Journal of Applied Mathematics and Physics, 2016, 4, 925-929
Published Online May 2016 in SciRes. http://www.scirp.org/journal/jamp
http://dx.doi.org/10.4236/jamp.2016.45101

Scientific Research Publishing

Fixed Points Associated to Power of Normal Completely Positive Maps*

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Received 1 April 2016; accepted 21 May 2016; published 24 May 2016

Abstract

Let φ_{A} be a normal completely positive map with Kraus operators $\mathrm{A}=\{E_k\}_{k=1}^{n}$. An operator X is said to be a fixed point of φ_{A}, if $\varphi_{\mathrm{A}}(X)=X$. Let $B(H)^{\varphi_{\mathrm{A}}}$ be the fixed points set of φ_{A}. In this paper, fixed points of φ_{A}^{j} are considered for $1<j<+\infty$, where φ_{A}^{j} means j-power of φ_{A}. We obtain that $B(H)^{\varphi_{\mathrm{A}}^{2j}}=B(H)^{\varphi_{\mathrm{A}}}+B(H)^{-\varphi_{\mathrm{A}}}$ and $B(H)^{\varphi_{\mathrm{A}}^{2j-1}}=B(H)^{\varphi_{\mathrm{A}}}$ for integral $j>1$ when A is self-adjoint and commutable. Moreover, $B(H)^{\varphi_{\mathrm{A}}^{j}}=B(H)^{\varphi_{\mathrm{A}}}$ holds under certain condition.

Keywords

Fixed Point, Power, Completely Positive Map

1. Introduction

Completely positive maps are founded to be very important in operator algebras and quantum information. Especially recent years, it has a great development since a quantum channel can be represented by a trace preserving completely positive map. Fixed points of completely positive map are useful in theory of quantum error correction and quantum measurement theory and have been studied in several papers from different aspects, many interesting results have been obtained (see [1]-[12]).

For the convenience of description, let H be a separable complex Hilbert space and $B(H)$ be the set of all bounded linear operators on H. Let Φ on $B(H)$ be a contractive $(\|\Phi\|)$ map. As we know, every contractive and normal (or weak * continuous) completely positive map Φ on $B(H)$ is determined by a row contraction on H in the sense that

*Fixed points of completely positive maps.

How to cite this paper: Zhang, H.Y. and Si, H.Y. (2016) Fixed Points Associated to Power of Normal Completely Positive Maps. *Journal of Applied Mathematics and Physics*, **4**, 925-929. http://dx.doi.org/10.4236/jamp.2016.45101

$$\Phi(X)=\sum_{k=1}^{n}E_kXE_k^*,\ \forall X\in B(H)$$

where if $n=+\infty$, the convergence is in the weak * topology (see [13] and [14]) and then denoting $\varphi_{\mathrm{A}}=\Phi$, we call φ_{A} a completely positive map associated with A.

Let $\mathrm{A}=\{E_k\}_{k=1}^{n}$ be an at most countable subset of $B(H)$ with $\sum_{k=1}^{n}E_kE_k^*\le I$, where the series is convergent in the strong operator topology. In this case, A is called a row contraction. Then $\varphi_{\mathrm{A}}(X)=\sum_{k=1}^{n}E_kXE_k^*$ is well defined on $B(H)$ and also a normal completely positive map. Moreover, we denote *j*-power for $1<j<+\infty$ by φ_{A}^j, that is $\varphi_{\mathrm{A}}^j=\varphi_{\mathrm{A}}\circ\varphi_{\mathrm{A}}\circ\cdots\circ\varphi_{\mathrm{A}}$. In addition, For a row contraction $\mathrm{A}=\{E_k\}_{k=1}^{n}$, we say that the operator sequence A is unital if $\sum_{k=1}^{n}E_kE_k^*=I$ is commutative, if $E_kE_j=E_jE_k$ for all $1\le k,j\le n$ is normal, if each E_k is normal and positive, and if every E_k is positive. If A is unital (resp. commutative) then we say that φ_{A} is unital (resp. commutative). Moreover, φ_{A} or A is called trace preserving if $\sum_{k=1}^{n}E_k^*E_k\le I$. For a subset $S\subset B(H)$, we denote the commutant of *S* in $B(H)$ by S'. We say that an $X\in B(H)$ is a fixed point of φ_{A} or a fixed point associated to the row contraction A if $\varphi_{\mathrm{A}}(X)=X$. Let $B(H)^{\varphi_{\mathrm{A}}}$ be the set of fixed points of φ_{A}. Some authors compared the commutant $\{E_k,E_k^*,1\le k\le n\}$ of $\mathrm{A}\cup\mathrm{A}^*$, where $\mathrm{A}^*=\{E_k^*\}_{k=1}^{n}$, and some conditions for which $B(H)^{\varphi_{\mathrm{A}}}=\{E_k,E_k^*,1\le k\le n\}$ are given (as in [1], [10]).

