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On the Nonlinear Difference Equation

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Abstract

In this paper, we investigate some qualitative behavior of the solutions of the difference equation $x_{n+1} = a + \frac{bx_{n-k}}{\sum_{i=0}^{k} c_i x_{n-i}}$, $n = 0,1,2,\cdots$, where the coefficients a, b and c_i are positive real numbers, $i,k \in \{0,1,2,\cdots\}$ and where the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_0$ are arbitrary positive real numbers.

Keywords

Difference Equation, Stability, Periodicity, Boundedness, Global Stability

1. Introduction

Our aim in this paper is to study with some properties of the solutions of the difference equation

$$x_{n+1} = a + \frac{bx_{n-k}}{\sum_{i=0}^{k} c_i x_{n-i}}, \quad n = 0,1,2,\cdots, \tag{1.1}$$

where the coefficients a, b and c_i are positive real numbers, $i,k \in \{0,1,2,\cdots\}$ and where the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_0$ are arbitrary positive real numbers. There is a class of nonlinear difference equations, known as the rational difference equations, each of which consists of the ratio of two polynomials in the sequence terms in the same form. There has been a lot of work concerning the global asymptotics of solutions of rational difference equations [1]-[8].

Many researchers have investigated the behavior of the solution of difference equation. For example:

Amleh *et al.* [9] has studied the global stability, boundedness and the periodic character of solutions of the equation

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$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}.$$

Our aim in this paper is to extend and generalize the work in [9], [10] and [11]. That is, we will investigate the global behavior of (1.1) including the asymptotical stability of equilibrium points, the existence of bounded solution, the existence of period two solution of the recursive sequence of Equation (1).

Now we recall some well-known results, which will be useful in the investigation of (1.1) and which are given in [12].

Let I be an interval of real numbers and let

$$F : I^{k+1} \to I,$$

where F is a continuous function. Consider the difference equation

$$y_{n+1} = F\left(y_n, y_{n-1}, \cdots, y_{n-k}\right), \quad n = 0,1,2,\cdots, \tag{1.2}$$

with the initial condition $y_{-k}, y_{-k+1}, \cdots, y_0 \in I$.

Definition 1. (Equilibrium Point)

A point $\overline{y} \in I$ is called an equilibrium point of Equation (1.2) if

$$\overline{y} = f\left(\overline{y}, \overline{y}, \cdots, \overline{y}\right).$$

That is, $y_n = \overline{y}$ for $n \geq 0$, is a solution of Equation (1.2), or equivalently, $\overline{y}$ is a fixed point of f.

Definition 2. (Stability)

Let $\overline{y} \in (0,\infty)$ be in equilibrium point of Equation (1.2) then

1) An equilibrium point $\overline{y}$ of Equation (1.2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $y_{-k}, y_{-k+1}, \cdots, y_0 \in (0,\infty)$ with $|y_{-k} - \overline{y}| + |y_{-k+1} - \overline{y}| + \cdots + |y_0 - \overline{y}| < \delta$, then $|y_n - \overline{y}| < \varepsilon$ for all $n \geq -k$.
2) An equilibrium point $\overline{y}$ of Equation (1.2) is called locally asymptotically stable if $\overline{y}$ is locally stable and there exists $\gamma > 0$ such that, if $y_{-k}, y_{-k+1}, \cdots, y_0 \in (0,\infty)$ with
$$|y_{-k} - \overline{y}| + |y_{-k+1} - \overline{y}| + \cdots + |y_0 - \overline{y}| < \gamma, \text{ then } \lim_{n\to\infty} y_n = \overline{y}.$$
3) An equilibrium point $\overline{y}$ of Equation (1.2) is called a global attractor if for all $y_{-k}, y_{-k+1}, \cdots, y_0 \in (0,\infty)$ we have
$$\lim_{n\to\infty} y_n = \overline{y}.$$
4) An equilibrium point $\overline{y}$ of Equation (1.2) is called globally asymptotically stable if $\overline{y}$ is locally stable and a global attractor.
5) An equilibrium point $\overline{y}$ of Equation (1.2) is called unstable if $\overline{y}$ is not locally stable.

