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Nonlinear Super Integrable Couplings of a Super Integrable Hierarchy

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Abstract

Nonlinear super integrable couplings of a super integrable hierarchy based upon an enlarged matrix Lie super algebra were constructed. And its super Hamiltonian structures were established by using super trace identity. As its reduction, special cases of this nonlinear super integrable coupling were obtained.

Keywords

Lie Super Algebra, Nonlinear Super Integrable Couplings, A Super Integrable Hierarchy, Super Hamiltonian Structures

1. Introduction

With the development of soliton theory, super integrable systems associated with Lie super algebra have aroused growing attentions by many mathematicians and physicists. It was known that super integrable systems contained the odd variables, which would provide more prolific fields for mathematical researchers and physical ones. Several super integrable systems including super AKNS hierarchy, super KdV hierarchy, super KP hierarchy, etc., have been studied in [1]-[4]. There are some interesting results on the super integrable systems, such as Darboux transformation in [5], super Hamiltonian structures in [6] [7], binary nonlinearization [8] and reciprocal transformation [9] and so on.

The research of integrable couplings of the well known integrable hierarchy has received considerable attention [10]-[12]. A few approaches to construct linear integrable couplings of the classical soliton equation are presented by permutation, enlarging spectral problem, using matrix Lie algebra [13] constructing new loop Lie algebra and creating semi-direct sums of Lie algebra. Recently, You [14] presented a scheme for constructing the nonlinear super integrable couplings for the super integrable hierarchy. Zhang [15] once constructed an integrable hierarchy and discussed Lax representation, Darboux transformation for its constrained flows. Shi [16] constructed the super extension of this hierarchy.

In this paper, we hope to construct nonlinear super integrable couplings of this super integrable hierarchy which was constructed in [16] through enlarging matrix Lie super algebra. We take the Lie algebra $B(0,1)$ as an example to illustrate the approach for extending Lie super algebras. Based on the enlarged Lie super algebra $gl(6,2)$, we work out nonlinear super integrable Hamiltonian couplings of this super integrable hierarchy. Finally, we will reduce the nonlinear super integrable couplings to some special cases.

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2. Enlargement of Lie Super Algebra *B*(0, 1)

Consider the Lie super algebra $B(0, 1)$. Its basis is

$$E_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},E_2=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},E_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},E_4=\begin{pmatrix}0&0&1\\0&0&0\\0&-1&0\end{pmatrix},E_5=\begin{pmatrix}0&0&0\\0&0&1\\1&0&0\end{pmatrix}. \quad (1)$$

where E_1,E_2,E_3 are even element and E_4,E_5 are odd elements. Their non-zero (anti) commutation relations are

$$\begin{aligned}&[E_1,E_2]=-2E_3,[E_1,E_3]=-2E_2,[E_1,E_4]=E_5,[E_1,E_5]=E_4,[E_2,E_3]=-2E_1,[E_2,E_4]=-E_5,[E_2,E_5]=E_4,\\&[E_3,E_4]=E_4,[E_3,E_5]=-E_5,[E_4,E_4]=-(E_1+E_2),[E_4,E_5]=[E_5,E_4]=E_3,[E_5,E_5]=E_1-E_2.\end{aligned} \quad (2)$$

Let us enlarge the Lie super algebra $B(0, 1)$ to the Lie super algebra $gl(6, 2)$ with a basis

$$\begin{aligned}&e_1=\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&0\end{pmatrix},e_2=\begin{pmatrix}0&1&0&0&0\\-1&0&0&0&0\\0&0&0&1&0\\0&0&-1&0&0\\0&0&0&0&0\end{pmatrix},e_3=\begin{pmatrix}1&0&0&0&0\\0&-1&0&0&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&0\end{pmatrix},e_4=\begin{pmatrix}0&0&0&1&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&0\end{pmatrix},\\&e_5=\begin{pmatrix}0&0&0&1&0\\0&0&-1&0&0\\0&0&0&1&0\\0&0&-1&0&0\\0&0&0&0&0\end{pmatrix},e_6=\begin{pmatrix}0&0&1&0&0\\0&0&0&-1&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&0\end{pmatrix},e_7=\begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&-1&0&1&0\end{pmatrix},e_8=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\1&0&-1&0&0\end{pmatrix}.\end{aligned} \quad (3)$$

where e_1,e_2,e_3,e_4,e_5,e_6 are even, and e_7,e_8 are odd.

