

The Boundedness of Fractional Integral with Variable Kernel on Variable Exponent Herz-Morrey Spaces

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Abstract

In this paper, we study the boundedness of the fractional integral with variable kernel. Under some assumptions, we prove that such kind of operators is bounded from the variable exponent Herz-Morrey spaces to the variable exponent Herz-Morrey spaces.

Keywords

Fractional Integral, Variable Kernel, Variable Exponent, Herz-Morrey Spaces

1. Introduction

Let $0 < \mu < n$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ ($r \geq 1$) is homogenous of degree zero on \mathbb{R}^n , S^{n-1} denotes the unit sphere in \mathbb{R}^n . If

i) For any $x, z \in \mathbb{R}^n$, one has $\Omega(x, \lambda z) = \Omega(x, z)$;

ii) $\|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})} := \sup_{x \in \mathbb{R}^n} \left(\int_{S^{n-1}} |\Omega(x, z')|^r d\sigma(z') \right)^{\frac{1}{r}} < \infty$

The fractional integral operator with variable kernel $T_{\Omega, \mu}$ is defined by

$$T_{\Omega, \mu} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} f(y) dy,$$

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When $\mu \equiv 1$, the above integral takes the Cauchy principal value. At this time $\mu \equiv 0$, $T_{\Omega, \mu}$ is much more close related to the elliptic partial equations of the second order with variable coefficients. Now we need the further assumption for $\Omega(x, z)$. It satisfies

$$\int_{S^{n-1}} \Omega(x, z') d\sigma(z') = 0, \forall x \in \mathbb{R}^n.$$

For $r \geq 1$, we say Kernel function $\Omega(x, z)$ satisfies the L^r -Dini condition if Ω meets the conditions i), ii) and

$$\int_0^1 \frac{\omega_r(\delta)}{\delta} d\delta < \infty$$

where $\omega_r(\delta)$ denotes the integral modulus of continuity of order r of Ω defined by

$$\omega_r(\delta) = \sup_{x \in \mathbb{R}^n, |\rho| < \delta} \left(\int_{S^{n-1}} |\Omega(x, \rho z') - \Omega(x, z')|^r d\sigma(z') \right)^{\frac{1}{r}}$$

where ρ is the a rotation in \mathbb{R}^n

$$|\rho| = \sup_{z' \in S^{n-1}} |\rho z' - z'|$$

when $\Omega \equiv 1$, $T_{\Omega, \mu}$ is the fraction integral operator

$$T_{\Omega, \mu} f(x) = \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\mu}} dy.$$

The corresponding fractional maximal operator with variable kernel is defined by

$$M_{\Omega, \mu} f(x) = \sup_{r>0} \frac{1}{r^{n-\mu}} \int_{|x-y|<r} |\Omega(x, x-y)| dy.$$

We can easily find that when $\Omega \equiv 1$, $M_{\Omega, \mu}$ is just the fractional maximal operator

$$M_{\Omega, \mu} f(x) = \sup_{r>0} \frac{1}{r^{n-\mu}} \int_{|x-y|<r} |f(y)| dy.$$

Especially, in the case $\mu \equiv 0$, the fractional maximal operator reduces the Hardy-Litelewood maximal operator.

Many classical results about the fractional integral operator with variable kernel have been achieved [1]-[4]. In 1971, Muckenhoupt and Wheeden [5] had proved the operator $T_{\Omega, \mu}$ was bounded from L^p to L^q . In 1991, Kováčik and Rákosník [6] introduced variable exponent Lebesgue and Sobolev spaces as a new method for dealing with nonlinear Dirichet boundary value problem. Then, variable problem and differential equation with variable exponent are intensively developed. In last years, more and more researchers have been interested in the theory of the variable exponent function space and its applications. The class of Herz-Morrey spaces with variable exponent is initially defined by the author [7], and the boundedness of vector-valued sub-linear operator and fractional integral on Herz-Morrey spaces with variable exponent was introduced by authors [7] and [8]. We also note that Herz-Morrey spaces with variable exponent are generalization of Morrey-Herz spaces [9] and Herz spaces with variable exponent [10]. Recently, Wang Zijian and Zhu Yueping [11] proved the boundedness of multilinear fractional integral operators on Herz-Morrey spaces with variable exponent.