Definition 3. (Permanence)

Equation (1.2) is called permanent if there exists numbers m and M with $0 < m < M < \infty$ such that for any initial conditions $y_{-k}, y_{-k+1}, \cdots, y_0 \in (0,\infty)$ there exists a positive integer N which depends on the initial conditions such that

$$m \leq y_n \leq M \quad \text{for all } n \geq -k.$$

Definition 4. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

The linearized equation of Equation (1.2) about the equilibrium point $\overline{y}$ is defined by the equation

$$z_{n+1} = \sum_{i=0}^{k} p_i z_{n-i}, \tag{1.3}$$

where

$$p_i = \frac{\partial F(\bar{y}, \bar{y}, \cdots, \bar{y})}{\partial y_{n-i}}, \quad i = 0, 1, \cdots, k.$$

The characteristic equation associated with Equation (1.3) is

$$\lambda^{k+1} - p_0 \lambda^k - p_1 \lambda^{k-1} - \cdots - p_{k-1} \lambda - p_k = 0. \tag{1.4}$$

Theorem 1.1. [13] Let $[p, q]$ be an interval of real numbers and assume that

$$g : [p, q]^{k+1} \to [p, q]$$

is a continuous function satisfying the following properties:

(a) $g(x_1, x_2, \cdots, x_{k+1})$ is non-increasing in the first (k) terms for each x_{k+1} in $[p, q]$ and non-decreasing in the last term for each x_i in $[p, q]$ for all $i = 1, 2, \cdots, k$.

(b) If $(m, M) \in [p, q] \times [p, q]$ is a solution of the system

$$M = g(m, m, m, \cdots, m, M) \quad \text{and} \quad m = g(M, M, M, \cdots, M, m),$$

implies

$$m = M.$$

Theorem 1.2. [12] Assume that F is a C^1-function and let $\bar{y}$ be an equilibrium point of Equation (1.2). Then the following statements are true:

1) If all roots of Equation (1.4) lie in the open unit disk $|\lambda| < 1$, then he equilibrium point $\bar{y}$ is locally asymptotically stable.
2) If at least one root of Equation (1.4) has absolute value greater than one, then the equilibrium point $\bar{y}$ is unstable.
3) If all roots of Equation (1.4) have absolute value greater than one, then the equilibrium point $\bar{y}$ is a source.

Theorem 1.3. [14] Assume that $p_i \in R, i = 1, 2, \cdots, k.$ Then

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotically stable of Equation (1.5)

$$y_{n+k} + p_1 y_{n+k-1} + \cdots + p_k y_n = 0, \ n = 0, 1, \cdots. \tag{1.5}$$

2. Local Stability of Equation (1.1)

In this section we investigate the local stability character of the solutions of Equation (1.1). Equation (1.1) has a unique nonzero equilibrium point

$$\bar{x} = a + \frac{b\bar{x}}{\sum_{i=0}^{k} c_i \bar{x}},$$

$$\bar{x} = a + \frac{b}{\sum_{i=0}^{k} c_i}.$$

Let

$$G = \sum_{i=0}^{k} c_i.$$

Then, we get

$$\bar{x} = a + \frac{b}{G}.$$

Let $f : (0, \infty)^{k+1} \to (0, \infty)$ be a function defined by

$$f\left(u_0,u_1,\cdots,u_k\right)=a+\frac{bu_k}{\sum_{i=0}^{k}c_iu_i}. \tag{2.1}$$

Therefore it follows that

$$\frac{\partial f\left(u_0,u_1,\cdots,u_k\right)}{\partial u_j}=\frac{-bu_kc_j}{\left[\sum_{i=0}^{k}c_iu_i\right]^2},\quad j=0,1,\cdots,k-1,$$

and

$$\frac{\partial f\left(u_0,u_1,\cdots,u_k\right)}{\partial u_k}=b\frac{\sum_{i=0}^{k-1}c_iu_i}{\left[\sum_{i=0}^{k}c_iu_i\right]^2}.$$

