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On Two Extension Formulas for Lauricella's Function of the Second Kind of Several Variables

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Abstract

The aim of this research paper is to derive two extension formulas for Lauricella's function of the second kind of several variables with the help of generalized Dixon's theorem on the sum of the series $_3F_2(1)$ obtained by Lavoie *et al.* [1]. Some special cases of these formulas are also deduced.

Keywords

Extension Formulas, Lauricella's Function, Dixon's Theorem, Hypergeometric Functions

1. Introduction

The Lauricella's function $F_B^{(r)}$ is defined and represented as follows [2]

$$F_B^{(r)}(a_1,\cdots,a_r,b_1,\cdots,b_r;c;x_1,\cdots,x_r) = \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(a_1)_{m_1}\cdots(a_r)_{m_r}(b_1)_{m_1}\cdots(b_r)_{m_r}}{(c)_{m_1+\cdots+m_r}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!} \tag{1.1}$$

$$\max\{|x_1|,\cdots,|x_r|\}<1;$$

where $(a)_n$ denotes the Pochhammer's symbol defined by

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$$(a)_n=\begin{cases}1, & \text{if } n=0\\ a(a+1)(a+2)\cdots(a+n-1), & \text{if } n=1,2,3,\cdots\end{cases} \tag{1.2}$$

$$=\frac{\Gamma(a+n)}{\Gamma(a)},\quad a\neq 0,-1,-2,\cdots. \tag{1.3}$$

Also, we note that

$$\Gamma\left(\frac{1}{2}\right)\Gamma(1+a)=2^a\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(1+\frac{1}{2}a\right) \tag{1.4}$$

$$(a)_{2n}=2^{2n}\left(\frac{1}{2}a\right)_n\left(\frac{1}{2}a+\frac{1}{2}\right)_n \tag{1.5}$$

$$\frac{\Gamma(a-n)}{\Gamma(a)}=\frac{(-1)^n}{(1-a)_n} \tag{1.6}$$

$$(2n)!=2^{2n}\left(\frac{1}{2}\right)_n n!. \tag{1.7}$$

The generalized Lauricella's function of several variables is defined as follows [2]

$$F^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}[z_1,\cdots,z_n]$$

$$\equiv F^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}\left(\begin{matrix}\left[(a):\theta',\cdots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\cdots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[(c):\psi',\cdots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\cdots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix} z_1,\cdots,z_n\right) \tag{1.8}$$

$$=\sum_{m_1,\cdots,m_n=0}^{\infty}\Omega(m_1,\cdots,m_n)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!},$$

where

$$\Omega(m_1,\cdots,m_n)=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\cdots\prod_{j=1}^{B^{(n)}}\left(b_j^{(n)}\right)_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}\left(d_j^{(n)}\right)_{m_n\delta_j^{(n)}}} \tag{1.9}$$

the coefficients $\theta_j^{(k)}, j=1,2\cdots,A;\phi_j^{(k)}, j=1,2,\cdots,B^{(k)};\psi_j^{(k)}, j=1,2,\cdots,C;\delta_j^{(k)}, j=1,2,\cdots,D^{(k)}$, for all $k\in\{1,2,\cdots,n\}$ are real and positive; (a) abbreviates the array of A parameters; $a_1,\cdots a_A,\left(b^{(k)}\right)$ abbreviate the array of $B^{(k)}$ parameters $b_j^{(k)}, j=1,2,\cdots,B^{(k)}$ for all $k\in\{1,2,\cdots,n\}$ with similar inter pretations for (c) and $\left(d^{(k)}\right)$ $k\in 1,2,\cdots,n$; *et cetera*. Note that, when the coefficients in Equation (1.8) equal to 1, the generalized Lauricella function (1.8) reduces to the following multivariable extension of the Kamp'e de F'eriet function [2]:

$$F^{p:q_1;\cdots;q_n}_{l:m_1;\cdots;m_n}[z_1,\cdots,z_n]\equiv F^{p:q_1;\cdots;q_n}_{l:m_1;\cdots;m_n}\left(\begin{matrix}(a_p):(b'_{q_1});\cdots;\left(b_{q_n}^{(n)}\right);\\ (c_l):(d'_{m_1});\cdots;\left(d_{m_n}^{(n)}\right);\end{matrix} z_1,\cdots,z_n\right)$$