The generator of Lie super algebra $gl(6, 2)$, $e_i (1\le i\le 8)$ satisfy the following (anti) commutation relations:

$$\begin{aligned}&[e_1,e_2]=-2e_3,[e_1,e_3]=-2e_2,[e_1,e_5]=-2e_6,[e_1,e_6]=-2e_5,[e_1,e_7]=e_8,[e_1,e_8]=e_7,[e_2,e_3]=-2e_1,[e_2,e_4]=2e_6,\\&[e_2,e_6]=-2e_4,[e_2,e_7]=-e_8,[e_2,e_8]=e_7,[e_3,e_4]=2e_5,[e_3,e_5]=2e_4,[e_3,e_7]=e_7,[e_3,e_8]=-e_8,[e_4,e_5]=-2e_6,\\&[e_4,e_6]=-2e_5,[e_5,e_6]=-2e_4,[e_7,e_7]=-e_1-e_2+e_4+e_5,[e_7,e_8]=[e_8,e_7]=e_3-e_6,[e_8,e_8]=e_1-e_2-e_4+e_5,\\&[e_1,e_4]=[e_2,e_5]=[e_3,e_6]=[e_4,e_7]=[e_4,e_8]=[e_5,e_7]=[e_5,e_8]=[e_6,e_7]=[e_6,e_8]=0.\end{aligned} \quad (4)$$

Define a loop super algebra corresponding to the Lie super algebra $gl(6, 2)$, denote by

$$\widetilde{gl}(6,2)=gl(6,2)\otimes\mathbb{C}\left[\lambda,\lambda^{-1}\right]=\{e_i\lambda^m,e_i\in gl(6,2),i=1,2,\cdots,8;m=0,\pm1,\pm2,\cdots. \quad (5)$$

The corresponding (anti)commutative relations are given as

$$[e_i\lambda^m,e_j\lambda^n]=[e_i,e_j]\lambda^{m+n},\forall e_i,e_j\in gl(6,2). \quad (6)$$

3. Nonlinear Super Integrable Couplings of a Super Integrable Hierarchy

If Let us start from an enlarged spectral problem associated with $gl(6, 2)$,

$$\begin{aligned}&\phi_x=U(u,\lambda)\phi,U=e_1(1)+qe_2(0)+re_3(0)+u_1e_5(0)+u_2e_6(0)+\alpha e_7(0)+\beta e_8(0)\\&=\begin{pmatrix}r&\lambda+q&u_2&u_1&\alpha\\\lambda-q&-r&-u_1&-u_2&\beta\\0&0&r+u_2&\lambda+q+u_1&0\\0&0&\lambda-q-u_1&-r-u_2&0\\\beta&-\alpha&-\beta&\alpha&0\end{pmatrix}.\end{aligned} \quad (7)$$

where q,r,u_1,u_2 are even potentials, but α,β are odd ones.

In order to obtain super integrable couplings of super integrable hierarchy, we solve the adjoint representation of (7),

$$V_x=[U,V], \tag{8}$$

with

$$V=Ae_1(0)+Be_2(0)+Ce_3(0)+Ee_4(0)+Fe_5(0)+Ge_6(0)+\rho e_7(0)+\delta e_8(0)$$
$$=\begin{pmatrix} C & A+B & G & E+F & \rho \\ A-B & -C & E-F & -G & \delta \\ 0 & 0 & C+G & A+B+E+F & 0 \\ 0 & 0 & A-B+E-F & -A-E & 0 \\ \delta & -\rho & -\delta & \rho & 0 \end{pmatrix}. \tag{9}$$

where A,B,C,E,F and G are commuting fields, and ρ,δ are anti-commuting fields.