Then we see that

$$\frac{\partial f\left(\bar{x},\cdots,\bar{x}_k\right)}{\partial uj}=-b\frac{c_j}{\left(aG+b\right)G}=-P_j,\ \ j=0,1,\cdots,k-1,$$

and

$$\frac{\partial f\left(\bar{x},\cdots,\bar{x}_k\right)}{\partial u_k}=b\frac{G-c_k}{\left(aG+b\right)G}=-P_k.$$

Then the linearized equation of (1.1) about $\bar{x}$ is

$$z_{n+1}=\sum_{i=0}^{k}p_iz_{n-i}. \tag{2.2}$$

Theorem 2.1. Assume that

$$\left(b-aG\right)G<2bc_k.$$

Then the equilibrium point of Equation (1.1) is locally stable.
Proof. It is follows by Theorem (1.3) that, Equation (2.2) is locally stable if

$$\left|p_k\right|+\cdots+\left|p_1\right|+\left|p_0\right|<1.$$

That is

$$\left|\frac{bc_0}{\left(aG+b\right)G}\right|+\left|\frac{bc_1}{\left(aG+b\right)G}\right|+\cdots+\left|\frac{bc_{k-1}}{\left(aG+b\right)G}\right|+\left|\frac{b\left(G-c_k\right)}{\left(aG+b\right)G}\right|<1.$$

This implies that

$$\frac{b\left[\sum_{i=0}^{k-1}c_i\right]}{\left(aG+b\right)G}+\left|\frac{b\left(G-c_k\right)}{\left(aG+b\right)G}\right|<1,$$

then

$$\frac{2b\left(G-c_k\right)}{\left(aG+b\right)G}<1.$$

Thus

$$\left(b-aG\right)G<2bc_k.$$

Hence, the proof is completed.

3. Periodic Solutions

In this section we investigate the periodic character of the positive solutions of Equation (1.1).

Theorem 3.1. Equation (1.1) has positive prime period-two solution only if

$$k-odd \text{ and } (\alpha-\beta)(a\beta-a\alpha+b) > 4a\alpha\beta. \tag{3.1}$$

Proof. Assume that there exists a prime period-two solution

$$\cdots, p, q, p, q, \cdots$$

of (1.1). Let $x_n = q, x_{n+1} = p$. Since $k-odd$, we have $x_{n-k} = p$. Thus, from Equation (1.1), we get

$$p = a + \frac{bp}{c_0 q + c_1 p + c_2 q + \cdots + c_k p},$$

and

$$q = a + \frac{bq}{c_0 p + c_1 q + c_2 p + \cdots + c_k q}.$$

Let

$$c_0 + c_2 + \cdots + c_{k-1} = \alpha,$$

and

$$c_1 + c_3 + \cdots + c_k = \beta.$$

Then

$$p = a + \frac{bp}{\alpha q + \beta p},$$

and

$$q = a + \frac{bq}{\alpha p + \beta q}.$$

Then

$$\alpha pq + \beta p^2 = a\alpha q + a\beta p + bp, \tag{3.2}$$

and

$$\alpha pq + \beta q^2 = a\alpha p + a\beta q + bq. \tag{3.3}$$

Subtracting (3.2) from (3.3) gives

$$\beta\left(p^2 - q^2\right) = (a\beta - a\alpha + b)(p-q).$$

Since $p \neq q$, we have

$$p + q = \frac{a\beta - a\alpha + b}{\beta}. \tag{3.4}$$

Also, since p and q are positive, $(a\beta - a\alpha + b)$ should be positive. Again, adding (3.2) and (3.3) yields

$$2\alpha pq + \beta\left(p^2 + q^2\right) = (a\alpha + a\beta + b)(p+q). \tag{3.5}$$

It follows by (3.4), (3.5) and the relation

$$p^2 + q^2 = (p+q)^2 - 2pq, \quad \forall p, q \in \mathrm{R},$$

that

$$pq = \frac{a\alpha(a\beta - a\alpha + b)}{\beta(\alpha - \beta)}. \tag{3.6}$$

Assume that p and q are two distinct real roots of the quadratic equation

$$\beta t^2 - (a\beta - a\alpha + b)t + \frac{a\alpha(a\beta - a\alpha + b)}{(\alpha - \beta)} = 0,$$

and so

$$(a\beta - a\alpha + b)^2 - 4\beta\frac{a\alpha(a\beta - a\alpha + b)}{(\alpha - \beta)} > 0,$$

which is equivalent to

$$(\alpha - \beta)(a\beta - a\alpha + b) > 4a\alpha\beta.$$

Thus, the proof is completed.

