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Existence and Uniqueness of Positive Solution for 2mth-Order Nonlinear Differential Equation with Boundary Conditions

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Abstract

In this paper, we study the existence and uniqueness of positive solution for 2mth-order nonlinear differential equation with boundary conditions, by using the fixed point theorems on compression and expansion of cones.

Keywords

2mth-Order, Uniqueness, Existence, Fixed Point Theorems on Compression and Expansion of Cones

1. Introduction

Recently, many authors studied the existence and multiplicity of positive solutions for the boundary value problem of even-order differential equations since it arose naturally in many different areas of applied mathematics and physics (see [1]-[3]).

In [4] by applying the theory of differential inequalities, the author established the existence of positive solution for the third-order differential equation. In [5], the authors derived the Green function of the 2mth-order nonlinear differential equation, and established the existence of positive solutions for BVP, by using the fixed point theorems on compression and expansion of cones. However, there are a few articles devoted to the uniqueness problem by using the fixed point theorem. In [6], the authors studied the existence and multiplicity of positive periodic solutions for second-order nonlinear damped differential equations by combing the analysis of positiveness of the Green function for a linear damped equation. Our nonlinearity may be singular in its depen-

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dent variable. The proof of the main result relies on the Guo-Krasnosel' skii fixed point theorem on compression and expansion of cones.

In this paper, we consider 2mth-order nonlinear differential equation

$$\begin{cases} (-1)^m y^{(2m)}(x) = f(x, y(x)), & x \in (0,1) \\ y^{(i)}(0) = y^{(i)}(1) = 0, & 0 \le i \le m-1 \end{cases}, \tag{1}$$

The existence and the uniqueness of positive solution are obtained, by means of the fixed point theorems on compression and expansion of cones.

Throughout this paper, we always suppose that

(H_1) $f(x,y):[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous;

(H_2) $f(x,y)\ne 0$, for any compact subinterval in $[0,1]\times[0,+\infty)$, $f(x,y)$ is nonincreasing in $y>0$, a.e. $x\in[0,1]$;

(H_3) $\lim_{y\to 0^+}\max_{x\in[0,1]}\frac{f(x,y)}{y}<\lambda_1$, $\lim_{y\to+\infty}\min_{x\in[0,1]}\frac{f(x,y)}{y}>\lambda_2$;

(H_4) $f(x,y)=a(x)h(y)$, $\lim_{y\to 0^+}\frac{h(y)}{y}>\mu_1$, $\lim_{y\to+\infty}\frac{h(y)}{y}<\mu_2$. where $h\in C([0,+\infty),[0,+\infty))$, $a\in C([0,1],[0,+\infty))$, $a(t)\ne 0$ for any compact subinterval in $[0,1]$. Here λ_1, λ_2, μ_1, μ_2 satisfied $\lambda_1\|\beta\|\int_0^1\gamma(s)\mathrm{d}s\le 1$, $\lambda_2\frac{\alpha_1^{\ 2}}{\|\beta\|}\int_{\sigma_1}^{1-\sigma_1}\gamma(s)\mathrm{d}s\ge 1$, $\mu_1\frac{\alpha_2^{\ 2}}{\|\beta\|}\int_{\sigma_2}^{1-\sigma_2}a(s)\gamma(s)\mathrm{d}s\ge 1$, $\mu_2\|\beta\|\int_0^1 a(s)\gamma(s)\mathrm{d}s\le 1$. $\sigma_1,\sigma_2\in\left(0,\frac{1}{2}\right)$, with $\|\beta\|=\max_{x\in[0,1]}\beta(x)$, $\alpha_1=\min_{x\in[\sigma_1,1-\sigma_1]}\alpha(x)$, $\alpha_2=\min_{x\in[\sigma_2,1-\sigma_2]}\alpha(x)$.

Definition $y(x)$ is the positive solution of boundary value problem (1), if $y(x)$ satisfied

1) $y\in C^{m-1}[0,1]$, $y(x)>0$, $x\in(0,1)$, and $y^{(i)}(0)=y^{(i)}(1)=0$, $0\le i\le m-1$;

2) $y^{(2m)}\in L^1_{loc}(0,1)$, and $(-1)^m y^{(2m)}(x)=f(x,y(x))$, $a.e.x\in(0,1)$.