Substituting

$$A=\sum_{m\ge0}A_m\lambda^{-m},B=\sum_{m\ge0}B_m\lambda^{-m},C=\sum_{m\ge0}C_m\lambda^{-m},E=\sum_{m\ge0}E_m\lambda^{-m},$$
$$F=\sum_{m\ge0}F_m\lambda^{-m},G=\sum_{m\ge0}G_m\lambda^{-m},\rho=\sum_{m\ge0}\rho_m\lambda^{-m},\delta=\sum_{m\ge0}\delta_m\lambda^{-m}. \tag{10}$$

into previous equation gives the following recursive formulas

$$\begin{cases} A_{m,x}=2rB_m-2qC_m-\alpha\rho_m+\beta\delta_m, \\ B_{m,x}=2rA_m-2C_{m+1}-\alpha\rho_m-\beta\delta_m, \\ C_{m,x}=2qA_m-2B_{m+1}+\beta\rho_m+\alpha\delta_m, \\ E_{m,x}=2u_2B_m-2u_1C_m+2rF_m+2u_2F_m-2qG_m-2u_1G_m+\alpha\rho_m-\beta\delta_m, \\ F_{m,x}=2u_2A_m+2rE_m+2u_2E_m-2G_{m+1}+\alpha\rho_m+\beta\delta_m, \\ G_{m,x}=2u_1A_m+2qE_m+2u_1E_m-2F_{m+1}-\beta\rho_m-\alpha\delta_m, \\ \rho_{m,x}=-\beta A_m-\beta B_m-\alpha C_m+r\rho_m+q\delta_m+\delta_{m+1}, \\ \delta_{m,x}=-\alpha A_m+\alpha B_m+\beta C_m-q\rho_m+\rho_{m+1}-r\delta_m. \end{cases} \tag{11}$$

