball B in $\mathbb{R}^n$,

$$|B|^{-1}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C. \tag{2.3}$$

Lemma 2.4. ([2]) Let $q_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ then for all measurable subsets $S \subset B$, and all ball B in $\mathbb{R}^n$

$$\frac{\|\chi_S\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2} \tag{2.4}$$

where δ_1, δ_2 are constants with $0<\delta_1,\delta_2<1$.

Lemma 2.5. ([4]) Let $b \in BMO(\mathbb{R}^n)$, and $i, j \in \mathbb{Z}$ with $i<j$ then

$$C^{-1}\|b\|_{BMO(\mathbb{R}^n)} \le \sup_B \frac{1}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\|(b-b_B)\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{BMO(\mathbb{R}^n)}$$

$$\left\|\left(b-b_{B_i}\right)\chi_{B_j}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C(j-i)\|b\|_{BMO(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.6. ([9]) Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n), b \in BMO$ function and T be a Calderón-Zygmund operator. Then

$$\|[b,T]f\|_{q(\cdot)}(\mathbb{R}^n) \le C\|b\|_{BMO(\mathbb{R}^n)}\|f\|_{q(\cdot)}(\mathbb{R}^n).$$

Theorem 2.1. Let $q_1(\cdot)$, $q_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $b \in Lip_\gamma$, $0<p_1<\infty$, $1/q_1(\cdot) - 1/q_2(\cdot) = \gamma/n$ and $-n\delta_1 < \alpha < n\delta_2$ where δ_1, δ_2 are a constants, then $[b,T]$ are bounded from $H\dot{K}^{\alpha,p}_{q_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,p}_{q_2(\cdot)}(\mathbb{R}^n)$.

Proof: we suffices to prove homogeneous case. Let $f(x) \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, $f = \sum_{j=-\infty}^{\infty}\lambda_j g_j$ in the $S'(\mathbb{R}^n)$ sense, where each g_j is a central $(\alpha, q(\cdot))$-atom with supp $g_j \subset B_j$. Write

$$\|f\|_{H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)} \approx inf\left\{\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\right\}^{\frac{1}{p_1}}.$$

We have

$$\|([b,T]f)\chi_k\|_{\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)} = \sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\|([b,T]f)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \tag{2.5}$$

$$\begin{aligned}\|([b,T]f)\chi_k\|^{p_1}_{\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)} &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\|([b,T]g_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1} \\ &+ C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k-1}^{k+1}|\lambda_j|\|([b,T]g_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1} \\ &+ C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k+1}^{\infty}|\lambda_j|\|([b,T]g_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1} \\ &= F_1 + F_2 + F_3\end{aligned} \tag{2.6}$$

By virtue of Lemma 2.1, we can easily see that

$$F_2 \le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.$$

First we estimate F_1. For each $j \le k-2$ and we shall get

$$\begin{aligned}
\left|[b,T]g_j\right| &\le \int_{\mathbb{R}^n} \left|K(x,y)(b(x)-b(y))g_j(y)\right| \mathrm{d}y \\
&\le C\int_{\mathbb{R}^n} \frac{\left|(b(x)-b(y))g_j(y)\right|}{|x-y|^n} \mathrm{d}y \\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|x-y|^\gamma |g_j(y)|}{|x-y|^n} \mathrm{d}y \\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|g_j(y)|}{|x-y|^{n-\gamma}} \mathrm{d}y \\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} 2^{-(k-2)(n-\gamma)} \|g_j\|_{L^1(\mathbb{R}^n)} \\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} 2^{-(k-2)(n-\gamma)} \|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}
\end{aligned} \tag{2.7}$$

$$\left\|([b,T]g_j)\chi_k\right\|_{\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{lip_\beta(\mathbb{R}^n)} 2^{-(k-2)(n-\gamma)} \|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}$$