4. Bounded Solution

Our aim in this section we investigate the boundedness of the positive solutions of Equation (1.1).

Theorem 4.1. The solutions $\{x_n\}_{n=-k}^{\infty}$ of Equation (1.1) are bounded.

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Equation (1.1). We see from Equation (1.1) that

$$x_{n+1} = a + \frac{bx_{n-k}}{\sum_{i=0}^{k} c_i x_{n-i}} = a + \frac{bx_{n-k}}{c_0 x_n + c_1 x_{n-1} + \cdots + c_k x_{n-k}}.$$

Then

$$x_n \le a + \frac{b}{c_k} = M \quad \text{for all } n \ge 1. \tag{4.1}$$

On the other hand, we see that the change of variables

$$x_n = \frac{1}{y_n},$$

transforms Equation (1.1) to the following form:

$$\frac{1}{y_{n+1}} = a + \frac{\frac{b}{y_{n-k}}}{\frac{c_0}{y_n} + \frac{c_1}{y_{n-1}} + \cdots + \frac{c_k}{y_{n-k}}}.$$

Hence, we obtain

$$\frac{1}{y_{n+1}} = a + \frac{\frac{b}{y_{n-k}}\prod_{i=0}^{k} y_{n-i}}{c_0\prod_{i=1}^{k} y_{n-i} + c_1\prod_{i=0,i\ne1}^{k} y_{n-i} + \cdots + c_k\prod_{i=0}^{k-1} y_{n-i}}.$$

Thus

$$y_{n+1} = \frac{c_0\prod_{i=1}^{k} y_{n-i} + c_1\prod_{i=0,i\ne1}^{k} y_{n-i} + \cdots + c_k\prod_{i=0}^{k-1} y_{n-i}}{ac_0\prod_{i=1}^{k} y_{n-i} + ac_1\prod_{i=0,i\ne1}^{k} y_{n-i} + \cdots + (ac_k + b)\prod_{i=0}^{k-1} y_{n-i}} = \frac{c_0\prod_{i=1}^{k} y_{n-i}}{ac_0\prod_{i=1}^{k} y_{n-i} + ac_1\prod_{i=0,i\ne1}^{k} y_{n-i} + \cdots + (ac_k + b)\prod_{i=0}^{k-1} y_{n-i}}$$

$$+ \frac{c_1\prod_{i=0,i\ne1}^{k} y_{n-i}}{ac_0\prod_{i=1}^{k} y_{n-i} + ac_1\prod_{i=0,i\ne1}^{k} y_{n-i} + \cdots + (ac_k + b)\prod_{i=0}^{k-1} y_{n-i}} + \cdots + \frac{c_k\prod_{i=0}^{k-1} y_{n-i}}{ac_0\prod_{i=1}^{k} y_{n-i} + ac_1\prod_{i=0,i\ne1}^{k} y_{n-i} + \cdots + (ac_k + b)\prod_{i=0}^{k-1} y_{n-i}},$$

and so,

$$y_{n+1} \le \frac{c_0 \prod_{i=1}^{k} y_{n-i}}{ac_0 \prod_{i=1}^{k} y_{n-i}} + \frac{c_1 \prod_{i=0, i\ne 1}^{k} y_{n-i}}{ac_1 \prod_{i=0, i\ne 1}^{k} y_{n-i}} + \cdots + \frac{c_k \prod_{i=0}^{k-1} y_{n-i}}{(ac_k + b) \prod_{i=0}^{k-1} y_{n-i}}$$

$$= \frac{k}{a} + \frac{c_k}{(ac_k + b)} = \frac{k(ac_k + b) + ac_k}{a(ac_k + b)}.$$

$$y_{n+1} = \frac{1}{x_{n+1}} \le \frac{k(ac_k + b) + ac_k}{a(ac_k + b)}.$$

It follows that

$$\frac{1}{x_{n+1}} \le \frac{k(ac_k + b) + ac_k}{a(ac_k + b)} = E \quad \text{for all } n \ge 1.$$