For a trace preserving quantum operation φ_{A}, it was proved that $B(H)^{\varphi_{\mathrm{A}}}=\{E_k,E_k^*,1\le k\le n\}'$ if $\dim H<\infty$ in [1]. And $B(H)^{\varphi_{\mathrm{A}}}=\mathrm{A}'$, if Kraus operators A is a spherical unitary [10]. On the other hand, the authors [12] consider some conditions for a unital and commuting row contraction A to be normal and therefore $B(H)^{\varphi_{\mathrm{A}}}=\mathrm{A}'$ in those cases. Moreover, the fixed points set $B(H)^{\varphi_{\mathrm{A}}}$ of φ_{A} is represented when A is a commuting and trace preserving row contraction [15].

The purpose of this paper is to investigate fixed points of *j*-power of the completely positive map φ_{A}^j for $j\ge1$. It is obtained that $B(H)^{\varphi_{\mathrm{A}}^{2j}}=B(H)^{\varphi_{\mathrm{A}}}+B(H)^{-\varphi_{\mathrm{A}}}$ and $B(H)^{\varphi_{\mathrm{A}}^{2j-1}}=B(H)^{\varphi_{\mathrm{A}}}$ when A is self-adjoint and commutable. Furthermore, $B(H)^{\varphi_{\mathrm{A}}^{j}}=B(H)^{\varphi_{\mathrm{A}}}$ holds under certain condition.

2. Main Results

In this section, let A be a normal and commuting row contraction. To give main results, we begin with some notations and lemmas. Let $Q_{\mathrm{A}}=s.o.-\lim_{j\to\infty}\varphi_{\mathrm{A}}^j(I)$ be the strong operator topology limit of $\{\varphi_{\mathrm{A}}^j(I)\}_{j=1}^{\infty}$.

Lemma 1 ([10]) Let $\mathrm{A}=\{E_k\}_{k=1}^{n}$ be a unital and normal commuting row contractions. Then $B(H)^{\varphi_{\mathrm{A}}}=\mathrm{A}'$.

Lemma 2 ([10]) Let $\mathrm{A}=\{E_k\}_{k=1}^{n}\subset B(H)$ be a commuting row contraction. If $Q_{\mathrm{A}}\ne0$, then there exists a triple $\{K,\Gamma,\{U_k\}_{k=1}^{n}\}$ where *K* is a Hilbert space, Γ is a bounded operator from *K* to *H* and $\{U_k\}_{k=1}^{n}$ is a spherical unitary on *K* satisfying the following properties:

1) $\Gamma\Gamma^*=Q_{\mathrm{A}}$;
2) $E_k\Gamma=\Gamma U_k$ for all *k*;
3) *K* is the smallest reducing subspace for $\{U_k\}_{k=1}^{n}$ containing Γ^*H;
4) The mapping

$$\rho:\{U_k\}_{k=1}^{n}{}'\to B(H)^{\varphi_{\mathrm{A}}}$$

defined by

$$\rho(Y)=\Gamma Y\Gamma^*,\quad Y\in\{U_k\}_{k=1}^{n}$$

is a complete isometry from the commutant of $\{U_k\}_{k=1}^{n}$ onto the space $B(H)^{\varphi_{\mathrm{A}}}$;

5) There exists a *-homomorphism $\pi:C^*\{I_H,B(H)^{\varphi_{\mathrm{A}}}\}\to\{U_k\}_{k=1}^{n}$ such that

$$\pi(\rho(Y))=Y,\ Y\in\{U_k\}_{k=1}^{n}{}'.$$

Lemma 3 ([16]) (Fuglede-Putnam Theorem) Let $A, B, C \in B(H)$, if A and B are normal, then $AC = CB$ implies $A^*C = CB^*$.

In general, there is no concrete relation between $B(H)^{\varphi_{\mathrm{A}}^k}$ and $B(H)^{\varphi_{\mathrm{A}}^j}$ for different positive integers k and j.

Example 4 Let $E = \begin{pmatrix} \frac{1}{2}+\frac{\sqrt{3}}{2}i & 0 \\ 0 & \frac{1}{2}-\frac{\sqrt{3}}{2}i \end{pmatrix}$ and $\mathrm{A} = \{E\}$, then $\varphi_{\mathrm{A}}(X) = EXE^*$ is well defined and $B(H)^{\varphi_{\mathrm{A}}} = \left\{ \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} : a_{11}, a_{22} \in C \right\}$. However, by a direct computation, $\varphi_{\mathrm{A}}^3(X) = X$ and $B(H)^{\varphi_{\mathrm{A}}^3} = M_2(C)$. Hence, $B(H)^{\varphi_{\mathrm{A}}^3} \neq B(H)^{\varphi_{\mathrm{A}}}$.