From previous equations, we can successively deduce

$$\begin{aligned}
&A_0=1,B_0=C_0=F_0=G_0=\rho_0=\delta_0=0,E_0=\varepsilon=\text{const.},A_1=0,B_1=q,C_1=r,E_1=0,F_1=\varepsilon q+u_1+\varepsilon u_1,\\
&G_1=\varepsilon r+u_2+\varepsilon u_2,\rho_1=\alpha,\delta_1=\beta,A_2=\frac{1}{2}q^2-\frac{1}{2}r^2-\alpha\beta,B_2=-\frac{1}{2}r_x,C_2=-\frac{1}{2}q_x,E_2=\frac{1}{2}\varepsilon q^2-\frac{1}{2}\varepsilon r^2\\
&+(1+\varepsilon)(qu_1-ru_2+\frac{1}{2}u_1{}^2-\frac{1}{2}u_2{}^2)+\alpha\beta,F_2=-\frac{1}{2}\varepsilon r_x-\frac{1}{2}u_{2x}-\frac{1}{2}\varepsilon u_{2x},G_2=-\frac{1}{2}\varepsilon q_x-\frac{1}{2}u_{1x}-\frac{1}{2}\varepsilon u_{1x},\\
&\rho_2=\beta_x,\delta_2=\alpha_x,A_3=\frac{1}{2}q_xr-\frac{1}{2}qr_x-\alpha\alpha_x+\beta\beta_x,B_3=\frac{1}{4}q_{xx}+\frac{1}{2}q^3-\frac{1}{2}qr^2-q\alpha\beta+\frac{1}{2}\alpha\alpha_x+\frac{1}{2}\beta\beta_x,\\
&C_3=\frac{1}{4}r_{xx}+\frac{1}{2}q^2r-\frac{1}{2}r^3-r\alpha\beta-\frac{1}{2}\alpha\beta_x+\frac{1}{2}\alpha_x\beta,E_3=\frac{1}{2}\varepsilon q_xr-\frac{1}{2}\varepsilon qr_x+(1+\varepsilon)(\frac{1}{2}ru_1+\frac{1}{2}u_{1x}u_2\\
&-\frac{1}{2}u_1u_{2x}-\frac{1}{2}r_xu_1+\frac{1}{2}q_xu_2-\frac{1}{2}qu_{2x})+\alpha\alpha_x-\beta\beta_x,F_3=\frac{1}{4}\varepsilon q_{xx}+\frac{1}{2}\varepsilon q^3-\frac{1}{2}\varepsilon qr^2+(1+\varepsilon)(\frac{1}{4}u_{1xx}\\
&-qru_2-ru_1u_2-\frac{1}{2}r^2u_1+\frac{3}{2}qu_1^2+\frac{3}{2}q^2u_1-\frac{1}{2}u_1u_2^2-\frac{1}{2}qu_2^2+\frac{1}{2}u_1^3)+q\alpha\beta-\frac{1}{2}\alpha\alpha_x-\frac{1}{2}\beta\beta_x,\\
&G_3=\frac{1}{4}\varepsilon r_{xx}+\frac{1}{2}\varepsilon q^2r-\frac{1}{2}\varepsilon r^3+(1+\varepsilon)(\frac{1}{4}u_{2xx}+qru_1+qu_1u_2-\frac{3}{2}ru_2^2-\frac{3}{2}r^2u_2+\frac{1}{2}q^2u_2+\frac{1}{2}u_1^2u_2\\
&+\frac{1}{2}ru_1^2-\frac{1}{2}u_2^3)+r\alpha\beta+\frac{1}{2}\alpha\beta_x-\frac{1}{2}\alpha_x\beta,\rho_3=\alpha_{xx}+\frac{1}{2}r_x\alpha+\frac{1}{2}q_x\beta+\frac{1}{2}q^2\alpha-\frac{1}{2}r^2\alpha+r\alpha_x+q\beta_x,\\
&\delta_3=\beta_{xx}-\frac{1}{2}q_x\alpha-\frac{1}{2}r_x\beta+\frac{1}{2}q^2\beta-\frac{1}{2}r^2\beta-q\alpha_x-r\beta_x.
\end{aligned}$$

Equations (11) can be written as

$$\begin{pmatrix} -2B_{m+1}-F_{m+1} \\ 2C_{m+1}+G_{m+1} \\ -B_{m+1}-F_{m+1} \\ C_{m+1}+G_{m+1} \\ \delta_{m+1} \\ -\rho_{m+1} \end{pmatrix} = L \begin{pmatrix} -2B_{m}-F_{m} \\ 2C_{m}+G_{m} \\ -B_{m}-F_{m} \\ C_{m}+G_{m} \\ \delta_{m} \\ -\rho_{m} \end{pmatrix}, \tag{12}$$