$$=\sum_{s_1,\cdots,s_n=0}^{\infty}\Omega(s_1,\cdots,s_n)\frac{z_1^{s_1}}{s_1!}\cdots\frac{z_n^{s_n}}{s_n!}, \tag{1.10}$$

where

$$\Omega(s_1,\cdots,s_n)=\frac{\prod_{j=1}^{p}(a_j)_{s_1+\cdots+s_n}\prod_{j=1}^{q_1}(b'_j)_{s_1}\cdots\prod_{j=1}^{q_n}(b_j^{(n)})_{s_n}}{\prod_{j=1}^{l}(c_j)_{s_1+\cdots+s_n}\prod_{j=1}^{m_1}(d'_j)_{s_1}\cdots\prod_{j=1}^{m_n}(d_j^{(n)})_{s_n}}. \tag{1.11}$$

In the theory of hypergeometric series, classical summation theorems such as Dixon, Watson and Whipple for the series ${}_3F_2$, have many generalizations and wide applications; see for example [1] [3]-[6]. In the present investigation, we shall require the following generalization of the classical Dixon's theorem for the series ${}_3F_2(1)$ [1]:

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix}a,b,c & ; \\ 1+a-b+i,1+a-c+i+j; & \end{matrix}1\right]\\
&=\frac{2^{-2c+i+j}\Gamma(1+a-b+i)\Gamma(1+a-c+i+j)\Gamma\left(b-\frac{1}{2}|i|-\frac{1}{2}i\right)\Gamma\left(c-\frac{1}{2}(i+j+|i+j|)\right)}{\Gamma(b)\Gamma(c)\Gamma(1+a-2c+i+j)\Gamma(1+a-b-c+i+j)}\\
&\times\left\{A_{i,j}\frac{\Gamma\left(\frac{1}{2}a-c+\frac{1}{2}+\left[\frac{i+j+1}{2}\right]\right)\Gamma\left(\frac{1}{2}a-b-c+1+i+\left[\frac{j+1}{2}\right]\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b+1+\left[\frac{i}{2}\right]\right)}\right.\\
&\left.+B_{i,j}\frac{\Gamma\left(\frac{1}{2}a-c+1+\left[\frac{i+j}{2}\right]\right)\Gamma\left(\frac{1}{2}a-b-c+\frac{3}{2}+i+\left[\frac{j}{2}\right]\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}+\left[\frac{i+1}{2}\right]\right)}\right\},\\
&\left(\begin{matrix}R(a-2b-2c)>-2-2i-j; \\ i=-3,-2,-1,0,1,2;\ j=0,1,2,3\end{matrix}\right),
\end{aligned} \tag{1.12}$$

where $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x. The coefficients $A_{i,j}$ and $B_{i,j}$ are given respectively in [1]. When $i=j=0$, (1.12) reduces immediately to the classical Dixon's theorem [3], (see also [6])

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix}a,b,c & ; \\ 1+a-b,\ 1+a-c; & \end{matrix}1\right]\\
&=\frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma(1+a-b-c)}\\
&\{R(a-2b-2c)>-2\}.
\end{aligned} \tag{1.13}$$

2. Extension Formulas

In this section, the following two extension formulas for Lauricella's function of the second kind of several variables will be established:

$$
\begin{aligned}
&F_B^{(2r)}\left(a_1-i,\,a_1,\cdots,a_r-i,\,a_r,b_1-i-j,b_1,\cdots,b_r-i-j,b_r;\,c\,;x_1,-x_1,\cdots,x_r,-x_r\right)\\
&=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a_1-i)_{2m_1}\cdots(a_r-i)_{2m_r}(b_1-i-j)_{2m_1}\cdots(b_r-i-j)_{2m_r}\,x_1^{2m_1}\cdots x_r^{2m_r}}{(c)_{2m_1+\cdots+2m_r}(2m_1)!\cdots(2m_r)!}\\
&\times I_1(a_1,b_1,2m_1,i,j)\left\{A_{i,j}^{(1)}\,A_1(a_1,b_1,2m_1,i,j)+B_{i,j}^{(1)}\,B_1(a_1,b_1,2m_1,i,j)\right\}\times\cdots\\
&\times I_r(a_r,b_r,2m_r,i,j)\left\{A_{i,j}^{(r)}\,A_r(a_r,b_r,2m_r,i,j)+B_{i,j}^{(r)}\,B_r(a_r,b_r,2m_r,i,j)\right\}+\cdots\\
&+\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a_1-i)_{2m_1+1}\cdots(a_r-i)_{2m_r+1}(b_1-i-j)_{2m_1+1}\cdots(b_r-i-j)_{2m_r+1}\,x_1^{2m_1+1}\cdots x_r^{2m_r+1}}{(c)_{2m_1+1+\cdots+2m_r+1}(2m_1+1)!\cdots(2m_r+1)!}\\
&\times I_1(a_1,b_1,2m_1+1,i,j)\left\{C_{i,j}^{(1)}\,A_1(a_1,b_1,2m_1+1,i,j)+D_{i,j}^{(1)}\,B_1(a_1,b_1,2m_1+1,i,j)\right\}\times\cdots\\
&\times I_r(a_r,b_r,2m_r+1,i,j)\left\{C_{i,j}^{(r)}\,A_r(a_r,b_r,2m_r+1,i,j)+D_{i,j}^{(r)}\,B_r(a_r,b_r,2m_r+1,i,j)\right\}
\end{aligned}
\tag{2.1}
$$

and

$$
\begin{aligned}
&F_B^{(2r+1)}\left(a,a_1-i,a_1,\cdots,a_r-i,a_r,\,b,b_1-i-j,b_1,\cdots,b_r-i-j,b_r;c\,;x,x_1,-x_1,\cdots,x_r,-x_r\right)\\
&=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_m(a_1-i)_{2m_1}\cdots(a_r-i)_{2m_r}(b)_m(b_1-i-j)_{2m_1}\cdots(b_r-i-j)_{2m_r}\,x^m x_1^{2m_1}\cdots x_r^{2m_r}}{(c)_{m+2m_1+\cdots+2m_r}\,m!(2m_1)!\cdots(2m_r)!}\\
&\times I_1(a_1,b_1,2m_1,i,j)\left\{A_{i,j}^{(1)}A_1(a_1,b_1,2m_1,i,j)+B_{i,j}^{(1)}B_1(a_1,b_1,2m_1,i,j)\right\}\times\cdots\\
&\times I_r(a_r,b_r,2m_r,i,j)\left\{A_{i,j}^{(r)}A_r(a_r,b_r,2m_r,i,j)+B_{i,j}^{(r)}B_r(a_r,b_r,2m_r,i,j)\right\}+\cdots\\
&+\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_m(a_1-i)_{2m_1+1}\cdots(a_r-i)_{2m_r+1}}{(c)_{m+2m_1+1+\cdots+2m_r+1}}\\
&\times\frac{(b)_m(b_1-i-j)_{2m_1+1}\cdots(b_r-i-j)_{2m_r+1}\,x^m x_1^{2m_1+1}\cdots x_r^{2m_r+1}}{m!\,(2m_1+1)!\cdots(2m_r+1)!}\\
&\times I_1(a_1,b_1,2m_1+1,i,j)\left\{C_{i,j}^{(1)}A_1(a_1,b_1,2m_1+1,i,j)+D_{i,j}^{(1)}B_1(a_1,b_1,2m_1+1,i,j)\right\}\times\cdots\\
&\times I_r(a_r,b_r,2m_r+1,i,j)\left\{C_{i,j}^{(r)}A_r(a_r,b_r,2m_r+1,i,j)+D_{i,j}^{(r)}B_r(a_r,b_r,2m_r+1,i,j)\right\}
\end{aligned}
\tag{2.2}
$$

where

$$
\begin{aligned}
I_r(a_r,b_r,m_r,i,j)&=2^{-2b_r+i+j}\,\Gamma(1-m_r-a_r+i)\Gamma(1-m_r-b_r+i+j)\\
&\times\frac{\Gamma\left(a_r-\frac{1}{2}|i|-\frac{1}{2}i\right)\Gamma\left(b_r-\frac{1}{2}\left(i+j+|i+j|\right)\right)}{\Gamma(a_r)\Gamma(b_r)\Gamma(1-m_r-2b_r+i+j)\Gamma(1-m_r-a_r-b_r+i+j)}
\end{aligned}
\tag{2.3}
$$