But if A is self-adjoint and commutable, the following result holds.

Theorem 5 If A is unital, self-adjoint and commutable, then $B(H)^{\varphi_{\mathrm{A}}^{2j-1}} = B(H)^{\varphi_{\mathrm{A}}}$ and $B(H)^{\varphi_{\mathrm{A}}^{2j}} = B(H)^{\varphi_{\mathrm{A}}} + B(H)^{-\varphi_{\mathrm{A}}}$ for any $j \geq 1$.

Proof. For any $j \geq 1$, we first prove $B(H)^{\varphi_{\mathrm{A}}^{2j-1}} = B(H)^{\varphi_{\mathrm{A}}}$. For any $X \in B(H)^{\varphi_{\mathrm{A}}}$, then $\varphi_{\mathrm{A}}(X) = X$. So $\varphi_{\mathrm{A}}^j(X) = \varphi_{\mathrm{A}}^{j-1}(X) = \cdots = X$ for any j. It is only to prove $B(H)^{\varphi_{\mathrm{A}}^{2j-1}} \subset B(H)^{\varphi_{\mathrm{A}}}$. According to A is unital, self-adjoint and commutable, then $\mathrm{A} = \left\{ E_{k_1}E_{k_2}\cdots E_{k_j} : k_i \in \mathrm{A} \text{ for } 1 \leq i \leq j \right\}$ is so. For any operator $A \in B(H)^{\varphi_{\mathrm{A}}^{2j-1}}$, then A and E_k^{2j-1} are commutable for any k by lemma 1. By the function calculus, A and E_k are commutable since E_k is self-adjoint, and so $A \in B(H)^{\varphi_{\mathrm{A}}}$. Therefore,

$$B(H)^{\varphi_{\mathrm{A}}^{2j-1}} = B(H)^{\varphi_{\mathrm{A}}}.$$

Next, we prove $B(H)^{\varphi_{\mathrm{A}}^{2j}} = B(H)^{\varphi_{\mathrm{A}}} + B(H)^{-\varphi_{\mathrm{A}}}$. For any $A \in B(H)^{\varphi_{\mathrm{A}}^{2j}}$, then $AE_kE_{k_1}^2\cdots E_{k_{j-1}}^2E_l = E_kE_{k_1}^2\cdots E_{k_{j-1}}^2E_lA$, for any $k, k_1, k_2, \cdots, k_{j-1}, l \in \{1, 2, \cdots, n\}$.

So $AE_kE_l = E_kE_lA$ since $\sum_{k=1}^n E_k^2 = I$, thus $A \in B(H)^{\varphi_{\mathrm{A}}^2}$. It follows that $\frac{1}{2}(A + \varphi_{\mathrm{A}}(A)) \in B(H)^{\varphi_{\mathrm{A}}}$ and $\frac{1}{2}(A - \varphi_{\mathrm{A}}(A)) \in B(H)^{-\varphi_{\mathrm{A}}}$. So $B(H)^{\varphi_{\mathrm{A}}^{2j}} \subset B(H)^{\varphi_{\mathrm{A}}} + B(H)^{-\varphi_{\mathrm{A}}}$. Conversely, for any $X \in B(H)^{\varphi_{\mathrm{A}}}$, $Y \in B(H)^{-\varphi_{\mathrm{A}}}$, then $\varphi_{\mathrm{A}}(Y) = -Y$. So $\varphi_{\mathrm{A}}^{2j}(Y) = \varphi_{\mathrm{A}}^{2j-1}(-Y) = \varphi_{\mathrm{A}}^{2j-2}(Y) = \cdots = Y$ for any j. Therefore $\varphi_{\mathrm{A}}^{2j}(X+Y) = \varphi_{\mathrm{A}}^{2j-1}(X-Y) = \varphi_{\mathrm{A}}^{2j-2}(X+Y) = \cdots = Y$. So $B(H)^{\varphi_{\mathrm{A}}} + B(H)^{-\varphi_{\mathrm{A}}} \subset B(H)^{\varphi_{\mathrm{A}}^{2j}}$ and $B(H)^{\varphi_{\mathrm{A}}} + B(H)^{-\varphi_{\mathrm{A}}} = B(H)^{\varphi_{\mathrm{A}}^{2j}}$. Therefore, the result holds. This completes the proof.

Corollary 6 Let $\mathrm{A} = \{E_k\}_{k=1}^n$ be unital, self-adjoint and commutable, then

$$\mathrm{A} \cdot \mathrm{A} = \mathrm{A}' + Iw(\mathrm{A}, -\mathrm{A}),$$

where $Iw(\mathrm{A}, -\mathrm{A}) = \{X \in B(H) : E_kX + XE_k = 0, \text{ for any } k\}$.