2. Preliminary

By a direct calculation, we can easily obtain

$$y(x)=\int_0^1 G(x,s)f(s,y(s))\mathrm{d}s,$$

following from [5], $G(x,s)$ can be written by

$$G(x,s)=\begin{cases} \frac{1}{[(m-1)!]^2}\int_0^{s(1-x)}u^{m-1}(x-s+u)^{m-1}\,\mathrm{d}u, & 0\le s\le x\le 1 \\ \frac{1}{[(m-1)!]^2}\int_0^{x(1-s)}u^{m-1}(u-x+s)^{m-1}\,\mathrm{d}u, & 0\le x\le s\le 1 \end{cases}, \tag{2}$$

Define an operator $\Phi: C[0,1]\to C[0,1]$, $(\Phi y)(x)=\int_0^1 G(x,s)f(s,y(s))\mathrm{d}s$.

Lemma 1 The function $G(x,s)$ defined by (2) satisfied the following conditions

$$\alpha(x)\gamma(s)\le G(x,s)\le\beta(x)\gamma(s),$$

where

$$\alpha(x)=\frac{x^m(1-x)^m}{2m-1},\quad \beta(x)=\frac{x^{m-1}(1-x)^{m-1}}{m},\quad \gamma(s)=\frac{s^m(1-s)^m}{[(m-1)!]^2}.$$

Proof By Newton binomial formula, we have

$$\begin{aligned}(u+x-s)^{m-1} &= \sum_{i=0}^{m-1}\frac{(m-1)!}{i!(m-1-i)!}u^i(x-s)^{m-1-i}\\ (u+s-x)^{m-1} &= \sum_{i=0}^{m-1}\frac{(m-1)!}{i!(m-1-i)!}u^i(s-x)^{m-1-i}\end{aligned}, \tag{3}$$

Put (3) into (2), and integral by item

$$G(x,s)=\begin{cases}\dfrac{1}{\left[(m-1)!\right]^2}\displaystyle\sum_{i=0}^{m-1}\frac{(m-1)!(x-s)^{m-1-i}\left[s(1-x)\right]^{m+i}}{i!(m-1-i)!(m+i)}, & 0\le s\le x\le 1\\ \dfrac{1}{\left[(m-1)!\right]^2}\displaystyle\sum_{i=0}^{m-1}\frac{(m-1)!(s-x)^{m-1-i}\left[x(1-s)\right]^{m+i}}{i!(m-1-i)!(m+i)}, & 0\le x\le s\le 1\end{cases},$$

and we can get

$$G(x,s)\le\begin{cases}\dfrac{\left[s(1-x)\right]^m\left[x(1-s)\right]^{m-1}}{m\left[(m-1)!\right]^2}, & 0\le s\le x\le 1\\ \dfrac{\left[x(1-s)\right]^m\left[s(1-x)\right]^{m-1}}{m\left[(m-1)!\right]^2}, & 0\le x\le s\le 1\end{cases}$$

$$\le\frac{s^m(1-s)^m x^{m-1}(1-x)^{m-1}}{m\left[(m-1)!\right]^2},$$

$$G(x,s)\ge\begin{cases}\dfrac{\left[s(1-x)\right]^m\left[x(1-s)\right]^{m-1}}{(2m-1)\left[(m-1)!\right]^2}, & 0\le s\le x\le 1\\ \dfrac{\left[x(1-s)\right]^m\left[s(1-x)\right]^{m-1}}{(2m-1)\left[(m-1)!\right]^2}, & 0\le x\le s\le 1\end{cases}$$

$$\ge\frac{s^m(1-s)^m x^m(1-x)^m}{(2m-1)\left[(m-1)!\right]^2},$$

If

$$\alpha(x)=\frac{x^m(1-x)^m}{2m-1},\quad \beta(x)=\frac{x^{m-1}(1-x)^{m-1}}{m},\quad \gamma(s)=\frac{s^m(1-s)^m}{\left[(m-1)!\right]^2},$$

the upper and lower bound of $G(x,s)$ is

$$\alpha(x)\gamma(s)\le G(x,s)\le\beta(x)\gamma(s).$$

Lemma 2 Let E be a Banach space, and $K\subset E$ is a cone, satisfied

$$K=\left\{y\in C[0,1]; y(x)\ge\frac{\alpha(x)}{\|\beta\|}\|y\|\right\},$$

where $\|y\|=\max_{x\in[0,1]}|y(x)|$, then K is an closed convex cone.