Thus by Lemma 2.3, Lemma 2.4 and Proposition 1.2, we get

$$\begin{aligned}
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} 2^{-k\gamma} 2^{-(k-2)(n-\gamma)} \|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} 2^{2(n-\gamma)} 2^{-nk+k\gamma-k\gamma} \|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \left(|B_k| \|\chi_{B_k}\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\right) \\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} 2^{nk} 2^{-nk} \|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \left(\frac{\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}\right) \\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} 2^{(j-k)\delta_2 n - j\alpha}
\end{aligned} \tag{2.8}$$

When $1 < p_1 < \infty$ and $\alpha < \delta_2 n$, by Hölder's inequality and (2.8), we calculations

$$\begin{aligned}
F_1 &\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \left\|[b,T]g_j\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1} \\
&\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|b\|_{lip_\gamma(\mathbb{R}^n)} 2^{(j-k)\delta_2 n - j\alpha}\right)^{p_1} \\
&\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} 2^{[(j-k)(\delta_2 n-\alpha)]\frac{p_1}{2}}\right) \times \left(\sum_{j=-\infty}^{k-2} 2^{[(j-k)(\delta_2 n-\alpha)]\frac{p_1'}{2}}\right)^{\frac{p_1}{p_1'}} \\
&\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \left(\sum_{k=j+2}^{\infty} 2^{[(j-k)(\delta_2 n-\alpha)]\frac{p_1}{2}}\right) \\
&\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}
\end{aligned} \tag{2.9}$$

where $0 < p_1 \le 1$ by $\alpha < \delta_2 n$, we get

$$
\begin{aligned}
F_1 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{(j-k)\delta_2 n-j\alpha}\right)^{p_1}\\
&\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|2^{(j-k)\delta_2 n-j\alpha}\right)^{p_1}\\
&\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\delta_2 n-\alpha)p_1}\right)\\
&\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\left(\sum_{k=j+2}^{\infty}2^{(j-k)(\delta_2 n-\alpha)p_1}\right)\\
&\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}
\end{aligned}
\tag{2.10}
$$

Now we estimate F_3. For each $k \ge j+2$, we shall get

$$
\begin{aligned}
\left|[b,T]g_j\right| &\le \int_{\mathbb{R}^n}\left|K(x,y)\left(b(x)-b(y)\right)g_j(y)\right|\mathrm{d}y\\
&\le C\int_{\mathbb{R}^n}\frac{\left|\left(b(x)-b(y)\right)g_j(y)\right|}{|x-y|^n}\mathrm{d}y\\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}\int_{R^n}\frac{|x-y|^\gamma|g_j(y)|}{|x-y|^n}\mathrm{d}y\\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|g_j(y)|}{|x-y|^{n-\gamma}}\mathrm{d}y\\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{-(k-2)(n-\gamma)}\|g_j\|_{L^1(\mathbb{R}^n)}\\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{-(j-2)(n-\gamma)}\|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}
\end{aligned}
\tag{2.11}
$$

$$
\left\|\left([b,T]g_j\right)\chi_k\right\|_{\dot K^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{lip_\beta(\mathbb{R}^n)}2^{-(j-2)(n-\gamma)}\|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}
$$

Using the Lemma 2.3 and Lemma 2.4 and Proposition 1.2, we obtain

$$
\begin{aligned}
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{-j\gamma}2^{-(j-2)(n-\gamma)}\|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{2(n-\gamma)}2^{-nj+j\gamma-j\gamma}\|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left(|B_j|\|\chi_{B_j}\|^{-1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)\\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{nj}2^{-nj}\|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left(\frac{\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}\right)\\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{(k-j)\delta_1 n-j\alpha}
\end{aligned}
\tag{2.12}
$$