Thus we obtain

$$x_n = \frac{1}{y_n} \ge \frac{1}{E} = m \quad \text{for all } n \ge 1. \tag{4.2}$$

From (4.1) and (4.2) we see that

$$m \le x_n \le M \quad \text{for all } n \ge 1.$$

Therefore every solution of Equation (1.1) is bounded.

5. Global Stability of Equation (1.1)

Our aim in this section we investigate the global asymptotic stability of Equation (1.1).

Theorem 5.1. If $aG = 2ac_k + b$, then the equilibrium point $\bar{x}$ of Equation (1.1) is global attractor.

Proof. Let $f : (0,\infty)^{k+1} \to (0,\infty)$ be a function defined by

$$f(u_0, u_1, \cdots, u_k) = a + \frac{bu_k}{\sum_{i=0}^{k} c_i u_i},$$

then we can see that the function $f(u_0, u_1, \cdots, u_k)$ is decreasing in the rest of arguments and increasing in u_k.

Suppose that (m, M) is a solution of the system

$$m = f(M, M, M, \cdots, M, m) \quad \text{and} \quad M = f(m, m, m, \cdots, m, M).$$

Then from Equation (2.1), we see that

$$m = a + \frac{bm}{\sum_{i=0}^{k-1} c_i M + c_k m}, \quad M = a + \frac{bM}{\sum_{i=0}^{k-1} c_i m + c_k M},$$

$$m = a + \frac{bm}{(G - c_k)M + c_k m}, \quad M = a + \frac{bM}{(G - c_k)m + c_k M},$$

$$(G - c_k)Mm + c_k m^2 = a(G - c_k)M + ac_k m + bm,$$

$$(G - c_k)Mm + c_k M^2 = a(G - c_k)m + ac_k M + bM,$$

then

$$c_k(m^2 - M^2) = (2ac_k + b - aG)(m - M).$$

Thus

$$m = M.$$

It follows by Theorem (1.1) that $\bar{x}$ is a global attractor of Equation (1.1) and then the proof is complete.

6. Numerical Examples

For confirming the results of this section, we consider numerical examples which represent different types of solution of Equation (1.1).

Example 6.1. Consider the difference equation

$$x_{n+1} = 0.5 + \frac{0.9x_{n-1}}{0.6x_n + 0.5x_{n-1}},$$

where $k = 1, a = 0.5, b = 0.9, \alpha = c_0 = 0.6, \beta = c_1 = 0.5$. **Figure 1** shows that the equilibrium point of Equation (1.1) has locally stable, with initial data $x_{-1} = 2.2, x_0 = 0.3$ (see **Table 1**).

Example 6.2. Consider the difference equation

$$x_{n+1} = 0.125 + \frac{4x_{n-1}}{2x_n + x_{n-1}},$$

where $k - odd = 1, a = 0.125, b = 4, \alpha = c_0 = 2, \beta = c_1 = 1$. **Figure 2**, shows that Equation (1.1) which is periodic with period two. Where the initial data satisfies condition (3.1) of Theorem (3.1) $x_{-1} = 0.1, x_0 = 0.3$ (see **Table 2**).

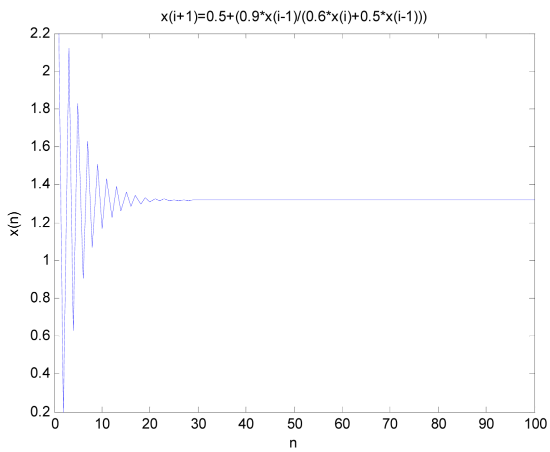

Figure 1. Ref. b1.