Proof 1) Let $y\in K$, $\lambda\ge 0$, we have

$$\lambda y\ge\lambda\frac{\alpha(x)}{\|\beta\|}\|y\|=\frac{\alpha(x)}{\|\beta\|}\|\lambda y\|,$$

i.e.

$$\lambda y \ge \frac{\alpha(x)}{\|\beta\|}\|\lambda y\|,$$

so

$$\lambda y \in K.$$

2) Because $y \in C[0,1]$, if $y \in K$, $-y \in K$, and $\alpha(x) \ge 0$, then $y = 0$. from 1) and 2), we prove that $K = \left\{ y \in C[0,1]; y(x) \ge \frac{\alpha(x)}{\|\beta\|}\|y\| \right\}$ is an closed convex cone.

Lemma 3 $\Phi : C[0,1] \to C[0,1]$ is completely continuous.

Proof Let $D \subset C[0,1]$ is bounded, then $\exists M > 0$, $\forall y \in D$, we have

$$\|y\| \le M,$$

and

$$\|\Phi y(x)\| \le \max_{\substack{0\le x\le 1\\ 0\le y\le M}} f(x,y)\int_0^1 G(x,s)\mathrm{d}s \le \max_{\substack{0\le x\le 1\\ 0\le y\le M}} f(x,y)\max_{0\le x\le 1}\beta(x)\int_0^1 \gamma(s)\mathrm{d}s,$$

Hence, $\Phi(D)$ is bounded.

Next, we show that $\Phi(D)$ is compact set. In fact

$$\frac{\partial G}{\partial x} = \begin{cases} \frac{1}{\left[(m-1)!\right]^2}\left[s(1-s)x(1-x)\right]^{m-1}(-s) + \frac{1}{\left[(m-1)!\right]^2}\int_0^{s(1-x)}(m-1)t^{m-1}(t+x-s)^{m-2}\,\mathrm{d}t, & 0\le s\le x\le 1\\ \frac{1}{\left[(m-1)!\right]^2}\left[s(1-s)x(1-x)\right]^{m-1}(-s) + \frac{-1}{\left[(m-1)!\right]^2}\int_0^{x(1-s)}(m-1)t^{m-1}(t+s-x)^{m-2}\,\mathrm{d}t, & 0\le x\le s\le 1 \end{cases},$$

then

$$\left|\frac{\partial G}{\partial x}\right| \le \frac{1}{\left[(m-1)!\right]^2} + \frac{1}{(m-1)!(m-2)!}\int_0^1 t^{m-1}(1+t)^{m-2}\mathrm{d}t$$
$$\le \frac{1}{\left[(m-1)!\right]^2} + \frac{1}{\left[(m-1)!\right]^2}\left[(1+t)^{m-1}\right]_0^1 = \frac{2^{m-1}}{\left[(m-1)!\right]^2},$$

For $\forall 0 \le x_1 \le x_2 \le 1$, $\forall y \in D$, then

$$\left|\Phi y(x_2) - \Phi y(x_1)\right| = \int_{x_1}^{x_2}\left|(\Phi y)'(x)\right|\mathrm{d}x = \int_{x_1}^{x_2}\mathrm{d}x\int_0^1\left|\frac{\partial G(x,s)}{\partial x}\right| f(s,y(s))\mathrm{d}s$$
$$\le \frac{2^{m-1}}{\left[(m-1)!\right]^2}\max_{\substack{0\le x\le 1\\ 0\le y\le M}} f(x,y)\left|x_2 - x_1\right|,$$

So $\Phi(D)$ is equicontinuous. By means of the Ascoli-Arzela theorem, $\Phi(D)$ is compact set, Φ is an compact operator.