$$
A_r(a_r,b_r,m_r,i,j)=\frac{\Gamma\left(\frac{1}{2}-\frac{1}{2}m_r-b_r+\left[\frac{i+j+1}{2}\right]\right)\Gamma\left(1-\frac{1}{2}m_r-a_r-b_r+i+\left[\frac{j+1}{2}\right]\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}m_r\right)\Gamma\left(1-\frac{1}{2}m_r-a_r+\left[\frac{i}{2}\right]\right)}
\tag{2.4}
$$

$$B_r\left(a_r,b_r,m_r,i,j\right)=\frac{\Gamma\left(1-\frac{1}{2}m_r-b_r+\left[\frac{i+j}{2}\right]\right)\Gamma\left(\frac{3}{2}-\frac{1}{2}m_r-a_r-b_r+i+\left[\frac{j}{2}\right]\right)}{\Gamma\left(-\frac{1}{2}m_r\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}m_r-a_r+\left[\frac{i+1}{2}\right]\right)} \tag{2.5}$$

for $i=-3,-2,-1,0,1,2;\ j=0,1,2,3;\ r=1,2,3,\cdots$.

The coefficients $A_{i,j}^{(r)}, B_{i,j}^{(r)}, C_{i,j}^{(r)}$ and $D_{i,j}^{(r)}$ can be obtained from the tables of $A_{i,j}$ and $B_{i,j}$ given in [1] by replacing a by $-2m_r$ and $-2m_r-1$ respectively.

Proof of (2.1): Denoting the left hand side of (2.1) by S, expanding $F_B^{(2r)}$ in a power series and using the results [2]:

$$(a)_{m+n}=(a)_m(a+m)_n \tag{2.6}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A(n,m)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}A(n,m-n) \tag{2.7}$$

$$(a)_{m-n}=\frac{(-1)^n(a)_m}{(1-a-m)_n},0\le n\le m \quad \text{and} \quad (m-n)!=\frac{(-1)^n m!}{(-m)_n},\ 0\le n\le m, \tag{2.8}$$

we get

$$S=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a_1-i)_{m_1}\cdots(a_r-i)_{m_r}(b_1-i-j)_{m_1}\cdots(b_r-i-j)_{m_r}x_1^{m_1}\cdots x_r^{m_r}}{(c)_{m_1+\cdots+m_r}\,m_1!\cdots m_r!}$$
$$\times f_1(a_1,b_1,m_1,i,j)\times\cdots\times f_r(a_r,b_r,m_r,i,j) \tag{2.9}$$

where

$$f_r(a_r,b_r,m_r,i,j)={}_3F_2\left[\begin{matrix}-m_r,a_r,b_r & ; \\ 1-m_r-a_r+i,1-m_r-b_r+i+j & ;\end{matrix}1\right]. \tag{2.10}$$

Separating (2.9) into its even and odd terms, we have

$$S=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a_1-i)_{2m_1}\cdots(a_r-i)_{2m_r}(b_1-i-j)_{2m_1}\cdots(b_r-i-j)_{2m_r}x_1^{2m_1}\cdots x_r^{2m_r}}{(c)_{2m_1+\cdots+2m_r}\,2m_1!\cdots 2m_r!}$$
$$\times f_1(a_1,b_1,2m_1,i,j)\times\cdots\times f_r(a_r,b_r,2m_r,i,j)+\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a_1-i)_{2m_1+1}(a_2-i)_{2m_2}\cdots(a_r-i)_{2m_r}}{(c)_{2m_1+1+2m_2+\cdots+2m_r}}$$
$$\times\frac{(b_1-i-j)_{2m_1+1}(b_2-i-j)_{2m_2}\cdots(b_r-i-j)_{2m_r}x_1^{2m_1+1}x^{2m_2}\cdots x_r^{2m_r}}{(2m_1+1)!(2m_2)!\cdots(2m_r)!}\times f_1(a_1,b_1,2m_1+1,i,j)$$
$$\times f_2(a_2,b_2,2m_2,i,j)\times\cdots\times f_r(a_r,b_r,2m_r,i,j)+\cdots+\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a_1-i)_{2m_1}\cdots(a_{r-1}-i)_{2m_{r-1}}(a_r-i)_{2m_r+1}}{(a)_{2m_1+\cdots+2m_{r-1}+2m_r+1}}$$
$$\times\frac{(b_1-i-j)_{2m_1}\cdots(b_{r-1}-i-j)_{2m_{r-1}}(b_r-i-j)_{2m_r+1}x_1^{2m_1}\cdots x_{r-1}^{2m_{r-1}}x_r^{2m_r+1}}{(2m_1)!\cdots(2m_{r-1})!(2m_r+1)!}$$
$$\times f_1(a_1,b_1,2m_1,i,j)\times\cdots\times f_{r-1}(a_{r-1},b_{r-1},2m_{r-1},i,j)\times f_r(a_r,b_r,2m_r+1,i,j)$$
$$+\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a_1-i)_{2m_1+1}\cdots(a_r-i)_{2m_r+1}(b_1-i)_{2m_1+1}(b_r-i)_{2m_r+1}x_1^{2m_1+1}\cdots x_r^{2m_r+1}}{(c)_{2m_1+1+\cdots+2m_r+1}(2m_1+1)!\cdots(2m_r+1)!}$$
$$\times f_1(a_1,b_1,2m_1+1,i,j)\times\cdots\times f_r(a_r,b_r,2m_r+1,i,j) \tag{2.11}$$