When $1 < p_1 < \infty$ and $\alpha > -\delta_1 n$, by Hölder's inequality and (2.12), we have

$$\begin{aligned}F_3 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k+2}^{\infty}|\lambda_j|\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\\ &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k+2}^{\infty}|\lambda_j|\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{(k-j)\delta_1 n-j\alpha}\right)^{p_1}\\ &\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}|\lambda_j|^{p_1}2^{[(k-j)(\delta_1 n+\alpha)]\frac{p_1}{2}}\right)\times\left(\sum_{j=k+2}^{\infty}2^{[(k-j)(\delta_1 n+\alpha)]\frac{p_1'}{2}}\right)^{\frac{p_1}{p_1'}}\\ &\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\left(\sum_{k=-\infty}^{j-2}2^{[(k-j)(\delta_1 n+\alpha)]\frac{p_1}{2}}\right),\\ &\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\end{aligned}\tag{2.13}$$

When $1<p_1\le 1$ by $\alpha>-n\delta_1$, we have

$$\begin{aligned}F_3 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k+2}^{\infty}|\lambda_j|\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k+2}^{\infty}|\lambda_j|\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{(k-j)\delta_1 n-j\alpha}\right)^{p_1}\\ &\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}|\lambda_j|^{p_1}2^{[(k-j)(\delta_1 n+\alpha)]p_1}\right)\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\left(\sum_{k=-\infty}^{j-2}2^{[(k-j)(\delta_1 n+\alpha)]p_1}\right)\\ &\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\end{aligned}\tag{2.14}$$

Combining (2.10)-(2.14), we get

$$\left\|[b,T]f\right\|^{p_1}_{\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)}\le C\|f\|^{p_1}_{H\dot{K}^{\alpha,p}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

Theorem 2.2. Let $q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $b\in BMO(\mathbb{R}^n)$, $0<p<\infty$, and $-n\delta_1<\alpha<n\delta_2$ where $\delta_1,\delta_2>0$ are a constants, then $[b,T]$ are bounded from $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Proof: we suffices to prove homogeneous case. Let $f(x)\in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, $f=\sum_{j=-\infty}^{\infty}\lambda_j g_j$ in the $S'(\mathbb{R}^n)$ sense, where each g_j is a central $(\alpha,q(\cdot))$-atom with supp $g_j\subset B_j$. Write

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\approx inf\left\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right\}^{\frac{1}{p}}.$$

We have

$$\left\|\left([b,T]f\right)\chi_k\right\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}=\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left\|\left([b,T]f\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

By inequality (2.5)we have

$$\begin{aligned}\left\|\left([b,T]f\right)\chi_k\right\|^p_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\\ &\quad+C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{k+1}|\lambda_j|\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\\ &\quad+C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+1}^{\infty}|\lambda_j|\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\\ &=F_1+F_2+F_3\end{aligned}$$

Firstly we estimate F_2 by Lemma 2.6 we can see

$$F_2 \le C\|b\|^p_{BMO(\mathbb{R}^n)} \sum_{j=-\infty}^{\infty} |\lambda_j|^p$$

Now we consider the estimates of F_1. Note that for each $x \in A_k$, $y \in A_j$, and $j \le k-2$, by generalized Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned}\left|[b,T]g_j\right| &\le \int_{A_j} \left|K(x,y)(b(x)-b(y))g_j(y)\right| \mathrm{d}y \le C\int_{A_j} \frac{\left|(b(x)-b(y))g_j(y)\right|}{|x-y|^n}\mathrm{d}y \\ &\le C2^{-nk}\left|b(x)-b_{B_j}\right|\int_{A_j}\left|g_j(y)\right|\mathrm{d}y + \int_{A_j}\left|b_{B_j}-b(y)\right|\left|g_j(y)\right|\mathrm{d}y \\ &\le C2^{-nk}\left\|g_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left[\left|b(x)-b_{B_j}\right|\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} + \left\|\left(b_{B_j}-b\right)\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right]\end{aligned}$$