The main purpose of this paper is to establish the boundedness of the fractional integral with variable kernel from $M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)$ to $M\dot{K}_{q_2, p_2(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)$. Throughout this paper $|E|$ denotes the Lebesgue measure, χ_E means the characteristic function of a measurable set $S \subset \mathbb{R}^n$. C always means a positive constant independent of the main parameters and may change from one occurrence to another.

2. Definition of Function Spaces with Variable Exponent

In this section we define Lebesgue spaces and Herz-Morrey spaces with variable exponent.

Let E be a measurable set in \mathbb{R}^n with $|E| > 0$. We first defined Lebesgue spaces with variable exponent.

Definition 2.1. Let $p(\cdot) : E \rightarrow [1, \infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by

$$L^{p(\cdot)}(E) = \left\{ f \text{ is measurable} : \int_E \left(\frac{|f(x)|}{\eta} \right)^{p(x)} dx < \infty \text{ for some constant } \eta > 0 \right\}$$

The space $L_{Loc}^{p(\cdot)}(E)$ is defined by

$$L_{loc}^{p(\cdot)}(E) = \left\{ f \text{ is measurable} : f \in L^{p(\cdot)}(K) \text{ for all compact } K \subset E \right\}$$

The Lebesgue spaces $L^{p(\cdot)}(E)$ is a Banach spaces with the norm defined by

$$\|f\|_{L^{p(\cdot)}(E)} = \inf \left\{ \eta > 0 : \int_E \left(\frac{|f(x)|}{\eta} \right)^{p(x)} dx \leq 1 \right\}$$

We denote

$$p_- = \operatorname{ess\,inf} \{ p(x) : x \in E \}, \quad p_+ = \operatorname{ess\,sup} \{ p(x) : x \in E \}.$$

Then $\mathcal{P}(E)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Let M be the Hardy-Littlewood maximal operator. We denote $\mathfrak{B}(E)$ to be the set of all function $p(\cdot) \in \mathcal{P}(E)$ satisfying the M is bounded on $L^{p(\cdot)}(E)$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $C_k = B_k \setminus B_{k-1}$, $\chi_k = \chi_{B_k}$, $k \in \mathbb{Z}$

Definition 2.2. Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $0 \leq \lambda < \infty$. The Herz- Morrey spaces with variable exponent $M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$ is defined by

$$M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n) = \left\{ f \in L_{Loc}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} < \infty \right\}$$

$$\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} := \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q} \|f \chi_k\|_{L^{p(\cdot)}}^q \right\}^{1/q}$$

Remark 2.1. (See [6]) Comparing the Homogeneous Herz-Morrey Spaces with variable exponent with the homogeneous Herz spaces with variable exponent, where $\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L_{Loc}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{\alpha kp} \|f \chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} < \infty \right\}$$

Obviously, $\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) = M\dot{K}_{p,q(\cdot)}^{\alpha,0}(\mathbb{R}^n)$

3. Properties of Variable Exponent

In this section we state some properties of variable exponent belonging to the class $\mathfrak{B}(\mathbb{R}^n)$ and $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$.

Proposition 3.1. (See [12]) If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies

$$|p(x) - p(y)| \leq \frac{-C}{\operatorname{Log}(|x - y|)}, \quad |x - y| \leq 1/2$$

$$|p(x) - p(y)| \leq \frac{C}{\operatorname{Log}(e + |x|)}, \quad |y| \geq |x|$$

then, we have $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$.