Table 1. The equilibrium point of Equation (1.1).

n	x(n)	n	x(n)	n	x(n)	n	x(n)
1	2.0000	16	1.2953	31	1.3187	46	1.3182
2	0.5000	17	1.3359	32	1.3178	47	1.3182
3	1.8846	18	1.3044	33	1.3185	48	1.3182
4	0.8259	19	1.3288	34	1.3179	49	1.3182
5	1.6796	20	1.3099	35	1.3184	50	1.3182
6	1.0232	21	1.3246	36	1.3180	51	1.3182
7	1.5399	22	1.3132	37	1.3183	52	1.3182
8	1.1415	23	1.3220	38	1.3181	53	1.3182
9	1.4526	24	1.3152	39	1.3182	44	1.3182
10	1.2123	25	1.3205	40	1.3181	45	1.3182
11	1.3993	26	1.3164	41	1.3182	46	1.3182
12	1.2547	27	1.3196	42	1.3182	57	1.3182
13	1.3671	28	1.3171	43	1.3182	58	1.3182
14	1.2801	29	1.3190	44	1.3182	59	1.3182
15	1.3476	30	1.3175	45	1.3182	60	1.3182

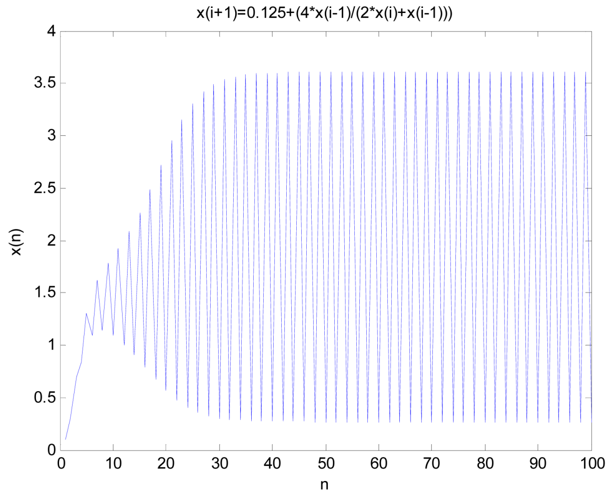

Figure 2. Ref. b4.

Table 2. The initial data satisfies condition (3.1) of Theorem (3.1).

n	$x(n)$	n	$x(n)$	n	$x(n)$	n	$x(n)$	n	$x(n)$
1	0.1000	17	2.4834	33	3.5629	49	3.6057	65	3.6064
2	0.3000	18	0.6734	34	0.2809	50	0.2688	66	0.2686
3	0.6964	19	2.7184	35	3.5803	51	3.6060	67	3.6064
4	0.8339	20	0.5658	36	0.2760	52	0.2687	68	0.2686
5	1.3033	21	2.9493	37	3.5907	53	3.6061	69	3.6064
6	1.0945	22	0.4751	38	0.2730	54	0.2687	70	0.2686
7	1.6178	23	3.1503	39	3.5970	55	3.6062	71	3.6064
8	1.1360	24	0.4055	40	0.2713	56	0.2687	72	0.2686
9	1.7886	25	3.3061	41	3.6008	57	3.6063	73	3.6064
10	1.0891	26	0.3561	42	0.2702	58	0.2686	74	0.2686
11	1.9286	27	3.4160	43	3.6030	59	3.6063	75	3.6064
12	1.0058	28	0.3232	44	0.2696	60	0.2686	76	0.2686
13	2.0829	29	3.4886	45	3.6044	61	3.6063	77	3.6064
14	0.9029	30	0.3021	46	0.2692	62	0.2686	78	0.2686
15	2.2675	31	3.5345	47	3.6052	63	3.6064	79	3.6064
16	0.7892	32	0.2889	48	0.2690	64	0.2686	80	0.2686

Remark 6.1. Note that the special cases of Equation (1.1) have been studied in [9] when $k=1, b=1, c_0=1, c_i=0, i\geq 1$ and in [10] when $k=1, b=1, c_0=1, c_i=0, i\geq 1$ and in [11] when $b=1, c_0=1, c_i=0, i\geq 1$.