where

$$L=\begin{pmatrix} 2q\partial^{-1}r & 2q\partial^{-1}q+\frac{1}{2}\partial & 2u_1\partial^{-1}r+2q\partial^{-1}u_2+2u_1\partial^{-1}u_2 & 2u_1\partial^{-1}q+2q\partial^{-1}u_1+2u_1\partial^{-1}u_1 & -\frac{1}{2}\partial-q\partial^{-1}\beta & \frac{1}{2}\beta-q\partial^{-1}\alpha \\ \frac{1}{2}\partial-2r\partial^{-1}r & -2r\partial^{-1}q & -2r\partial^{-1}u_2-2u_2\partial^{-1}r-2u_2\partial^{-1}u_2 & -2r\partial^{-1}u_1-2u_2\partial^{-1}q-2u_2\partial^{-1}u_1 & r\partial^{-1}\beta-\frac{1}{2}\beta & \frac{1}{2}\alpha+r\partial^{-1}\alpha \\ 0 & 0 & 2q\partial^{-1}r+2q\partial^{-1}u_2+2u_1\partial^{-1}r+2u_1\partial^{-1}u_2 & \frac{1}{2}\partial+2q\partial^{-1}u_1+2q\partial^{-1}q+2u_1\partial^{-1}q+2u_1\partial^{-1}u_1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\partial-2r\partial^{-1}r-2r\partial^{-1}u_2-2u_2\partial^{-1}r-2u_2\partial^{-1}u_2 & -2r\partial^{-1}q-2r\partial^{-1}u_1-2u_2\partial^{-1}q-2u_2\partial^{-1}u_1 & 0 & 0 \\ -\beta-2\beta\partial^{-1}r & \alpha-2\beta\partial^{-1}q & \beta+2\beta\partial^{-1}r & -\alpha+2\beta\partial^{-1}q & \beta\partial^{-1}\beta-q & r-\partial+\beta\partial^{-1}\alpha \\ -\alpha+2\alpha\partial^{-1}r & \beta+2\alpha\partial^{-1}q & \alpha-2\alpha\partial^{-1}r & -\beta-2\alpha\partial^{-1}q & -r-\partial-\alpha\partial^{-1}\beta & q-\alpha\partial^{-1}\alpha \end{pmatrix}. \tag{13}$$

Then, let us consider the spectral problem (7) with the following auxiliary problem

$$\phi_{t_n}=V^{(n)}\phi=\sum_{j=0}^{n}\begin{pmatrix} C_j & A_j+B_j & G_j & E_j+F_j & \rho_j \\ A_j-B_j & -C_j & E_j+F_j & -G_j & \delta_j \\ 0 & 0 & C_j+G_j & A_j+B_j+E_j+F_j & 0 \\ 0 & 0 & A_j-B_j+E_j-F_j & -C_j-G_j & 0 \\ \delta_j & -\rho_j & -\delta_j & \rho_j & 0 \end{pmatrix}\lambda^{n-j}\phi, \tag{14}$$

From the compatible condition $\phi_{x,t_n}=\phi_{t_n,x}$, according to (7) and (14), we get the zero curvature equation

$$U_{t_n}-V_x^{(n)}+[U,V^{(n)}]=0. \tag{15}$$

which gives a nonlinear Lax super integrable hierarchy

$$u_{t_n}=\begin{pmatrix} q \\ r \\ u_1 \\ u_2 \\ \alpha \\ \beta \end{pmatrix}_{t_n}=\begin{pmatrix} -2C_{n+1} \\ -2B_{n+1} \\ -2G_{n+1} \\ -2F_{n+1} \\ \delta_{n+1} \\ \rho_{n+1} \end{pmatrix}. \tag{16}$$

The super integrable hierarchy (16) is a nonlinear super integrable couplings for the integrable hierarchy in [16]

$$\tilde{u}_{t_n}=\begin{pmatrix} q \\ r \\ \alpha \\ \beta \end{pmatrix}_{t_n}=\begin{pmatrix} -2C_{n+1} \\ -2B_{n+1} \\ \delta_{n+1} \\ \rho_{n+1} \end{pmatrix}. \tag{17}$$

4. Super Hamiltonian Structure

A direct calculation reads

$$\begin{aligned} &\mathrm{Str}(U_\lambda,V)=4A+2E,\mathrm{Str}(U_q,V)=-4B-2F,\mathrm{Str}(U_r,V)=4C+2G, \\ &\mathrm{Str}(U_{u_1},V)=-2B-2F,\mathrm{Str}(U_{u_2},V)=2C+2G,\mathrm{Str}(U_\alpha,V)=2\delta,\mathrm{Str}(U_\beta,V)=-2\rho. \end{aligned} \tag{18}$$