Proposition 3.2. (see [13]) Suppose that $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. Let $0 < \mu \leq \frac{n}{(p_1)_+}$, and define the variable exponent $p_2(\cdot)$ by: $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\mu}{n}$. Then we have that for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$,

$$\|T_{\Omega, \mu} f\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} = \|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}$$

Now, we need recall some lemmas

Lemma 3.1. (See [14]) Given $p(\cdot): \mathbb{R}^n \rightarrow [1, \infty)$ have that for all function f and g ,

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$

Lemma 3.2. (See [15]) Suppose that $0 < \mu < n$, $r > 1$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ satisfies the L^r -Dini condition. If there exists an $0 < \alpha < 1/2$ such that $|y| < \alpha_0 R$ then

$$\left(\int_{R < |x| < 2R} \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right|^r dx \right)^{\frac{1}{r}} \leq CR^{\left(\frac{n-n+\mu}{r}\right)} \left(\frac{|y|}{R} + \int_{|y|/2R}^{|y|/R} \frac{\omega_r(\delta)}{\delta} d\delta \right)$$

Lemma 3.3. (See [16]) Suppose that $x \in \mathbb{R}^n$, the variable function $\tilde{q}(x)$ is defined by $\frac{1}{p(x)} = \frac{1}{q} + \frac{1}{\tilde{q}(x)}$, then for all measurable function f and g , we have

$$\|f(x)g(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|g(x)\|_{L^q(\mathbb{R}^n)} \|f(x)\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}$$

Lemma 3.4. (See [17]) Suppose that $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ and $0 < p^- \leq p^+ < \infty$.

- 1) For any cube and $|Q| \leq 2^n$, all the $\chi \in Q$, then: $\|\chi_Q\|_{L^{p(\cdot)}} \approx |Q|^{1/p(x)}$
- 2) For any cube and $|Q| \geq 1$, then $\|\chi_Q\|_{L^{p(\cdot)}} \approx |Q|^{1/p_\infty}$ where

$$p_\infty = \lim_{x \rightarrow \infty} p(x)$$

Lemma 3.5. (See [18]) If $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, then there exist constant $C > 0$ such that for all balls B in \mathbb{R}^n and all measurable subset $S \subset B$

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|S|}{|B|} \right)^\delta, \quad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|S|}{|B|} \right)^{\delta_1}$$

such that δ, δ_1 is constants satisfying $0 < \delta, \delta_1 < 1$

Lemma 3.6. (See [14]) If $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, there exist a constant $C > 0$ such that for any balls B in \mathbb{R}^n . we have

$$\frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C$$

4. Main Theorem and Its Proof

In this section we prove the boundedness of fractional integral with variable kernel on variable exponent Herz-Morrey spaces under some conditions.

Theorem A. Suppose that $0 < \mu < n, 0 < \beta \leq 1, \lambda < \alpha < n\delta_1 + \beta, 0 < q_1 \leq q_2 < \infty$. Let $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1}) (r > p_2^+)$, and the integral modulus of continuity $\omega_r(\delta)$ satisfies

$$\int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\beta}} d\delta < \infty$$

And let $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ satisfy $0 < \mu \leq \frac{n}{(p_1)_+}$ and define the variable exponent $p_2(x)$ by

$$\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\mu}{n}, \text{ then we have}$$

$$\|T_{\Omega,\mu} f\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} \leq C \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}.$$

For all $f \in M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$.

Proof If $f \in M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$ arbitrarily, we apply inequality $\left[\sum_{j=1}^{\infty} a_j\right]^{q_1/q_2} \leq \sum_{j=1}^{\infty} a_j^{q_1/q_2}$ $q_1, q_2, \dots \geq 0$

$$\begin{aligned} \|T_{\Omega,\mu} f\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1} &= \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{K=-\infty}^L 2^{k\alpha q_2} \|T_{\Omega,\mu}(f) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_2} \right\}^{q_1/q_2} \\ &\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{K=-\infty}^L 2^{k\alpha q_1} \|T_{\Omega,\mu}(f) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1} \right\} \end{aligned}$$

If we denote

$$f(x) = \sum_{k=-\infty}^{\infty} f(x) \chi_k = \sum_{k=-\infty}^{\infty} f_j(x),$$