Thus by Lemma 2.5 we get

$$\begin{aligned}&\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C2^{-nk}\left\|g_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left[\left\|\left(b-b_{B_j}\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} + \left\|\left(b_{B_j}-b\right)\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left\|\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right] \\ &\le C2^{-nk}\left\|g_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left[(k-j)\|b\|_{BMO(\mathbb{R}^n)}\left\|\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} + \|b\|_{BMO(\mathbb{R}^n)}\left\|\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right] \\ &\le C(k-j)2^{-nk}\left\|g_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|b\|_{BMO(\mathbb{R}^n)} \times \left\|\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\end{aligned} \tag{2.16}$$

Thus by Lemma 2.3, Lemma 2.4 and noting that $\left\|\chi_i\right\|_{L^{s(\cdot)}(\mathbb{R}^n)} \le \left\|\chi_{B_i}\right\|_{L^{s(\cdot)}(\mathbb{R}^n)}$ we get

$$\begin{aligned}&\le C(k-j)2^{-nk}\left\|g_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|b\|_{BMO(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left(|B_k|\left\|\chi_{B_k}\right\|^{-1}_{L^{q'(\cdot)}(\mathbb{R}^n)}\right) \\ &\le C\|b\|_{BMO(\mathbb{R}^n)}2^{nk}2^{-nk}\left\|g_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\frac{\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B_k}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\right) \\ &\le C(k-j)\|b\|_{BMO(\mathbb{R}^n)}2^{(j-k)n\delta_2-j\alpha}\end{aligned} \tag{2.17}$$

When $1 < p < \infty$ and $\alpha < \delta_2 n$, by Hölder's inequality and (2.17), we calculations

$$\begin{aligned}F_1 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\left\|[b,T]g_j\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\ &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|(k-j)\|b\|_{BMO(\mathbb{R}^n)}2^{(j-k)n\delta_2-j\alpha}\right)^p \\ &\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^p 2^{(k-j)[-n\delta_2+\alpha]\frac{p}{2}}\right)\times\left(\sum_{j=-\infty}^{k-2}(k-j)^{p'}2^{(k-j)[-n\delta_2+\alpha]\frac{p'}{2}}\right)^{\frac{p_1}{p_1'}} \\ &\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^p 2^{(k-j)[-n\delta_2+\alpha]\frac{p}{2}}\right) \\ &\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=j+1}^{\infty}2^{(k-j)[-n\delta_2+\alpha]\frac{p}{2}}\right) \\ &\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}|\lambda_j|^p\end{aligned} \tag{2.18}$$

when $0 < p_1 \le 1$ by $\alpha < \delta_2 n$, we get

$$\begin{aligned} F_1 &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \left\| \left([b,T] g_j \right) \chi_k \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \\ &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| (k-j) \|b\|_{BMO(\mathbb{R}^n)} 2^{(j-k)n\delta_2 - j\alpha} \right)^p \\ &\le C \|b\|^p_{BMO(\mathbb{R}^n)} \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=-\infty}^{k-2} |\lambda_j|^p (k-j) 2^{[(j-k)n\delta_2 - j\alpha]p} \\ &\le C \|b\|^p_{BMO(\mathbb{R}^n)} \sum_{k=-\infty}^{\infty} |\lambda_j|^p \sum_{j=-\infty}^{k-2} (k-j) 2^{(k-j)[-n\delta_2 + \alpha]p} \\ &\le C \|b\|^p_{BMO(\mathbb{R}^n)} \sum_{j=-\infty}^{\infty} |\lambda_j|^p \end{aligned} \tag{2.19}$$

Finally we consider the estimates of F_3. Note that for each $x \in A_k$, $y \in A_j$, and $k \ge j+2$, by generalized Hölder's inequality and Lemma 2.2. we have

$$\begin{aligned} \left| [b,T] g_j \right| &\le \int_{A_j} \left| K(x,y) \left(b(x) - b(y) \right) g_j(y) \right| \mathrm{d}y \\ &\le C \int_{A_j} \frac{\left| \left(b(x) - b(y) \right) g_j(y) \right|}{|x-y|^n} \mathrm{d}y \\ &\le C 2^{-nj} \left| b(x) - b_{B_k} \right| \int_{A_j} |g_j(y)| \mathrm{d}y + \int_{A_j} \left| b_{B_k} - b(y) \right| |g_j(y)| \mathrm{d}y \\ &\le C 2^{-nk} \|g_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left[\left| b(x) - b_{B_k} \right| \left| \chi_j \right|_{L^{q'(\cdot)}(\mathbb{R}^n)} + \left\| \left(b_{B_k} - b \right) \chi_j \right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \right] \end{aligned} \tag{2.20}$$