Substituting above results into the super trace identity [7]

$$\frac{\delta}{\delta u}\int \mathrm{Str}\left(\frac{\delta U}{\delta \lambda}V\right)\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{Str}\left(\frac{\delta U}{\delta u}V\right), \tag{19}$$

and comparing the coefficients of λ^{-n-1} on both side of (19)

$$\frac{\delta}{\delta u}\int (4A_{n+1}+2E_{n+1})\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{pmatrix}-4B_n-2F_n\\ 4C_n+2G_n\\ -2B_n-2F_n\\ 2C_n+2G_n\\ 2\delta_n\\ -2\rho_n\end{pmatrix}, n\geq 0. \tag{20}$$

From the initial values in (11), we obtain $\gamma=0$. Thus we have

$$\frac{\delta H_n}{\delta u}=\begin{pmatrix}-2B_n-F_n\\ 2C_n+G_n\\ -B_n-F_n\\ C_n+G_n\\ \delta_n\\ -\rho_n\end{pmatrix}, H_n=-\int\frac{2A_{n+1}+E_{n+1}}{n+1}\mathrm{d}x, n\geq 0. \tag{21}$$

It then follows that the nonlinear super integrable couplings (16) possess the following super Hamiltonian form

$$u_{t_n}=K_n(u)=J\frac{\delta H_n}{\delta u}. \tag{22}$$

where

$$J=\begin{pmatrix}0&-2&0&2&0&0\\ 2&0&-2&0&0&0\\ 0&2&0&-4&0&0\\ -2&0&4&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&-1\end{pmatrix}. \tag{23}$$

is a super Hamiltonian operator and $H_n(n\geq 0)$ are Hamiltonian functions.

5. Reductions

Taking $\alpha=\beta=0,$ (16) reduces to a nonlinear integrable couplings of the integrable hierarchy in [15].

When $n=2$ in (16), we obtain the nonlinear super integrable couplings of the second order super integrable equations

$$\begin{cases} q_{t_2}=-\frac{1}{2}r_{xx}-q^2r+r^3+2r\alpha\beta+\alpha\beta_x-\alpha_x\beta,\\ r_{t_2}=-\frac{1}{2}q_{xx}-q^3+qr^2+2q\alpha\beta-\alpha\alpha_x-\beta\beta_x,\\ u_{1,t_2}=-\frac{1}{2}\varepsilon r_{xx}-\varepsilon q^2r+\varepsilon r^3+(\varepsilon+1)(-\frac{1}{2}u_{2xx}-2qru_1-2qu_1u_2+3ru_2^2\\ \qquad +3r^2u_2-ru_1^2-u_1^2u_2-q^2u_2+u_2^3)-2r\alpha\beta-\alpha\beta_x+\alpha_x\beta,\\ u_{2,t_2}=-\frac{1}{2}\varepsilon q_{xx}-\varepsilon q^3+\varepsilon qr^2+(\varepsilon+1)(-\frac{1}{2}u_{1xx}+2qru_2+2ru_1u_2-3qu_1^3\\ \qquad +r^2u_1+qu_2^2+u_1u_2^2-3q^2u_1-u_1^3)-2q\alpha\beta+\alpha\alpha_x+\beta\beta_x,\\ \alpha_{t_2}=\beta_{xx}-\frac{1}{2}q_x\alpha-\frac{1}{2}r_x\beta+\frac{1}{2}q^2\beta-\frac{1}{2}r^2\beta-q\alpha_x-r\beta_x,\\ \beta_{t_2}=\alpha_{xx}+\frac{1}{2}r_x\alpha+\frac{1}{2}q_x\beta+\frac{1}{2}q^2\alpha-\frac{1}{2}r^2\alpha+r\alpha_x+q\beta_x. \end{cases} \tag{24}$$