Finally, in (2.11) if we use the result (1.12), then we obtain the right hand side of (2.1). This completes the proof of (2.1). The result (2.2) can be proved by the similar manner.

3. Special Cases

1) In (2.1), if we take $i = j = 0$ and use the results (1.3)-(1.7), then after some simplification we obtain the following transformation formula:

$$F_B^{(2r)}\left(a_1, a_1, \cdots, a_r, a_r, b_1, b_1, \cdots, b_r, b_r; c; x_1, -x_1, x_2, -x_2, \cdots, x_r, -x_r\right)$$

$$= F^{0:4;\cdots;4}_{2:1;\cdots;1}\left[\begin{matrix} & : a_1, b_1, \frac{1}{2}(a_1+b_1), \frac{1}{2}(a_1+b_1+1); \\ \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2}: & a_1+b_1 \quad ; \end{matrix}\right. \tag{3.1}$$

$$\left.\begin{matrix} \cdots; a_r, b_r, \frac{1}{2}(a_r+b_r)\frac{1}{2}(a_r+b_r+1); \\ \cdots; \quad a_r+b_r \quad ; \end{matrix} x_1^2, \cdots, x_r^2 \right]$$

which for $r = 1$, reduces immediately to a known result of Bailey [7]

$$F_3\left[a, a, b, b; c; x, -x\right] = {}_4F_3\left[\begin{matrix} a, b, \frac{1}{2}(a+b), \frac{1}{2}(a+b+1); \\ \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2}, a+b \quad ; \end{matrix} x^2\right] \tag{3.2}$$

where F_3 is Appell's function [2].

2) Similarly, in (2.2) if we take $i = j = 0$ and use the results (1.3)-(1.7), then we obtain the following transformation formula:

$$F_B^{(2r+1)}\left(a, a_1, a_1, \cdots, a_r, a_r, b, b_1, b_1, \cdots, b_r, b_r; c; x, x_1, -x_1, \cdots, x_r, -x_r\right)$$

$$= F^{0:2;4;\cdots;4}_{1:0;1;\cdots;1}\left[\begin{matrix} ---- & :(a:1),(b:1);(a_1:1),(b_1:1),\left(\frac{1}{2}(a_1+b_1):1\right),\left(\frac{1}{2}(a_1+b_1+1):1\right); \\ (c:1,2,\cdots,2): & ----\quad ; \quad (a_1+b_1:1) \quad ; \end{matrix}\right. \tag{3.3}$$

$$\left.\begin{matrix} \cdots;(a_r:1),(b_r:1),\left(\frac{1}{2}(a_r+b_r):1\right),\left(\frac{1}{2}(a_r+b_r+1):1\right); \\ \cdots; \quad (a_r+b_r:1) \quad ; \end{matrix} x, 4x_1^2, \cdots, 4x_r^2\right]$$

3) In (2.2) if we take $r = 1$, then we get a known extension formulas [8] for Lauricella's function of three variables $F_B^{(3)}\left(a, a_1 - i, a_1, b, b_1 - i - j, b_1; c; x, x_1, -x_1\right)$ for $\{i = -3, -2, -1, 0, 1, 2;\ j = 0, 1, 2, 3\}$.

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