Then we have

$$\begin{aligned} \|T_{\Omega,\mu} f\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1} &\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=-\infty}^{\infty} \|T_{\Omega,\mu}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\ &\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=-\infty}^{k-2} \|T_{\Omega,\mu}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\ &\quad + \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k-1}^{\infty} \|T_{\Omega,\mu}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\ &= U_1 + U_2. \end{aligned}$$

Below, we first estimate U_1 using size condition of f_j . Minkowski inequality when $j \leq k - 1$, we get

$$\begin{aligned} \|T_{\Omega,\mu}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &= \left(\int_{C_k} |T_{\Omega,\mu}(f_j)|^{p_2(\cdot)} dx \right)^{1/p_2(\cdot)} \\ |T_{\Omega,\mu}(f_j)| &= \left| \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} f_j(y) dy \right| \end{aligned}$$

Then we have

$$\|T_{\Omega,\mu}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} = \int_{B_j} f_j(y) \left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\| \chi_k \Big\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} dy$$

Since $r > p_2^+$ we define the variable exponent $\frac{1}{p_2(x)} = \frac{1}{r} + \frac{1}{\tilde{p}_2(x)}$ by Lemma 3.3 and we get

$$\begin{aligned} & \left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\| \chi_k \Big\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ & \leq \left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\|_{L^r(\mathbb{R}^n)} \|\chi_k\|_{L^{p_2(x)}(\mathbb{R}^n)} \\ & \leq \left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\|_{L^r(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{\tilde{p}_2(x)}(\mathbb{R}^n)} \end{aligned}$$

According to Lemma 3.4 and the formula $\frac{1}{\tilde{p}_2(x)} = \frac{1}{p_2(x)} - \frac{1}{r}$, then we have

$\|\chi_{B_k}\|_{L^{\tilde{p}_2(x)}(\mathbb{R}^n)} \approx \|\chi_{B_k}\|_{L^{p_1(x)}(\mathbb{R}^n)} |B|^{-\frac{1-\mu}{r}}$. Combining Lemma 3.2, note that $2^{j-k} \leq 2^{(j-k)\beta}$, we get

$$\begin{aligned} \left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\|_{L^r(\mathbb{R}^n)} & \leq CR^{\left(\frac{n}{r}-n+\mu\right)} \left(\frac{|y|}{2^{k-1}} + \int_{|y|/2^k}^{|y|/2^{k-1}} \frac{\omega_r(\delta)}{\delta} d\delta \right) \\ & \leq CR^{\left(\frac{n}{r}-n+\mu\right)} \left(2^{j-k} + 2^{(j-k)\beta} \int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\beta}} d\delta \right) \\ & \leq CR^{\left(\frac{n}{r}-n+\mu\right)} 2^{(j-k)\beta} \left(1 + \int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\beta}} d\delta \right) \\ & = C2^{(k-1)\left(\frac{n}{r}-n+\mu\right)} 2^{(j-k)\beta} \end{aligned}$$

It follows that

$$\|T_{\Omega, \mu}(f_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C2^{-kn+(j-k)\beta} \int_{B_j} f_j(y) dy \|\chi_{B_k}\|_{L^{p_1(x)}(\mathbb{R}^n)}$$

Using Lemma 3.1, Lemma 3.5 and Lemma 3.6, we obtain

$$\begin{aligned} \|T_{\Omega, \mu}(f_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} & \leq C2^{-kn+(j-k)\beta} \|f_j\|_{L^{p_1(x)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_1(x)}(\mathbb{R}^n)} \\ & \leq C2^{(j-k)\beta} \|f_j\|_{L^{p_1(x)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} 2^{-kn} \|\chi_{B_k}\|_{L^{p_1(x)}(\mathbb{R}^n)} \\ & \leq C2^{(j-k)\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}} \\ & \leq C2^{(j-k)\beta} 2^{(j-k)n\delta_1} \|f_j\|_{L^{p_1(x)}(\mathbb{R}^n)} \\ & \leq 2^{(j-k)(\beta+n\delta_1)} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \end{aligned}$$