Let $y_n \in C[0,1]$, $y_0 \in C[0,1]$, with $y_n \to y_0$, because of the convergence properties, $\exists M_0 > 0$ we have $\|y_0\| \le M_0$, $\|y_n\| \le M_0$, $(n = 1,2,\cdots)$. Now we show that $\Phi y_n \to \Phi y_0$. In fact $\forall x \in [0,1]$

$$\left|\Phi y_n(x) - \Phi y_0(x)\right| \le \int_0^1 G(x,s)\left|f(s,y_n(s)) - f(s,y_0(s))\right|\mathrm{d}s$$
$$\le \int_0^1 \max_{0\le x\le 1}\beta(x)\max_{s\in[0,1]}\gamma(s)\left|f(s,y_n(s)) - f(s,y_0(s))\right|\mathrm{d}s$$
$$\le 2\int_0^1\|\beta(x)\|\max_{\substack{0\le x\le 1\\ 0\le y\le M_0}} f(x,y)\max_{s\in[0,1]}\gamma(s)\,\mathrm{d}s,$$

where

$$h_n(s) = \max_{0\le x\le 1}\beta(x)\max_{s\in[0,1]}\gamma(s)\left|f(s,y_n(s)) - f(s,y_0(s))\right|,$$

$$H = 2\|\beta(x)\|\max_{\substack{0\le x\le 1\\ 0\le y\le M_0}} f(x,y)\max_{s\in[0,1]}\gamma(s),$$

and

$$0 \le \int_0^1 H\mathrm{d}s < +\infty, \text{ and } |h_n(s)| \le H.$$

Because $\|\beta(x)\|\max_{s\in[0,1]}\gamma(s) f(s,y)$ is continuous on $[0,1]\times[0,+\infty)$, so $\|\beta(x)\|\max_{s\in[0,1]}\gamma(s) f(s,y)$ is uniformly continuous on $[0,1]\times[0,M_0]$. $\forall\varepsilon>0$, $\exists\delta>0$, $\forall s\in[0,1]$, when $y_1,y_2\in[0,M_0]$, and $|y_1-y_2|<\delta$, $\|\beta(x)\|\max_{s\in[0,1]}\gamma(s)\left|f(s,y_1(s)) - f(s,y_2(s))\right|<\varepsilon$, $y_n(s)\to y_0(s)$, then $\exists N>0$, $\forall n>N$, with $|y_n(s)-y_0(s)|<\delta$, as $s\in[0,1]$, also notice that

$$|h_n(s)-0| = \max_{0\le x\le 1}\beta(x)\max_{s\in[0,1]}\gamma(s)\left|f(s,y_n(s)) - f(s,y_0(s))\right| < \varepsilon,$$

i.e. $h_n(s)\to 0$, a.e. $[0,1]$. And

$$|\Phi y_n(x) - \Phi y_0(x)| \le 2\int_0^1 \max_{0\le x\le 1}\beta(x)\max_{s\in[0,1]}\gamma(s)\left|f(s,y_n(s)) - f(s,y_0(s))\right|\mathrm{d}s < \varepsilon,$$

By using Lebesgue control convergence theorem $\Phi y_0(x)\to\Phi y_n(x)$ as $n\to\infty$, $\forall x\in[0,1]$, so Φ is continuous operator on $C[0,1]$. In conclusion, Φ is completely continuous operator.

3. Main Results

Theorem 1 suppose (H_1)-(H_3) or (H_1), (H_2), (H_4) holds, BVP (1) has at least one positive solution.