Thus by Proposition 1.2, and Lemma 2.5, we get

$$\begin{aligned} &\left\| \left([b,T] g_j \right) \chi_k \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C 2^{-nj} \|g_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left[\left\| \left(b - b_{B_k} \right) \chi_k \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)} + \left\| \left(b_{B_k} - b \right) \chi_j \right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right] \\ &\le C 2^{-nj} \|g_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left[\|b\|_{BMO(\mathbb{R}^n)} \|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)} + (j-k) \|b\|_{BMO(\mathbb{R}^n)} \|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \right] \\ &\le C (j-k) 2^{-nj} \|g_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|b\|_{BMO(\mathbb{R}^n)} \times \|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \end{aligned} \tag{2.21}$$

Thus by Lemma 2.3, Lemma 2.4 and noting that $\|\chi_i\|_{L^{s(\cdot)}(\mathbb{R}^n)} \le \|\chi_{B_i}\|_{L^{s(\cdot)}(\mathbb{R}^n)}$ we get

$$\begin{aligned} &\le C (j-k) 2^{-nj} \|g_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|b\|_{BMO(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left(|B_j| \|\chi_{B_j}\|^{-1}_{L^{q(\cdot)}(\mathbb{R}^n)} \right) \\ &\le C (j-k) \|b\|_{BMO(\mathbb{R}^n)} 2^{nj} 2^{-nj} \|g_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left(\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \right) \\ &\le C (j-k) \|b\|_{BMO(\mathbb{R}^n)} 2^{(k-j)n\delta_1 - j\alpha} \end{aligned} \tag{2.22}$$

When $1 < p < \infty$ and $\alpha > -\delta_1 n$, by Hölder's inequality and (2.22),we calculations

$$
\begin{aligned}
F_3 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}|\lambda_j|\left\|[b,T]g_j\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}|\lambda_j|(j-k)\|b\|_{BMO(\mathbb{R}^n)}2^{(k-j)n\delta_1-j\alpha}\right)^p \\
&\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}|\lambda_j|^p 2^{(j-k)[-n\delta_2+\alpha]\frac{p}{2}}\right)\times\left(\sum_{j=k+2}^{\infty}(j-k)^{p'}2^{(j-k)[-n\delta_2+\alpha]\frac{p'}{2}}\right)^{\frac{p}{p'}} \\
&\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}|\lambda_j|^p 2^{(j-k)[-n\delta_2+\alpha]\frac{p}{2}}\right) \\
&\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=\infty}^{j-2}2^{(j-k)[-n\delta_2+\alpha]\frac{p}{2}}\right) \\
&\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{j=-\infty}^{\infty}|\lambda_j|^p
\end{aligned}
\tag{2.23}
$$

when $0<p\le 1$ by $\alpha>-\delta_1 n$, we get

$$
\begin{aligned}
F_3 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}|\lambda_j|\left\|([b,T]g_j)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}|\lambda_j|\|b\|_{BMO(\mathbb{R}^n)}2^{(k-j)n\delta_1-j\alpha}\right)^p \\
&\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}|\lambda_j|2^{(j-k)[-n\delta_1-\alpha]}\right)^p \\
&\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=\infty}^{j-2}2^{(j-k)[-n\delta_1-\alpha]p}\right) \\
&\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{j=-\infty}^{\infty}|\lambda_j|^p
\end{aligned}
\tag{2.24}
$$

combining (2.14)-(2.24) the prove is completed.

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