Proof. From Theorem 5 and Lemma 1, it is only to prove that $B(H)^{-\varphi_{\mathrm{A}}} = Iw(\mathrm{A}, -\mathrm{A})$. Let $F_k = \begin{pmatrix} E_k & 0 \\ 0 & -E_k \end{pmatrix}$ and $\mathrm{B} = \{F_k\}_{k=1}^n$, for any operator $X \in B(H)^{-\varphi_{\mathrm{A}}}$, then $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \in B(H \oplus H)^{\varphi_{\mathrm{B}}}$.

From Lemma 1, we have

$$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E_k & 0 \\ 0 & -E_k \end{pmatrix} = \begin{pmatrix} E_k & 0 \\ 0 & -E_k \end{pmatrix}\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.$$

It follows that $E_k X + XE_k = 0$ for any k and then $X \in Iw(\mathrm{A}, -\mathrm{A})$. This completes the proof.

Theorem 7 Let $\mathrm{A} = \{E_k\}_{k=1}^n$ be unital and commutable. Supposing that there is an k_0 such that $E_{k_0} \in \mathrm{A}$ is positive and invertible, then $B(H)^{\varphi_{\mathrm{A}}^j} = B(H)^{\varphi_{\mathrm{A}}}$, where $1 \le j < \infty$.

Proof. From Lemma 2, there exists a triple $\{K, \Gamma, \{U_k\}_{k=1}^n\}$ where K is a Hilbert space, $\Gamma : K \to H$ is aboounded operator and $\{U_k\}_{k=1}^n$ is a normal unital and commuting operator sequence on K having the properties 1) $\Gamma\Gamma^* = I$; 2) $E_k\Gamma = \Gamma U_k$; 3) K is the smallest reducing subspace for $\{U_k\}_{k=1}^n$ containing $\Gamma^* H$; 4) The mapping $\rho : \{U_k\}_{k=1}^{n\,\prime} \to B(H)^{\varphi_{\mathrm{A}}}$ defined by $\rho(Y) = \Gamma Y \Gamma^*$ and $Y \in \{U_k\}_{k=1}^{n\,\prime}$ is a complete isometry from the commutant of $\{U_k\}_{k=1}^n$ onto the space $B(H)^{\varphi_{\mathrm{A}}}$; also it is obtained that $U_k = \pi(E_k)$ for any k, where $\pi : C^*\{I_H, B(H)^{\varphi_{\mathrm{A}}}\} \to B(K)$ is a unital *-homomorphism. Then $U_{k_0} = \pi(E_{k_0})$ is positive and invertible since E_{k_0} is positive and invertible. Next, we write $\Lambda = \{\{k_1, k_2, \cdots, k_j\} : 1 \le k_1, k_2, \cdots, k_j \le n\}$ and $U_\alpha = U_{k_1} U_{k_2} \cdots U_{k_j}$ for any $\alpha = \{k_1, k_2, \cdots, k_j\} \in \Lambda$. In fact, $U_\alpha A = AU_\alpha$ if and only if $U_k A = AU_k$ for any $1 \le k \le n$. On one hand, $U_\alpha A = AU_\alpha$ if $U_k A = AU_k$; on the other hand, if $U_\alpha A = AU_\alpha$, then $U_{k_0}^j A = AU_{k_0}^j$ and so $U_{k_0} A = AU_{k_0}$ by the function calculus. Moreover, $AU_{k_0}^{j-1} U_k = U_{k_0}^{j-1} U_k A = U_{k_0}^{j-1} AU_k$, thus $U_k A = AU_k$ since $U_{k_0}^{j-1}$ is invertible. That is to say, $\{U_\alpha\}_{\alpha\in\Lambda}$ and $\{U_k\}_{k=1}^n$ have the same reducing subspace. It follows that K is also the smallest reducing subspace for the unital, normal and commuting operator sequence $\{U_\alpha\}_{\alpha\in\Lambda}$ containing $\Gamma^* H$. Thus $\rho(Y) = \Gamma Y \Gamma^*, Y \in \{U_\alpha\}_{\alpha\in\Lambda}'$ is also a complete isometry from the commutant of $\{U_\alpha\}_{\alpha\in\Lambda}$ onto the space $(H)^{\varphi_{\mathrm{A}}^j}$. Combining with $B(H)^{\varphi_{\mathrm{A}}} \subset B(H)^{\varphi_{\mathrm{A}}^j}$, it is easy to get $B(H)^{\varphi_{\mathrm{A}}^j} = B(H)^{\varphi_{\mathrm{A}}}$. The proof is completed.

Acknowledgements

This research was supported by the Natural Science Basic Research Plan of Henan Province (No. 14 B110010 and No. 1523000410221).

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