Proof We prove $\Phi(K)\subset K$. Since $\forall y\in K$, we have

$$\begin{aligned}(\Phi y)(x) &= \int_0^1 G(x,s) f(s,y(s))\mathrm{d}s \ge \int_0^1 \alpha(x)\gamma(s) f(s,y(s))\mathrm{d}s\\ &\ge \int_0^1 \frac{\alpha(x)}{\|\beta\|}\|\beta\|\gamma(s) f(s,y(s))\mathrm{d}s = \frac{\alpha(x)}{\|\beta\|}\max_{x\in[0,1]}\beta(x)\int_0^1\gamma(s) f(s,y(s))\mathrm{d}s\\ &= \frac{\alpha(x)}{\|\beta\|}\max_{x\in[0,1]}\int_0^1\beta(x)\gamma(s) f(s,y(s))\mathrm{d}s = \frac{\alpha(x)}{\|\beta\|}\max_{x\in[0,1]}\int_0^1 G(x,s) f(s,y(s))\mathrm{d}s\\ &= \frac{\alpha(x)}{\|\beta\|}\|\Phi y\|,\end{aligned}$$

then $\Phi y\in K$, for $\forall y\in K$, *i.e.* $\Phi(K)\subset K$.

It follows form (H_3), $\lim_{y\to 0^+}\max_{x\in[0,1]}\frac{f(x,y)}{y}<\lambda_1$, where $\lambda_1\|\beta\|\int_0^1\gamma(s)\mathrm{d}s\le 1$, there exist $\delta_1>0$, such that $\max_{x\in[0,1]}\frac{f(x,y)}{y}<\lambda_1$, $\forall\ 0<y<\delta_1$ *i.e.* $f(x,y)<\lambda_1 y$. Let $\Omega_1=\{y\in C[0,1];\|y\|<N_1, 0<N_1<\delta_1\}$, for any $y\in K\cap\partial\Omega_1$, and $\|y\|=N_1$, since $y\le\|y\|=N_1<\delta_1$, we have

$$\begin{aligned}\|(\Phi y)(x)\| &= \max_{x\in[0,1]}\int_0^1 G(x,s) f(s,y(s))\mathrm{d}s \le \max_{x\in[0,1]}\beta(x)\int_0^1\gamma(s) f(s,y(s))\mathrm{d}s\\ &< \|\beta\|\int_0^1\gamma(s)\lambda_1 y(s)\mathrm{d}s \le \lambda_1\|\beta\|\|y\|\int_0^1\gamma(s)\mathrm{d}s < \|y\|,\end{aligned}$$

From $\lim_{y\to+\infty}\min_{x\in[0,1]}\frac{f(x,y)}{y}>\lambda_2$, where $\lambda_2\frac{\alpha_1^2}{\|\beta\|}\int_{\sigma_1}^{1-\sigma_1}\gamma(s)\mathrm{d}s\ge 1$, $\sigma_1\in\left(0,\frac{1}{2}\right)$, there exists $M>0$, such that

$\min_{x\in[0,1]}\frac{f(x,y)}{y}>\lambda_2$, $\forall y>M$, *i.e.* $f(x,y)>\lambda_2 y$.

Let $\Omega_2=\{y\in[0,1];\|y\|<N_2\}$, and $N_2>\max\left\{\frac{M\|\beta\|}{\alpha_1},N_1\right\}$. Then for any, $y\in K\cap\partial\Omega_2$, and $y(x)\ge\frac{\alpha(x)}{\|\beta\|}\|y\|$, we have $\min_{x\in[\sigma_1,1-\sigma_1]}y(x)\ge\min_{x\in[\sigma_1,1-\sigma_1]}\frac{\alpha(x)\|y\|}{\|\beta\|}=\frac{\alpha_1 N_2}{\|\beta\|}>M$, and

$$\begin{aligned}\|(\Phi y)(x)\|&=\max_{x\in[0,1]}\int_0^1 G(x,s)f(s,y(s))\mathrm{d}s\ge\max_{x\in[0,1]}\int_0^1\alpha(x)\gamma(s)f(s,y(s))\mathrm{d}s\\&=\max_{x\in[0,1]}\alpha(x)\int_0^1\gamma(s)f(s,y(s))\mathrm{d}s\ge\|\alpha\|\int_{\sigma_1}^{1-\sigma_1}\gamma(s)f(s,y(s))\mathrm{d}s\\&>\|\alpha\|\int_{\sigma_1}^{1-\sigma}\gamma(s)\lambda_2 y(s)\mathrm{d}s\ge\|\alpha\|\lambda_2\int_{\sigma_1}^{1-\sigma_1}\gamma(s)\frac{\alpha(x)}{\|\beta\|}\|y\|\mathrm{d}s\\&\ge\lambda_2\|\alpha\|\min_{x\in[\sigma_1,1-\sigma_1]}\frac{\alpha(x)}{\|\beta\|}\|y\|\int_{\sigma_1}^{1-\sigma_1}\gamma(s)\mathrm{d}s\ge\frac{\lambda_2\alpha_1^{2}}{\|\beta\|}\|y\|\int_{\sigma_1}^{1-\sigma_1}\gamma(s)\mathrm{d}s\ge\|y\|,\end{aligned}$$