References

[1] Elabbasy, E.M., El-Metwally, H. and Elsayed, E.M. (2005) On the Periodic Nature of Some Max-Type Difference Equations. *International Journal of Mathematics and Mathematical Sciences*, **2005**, 2227-2239. http://dx.doi.org/10.1155/IJMMS.2005.2227

[2] Elabbasy, E.M., El-Metwally, H. and Elsayed, E.M. (2006) On the Difference Equation $x_{n+1}=ax_n-\frac{bx_n}{cx_n-dx_{n-1}}$. *Advances in Difference Equations*, 1-10(**2006**), Article ID: 82579.

[3] Elabbasy, E.M., El-Metwally, H. and Elsayed, E.M. (2007) Qualitative Behavior of Higher Order Difference Equation. *Soochow Journal of Mathematics*, **33**, 861-873.

[4] El-Moneam, M.A. and Zayed, E. (2014) Dynamics of the Rational Difference Equation $x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + \frac{bx_n x_{n-k} x_{n-l}}{dx_{n-k} - ex_{n-l}}$. *DCDIS Series A*: *Mathematical Analysis*, **21**, 317-331.

[5] Elaydi, S.N. (1996) An Introduction to Difference Equations, Undergraduate Texts in Mathematics. Springer, New York. http://dx.doi.org/10.1007/978-1-4757-9168-6

[6] Kocic, V.L. and Ladas, G. (1993) Global Behavior of Nonlinear Difference Equations of Higher Order with Applications. Kluwer Academic Publishers, Dordrecht.

[7] Stevic, S. (2005) On the Recursive Sequence $x_{n+1} = \frac{\alpha + \beta x_{n-k}}{f(x_n, \cdots, x_{n-k+1})}$. *Taiwanese Journal of Mathematics*, **9**, 583-593.

[8] Zayed, E. and EI-Moneam, M.A. (2010) On the Rational Recursive Sequence $x_{n+1} = \frac{\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-k}}{\beta_0 x_n + B_1 x_{n-l} + \beta_2 x_{n-k}}$. *Mathematica Bohemica*, **135**, 319-363.

[9] Amleh, A.M., Grove, E.A., Georgiou, D.A. and Ladas, G. (1999) On the Recursive Sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$. *Journal of Mathematical Analysis and Applications*, **233**, 790-798. http://dx.doi.org/10.1006/jmaa.1999.6346

[10] Hamza, A.E. (2006) On the Recursive Sequence $x_{n+1} = \alpha + \frac{x_{n+1}}{x_n}$. *Journal of Mathematical Analysis and Applications*, **322**, 668-674. http://dx.doi.org/10.1016/j.jmaa.2005.09.029

[11] Saleh, M. and Aloqeili, M. (2005) On the Rational Difference Equation $x_{n+1} = A + \frac{x_{n-k}}{x_n}$. *Applied Mathematics and Computation*, **171**, 862-869. http://dx.doi.org/10.1016/j.amc.2005.01.094

[12] Grove, E.A. and Ladas, G. (2005) Periodicities in Nonlinear Difference Equations. Vol. 4, Chapman and Hall/CRC, Boca Raton.

[13] Elabbasy, E.M., El-Metwally, H. and Elsayed, E.M. (2007) On the Difference Equations $x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}$. *Journal of Concrete and Applicable Mathematics*, **5**, 101-113.

[14] Kulenovic, M.R.S. and Ladas, G. (2001) Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures. Chapman & Hall/CRC, Florida. http://dx.doi.org/10.1201/9781420035384