Let $\varepsilon=0$ in (24), we have

$$\begin{cases} q_{t_2} = -\frac{1}{2}r_{xx} - q^2 r + r^3 + 2r\alpha\beta + \alpha\beta_x - \alpha_x\beta, \\ r_{t_2} = -\frac{1}{2}q_{xx} - q^3 + qr^2 + 2q\alpha\beta - \alpha\alpha_x - \beta\beta_x, \\ u_{1,t_2} = -\frac{1}{2}u_{2xx} - 2qru_1 - 2qu_1u_2 + 3ru_2^2 + 3r^2u_2 - ru_1^2 - u_1^2u_2 - q^2u_2 + u_2^3 - 2r\alpha\beta - \alpha\beta_x + \alpha_x\beta, \\ u_{2,t_2} = -\frac{1}{2}u_{1xx} + 2qru_2 + 2ru_1u_2 - 3qu_1^3 + r^2u_1 + qu_2^2 + u_1u_2^2 - 3q^2u_1 - u_1^3 - 2q\alpha\beta + \alpha\alpha_x + \beta\beta_x, \\ \alpha_{t_2} = \beta_{xx} - \frac{1}{2}q_x\alpha - \frac{1}{2}r_x\beta + \frac{1}{2}q^2\beta - \frac{1}{2}r^2\beta - q\alpha_x - r\beta_x, \\ \beta_{t_2} = \alpha_{xx} + \frac{1}{2}r_x\alpha + \frac{1}{2}q_x\beta + \frac{1}{2}q^2\alpha - \frac{1}{2}r^2\alpha + r\alpha_x + q\beta_x. \end{cases} \tag{25}$$

Especially, taking $\alpha=\beta=0$ in (24), we can obtain the nonlinear integrable couplings of the second order integrable equations

$$\begin{cases} q_{t_2} = -\frac{1}{2}r_{xx} - q^2 r + r^3, \\ r_{t_2} = -\frac{1}{2}q_{xx} - q^3 + qr^2, \\ u_{1,t_2} = -\frac{1}{2}\varepsilon r_{xx} - \varepsilon q^2 r + \varepsilon r^3 + (\varepsilon+1)(-\frac{1}{2}u_{2xx} - 2qru_1 - 2qu_1u_2 + 3ru_2^2 + 3r^2u_2 - ru_1^2 - u_1^2u_2 - q^2u_2 + u_2^3). \\ u_{2,t_2} = -\frac{1}{2}\varepsilon q_{xx} - \varepsilon q^3 + \varepsilon qr^2 + (\varepsilon+1)(-\frac{1}{2}u_{1xx} + 2qru_2 + 2ru_1u_2 - 3qu_1^3 + r^2u_1 + qu_2^2 + u_1u_2^2 - 3q^2u_1 - u_1^3). \end{cases} \tag{26}$$

If setting $\varepsilon=-1, u_1=-q, u_2=-r$ in (24), we obtain the second order super integrable equations of (17)

$$\begin{cases} q_{t_2} = -\frac{1}{2}r_{xx} - q^2 r + r^3 + 2r\alpha\beta + \alpha\beta_x - \alpha_x\beta, \\ r_{t_2} = -\frac{1}{2}q_{xx} - q^3 + qr^2 + 2q\alpha\beta - \alpha\alpha_x - \beta\beta_x, \\ \alpha_{t_2} = \beta_{xx} - \frac{1}{2}q_x\alpha - \frac{1}{2}r_x\beta + \frac{1}{2}q^2\beta - \frac{1}{2}r^2\beta - q\alpha_x - r\beta_x, \\ \beta_{t_2} = \alpha_{xx} + \frac{1}{2}r_x\alpha + \frac{1}{2}q_x\beta + \frac{1}{2}q^2\alpha - \frac{1}{2}r^2\alpha + r\alpha_x + q\beta_x. \end{cases} \tag{27}$$

6. Conclusion

In this paper, we introduced an approach for constructing nonlinear integrable couplings of super integrable hierarchy. Zhang [17] once employed two kinds of explicit Lie algebra F and G to obtain the nonlinear integrable couplings of the GJ hierarchy and Yang hierarchy, respectively. It is easy to see that Lie algebra F given in [17] is isomorphic to the Lie algebra span $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ in $gl(6, 2)$. So we can obtain nonlinear integrable couplings of super GJ and Yang hierarchy easily. The method in this paper can be applied to other super integrable systems for constructing their super integrable couplings.

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