Hence we have

$$\begin{aligned} U_1 & \leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=-\infty}^{k-2} 2^{\alpha k} 2^{(j-k)(\beta+n\delta_1)} \|f_j\|_{L^{p_1(x)}(\mathbb{R}^n)} \right)^{q_1} \\ & \leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=-\infty}^{k-2} 2^{\alpha j} 2^{(j-k)(\beta+n\delta_1-\alpha)} \|f_j\|_{L^{p_1(x)}(\mathbb{R}^n)} \right)^{q_1} \end{aligned}$$

Remark that $\alpha < n\delta_1 + \beta$. We consider the two cases $1 < q_1 < \infty$ and $0 < q_1 \leq 1$. If $0 < q_1 \leq 1$, then we use the Hölder inequality and obtain

$$\begin{aligned}
U_1 &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=-\infty}^{k-2} 2^{\alpha j q_1} 2^{(j-k)(\beta+n\delta_1-\alpha)q_1/2} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \right) \times \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(\beta+n\delta_1-\alpha)q_1/2} \right)^{q_1/q_1'} \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \sum_{j=-\infty}^{k-2} 2^{\alpha j q_1} 2^{(j-k)(\beta+n\delta_1-\alpha)q_1/2} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=j+2}^L 2^{(j-k)(\beta+n\delta_1-\alpha)q_1/2} \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\leq C \|f\|_{M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}^{q_1}
\end{aligned}$$

If $0 < q_1 \leq 1$, $\alpha < n\delta_1 + \beta$, we get

$$\begin{aligned}
U_1 &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \sum_{j=-\infty}^{k-2} 2^{\alpha j q_1} 2^{(j-k)(\beta+n\delta_1-\alpha)q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=j+2}^L 2^{(j-k)(\beta+n\delta_1-\alpha)q_1} \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\leq C \|f\|_{M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}^{q_1}
\end{aligned}$$

Next we estimate U_2 , by using Proposition 3.2 we have

$$\begin{aligned}
U_2 &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k-1}^{\infty} \|T_{\Omega, \mu}(f_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k-1}^{\infty} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=k-1}^{\infty} 2^{\alpha j} 2^{(k-j)\alpha} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=k-1}^{k+1} 2^{\alpha j} 2^{(k-j)\alpha} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&\quad + C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=k+2}^{\infty} 2^{\alpha j} 2^{(k-j)\alpha} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&= J_1 + J_2
\end{aligned}$$

First we estimate of J_1 , then we have

$$\begin{aligned}
J_1 &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=k-1}^{k+1} 2^{\alpha j} 2^{(k-j)\alpha} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{\alpha k q_1} \|f_k\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\leq C \|f\|_{M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}^{q_1}
\end{aligned}$$

To estimate of J_2 , when $\lambda < \alpha$, we have

$$\begin{aligned} J_2 &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=k+2}^{\infty} 2^{\alpha j} 2^{(k-j)\alpha} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\ &\leq C \sup_{L \in \mathbb{Z}} \sum_{k=-\infty}^L 2^{(K-L)\lambda q_1} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha-\lambda)} 2^{-j\lambda} 2^{j\alpha} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\ &\leq C \sup_{L \in \mathbb{Z}} \sum_{k=-\infty}^L 2^{(K-L)\lambda q_1} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha-\lambda)} 2^{-j\lambda} \left\{ \sum_{n=-\infty}^j 2^{n\alpha} \|f_n\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \right\}^{\frac{1}{q_1}} \right)^{q_1} \\ &\leq C \sup_{L \in \mathbb{Z}} \sum_{k=-\infty}^L 2^{(K-L)\lambda q_1} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha-\lambda)} \right)^{q_1} \|f\|_{M_{q_1, p_1(\cdot)}^{\lambda, \alpha}(\mathbb{R}^n)}^{q_1} \leq C \|f\|_{M_{q_1, p_1(\cdot)}^{\lambda, \alpha}(\mathbb{R}^n)}^{q_1} \end{aligned}$$

Complete prove Theorem A.

Competing Interests

The authors declare that they have no competing interests.

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