According to the theorems on compression and expansion of cones, Φ has at least a fixed point, *i.e.* $\Phi y=y$, $\forall y\in K\cap(\bar{\Omega}_2\setminus\Omega_1)$ an y satisfied integral equation $y(x)=\int_0^1 G(x,s)f(s,y(s))\mathrm{d}s$, so, y is the positive solution of (1).

From (H_4), we know $\lim_{y\to0^+}\frac{h(y)}{y}>\mu_1$, where $\mu_1\frac{\alpha_2^{2}}{\|\beta\|}\int_{\sigma_2}^{1-\sigma_2}a(s)\gamma(s)\mathrm{d}s\ge1$, $\sigma_2\in\left(0,\frac{1}{2}\right)$. There exists $\delta_2>0$, such that $\forall\ 0<y<\delta_2$, we have $h(y)>\mu_1 y$. Let $\Omega_1=\{y\in C[0,1];\|y\|<M_1,0<M_1<\delta_2\}$, for any $y\in K\cap\partial\Omega_1$, we have

$$\|y\|=M_1,\ y\le\|y\|=M_1<\delta_2,\text{ thus }\min_{x\in[\sigma_2,1-\sigma_2]}y(x)\ge\min_{x\in[\sigma_2,1-\sigma_2]}\frac{\alpha(x)\|y\|}{\|\beta\|}=\frac{\alpha_2 M_1}{\|\beta\|},$$

i.e.

$$\begin{aligned}\|(\Phi y)(x)\|&=\max_{x\in[0,1]}\int_0^1 G(x,s)a(s)h(y(s))\mathrm{d}s\ge\max_{x\in[0,1]}\alpha(x)\int_0^1\gamma(s)a(s)h(y(s))\mathrm{d}s\\&\ge\|\alpha\|\int_{\sigma_2}^{1-\sigma_2}\gamma(s)a(s)h(y(s))\mathrm{d}s>\|\alpha\|\int_{\sigma_2}^{1-\sigma_2}\gamma(s)\mu_1 a(s)y(s)\mathrm{d}s\\&\ge\|\alpha\|\mu_1\int_{\sigma_2}^{1-\sigma_2}a(s)\gamma(s)\frac{\alpha(x)}{\|\beta\|}\|y\|\mathrm{d}s\ge\mu_1\|\alpha\|\min_{x\in(\sigma_2,1-\sigma_2)}\frac{\alpha(x)}{\|\beta\|}\|y\|\int_{\sigma_2}^{1-\sigma_2}a(s)\gamma(s)\mathrm{d}s\\&\ge\frac{\mu_1\alpha_2^{2}}{\|\beta\|}\|y\|\int_{\sigma_2}^{1-\sigma_2}a(s)\gamma(s)\mathrm{d}s>\|y\|,\end{aligned}$$

From $\lim_{y\to+\infty}\frac{h(y)}{y}<\mu_2$, where $\mu_2\|\beta\|\int_0^1 a(s)\gamma(s)\mathrm{d}s\le1$. We know $\exists N>0$, $\forall y>N$, such that $h(y)<\mu_2 y$. In the following, we consider two cases:

1) If $h(y)$ bounded on $y\in[0,+\infty)$, Let $h(y)\le H$, $M_2=\max\left\{2M_1,H\|\beta\|\int_0^1 a(s)\gamma(s)\mathrm{d}s\right\}$, $\Omega_2=\{y\in C[0,1];\|y\|<M_2\}$, since $y\in K\cap\partial\Omega_2$ and $\|y\|=M_2$, so

$$\begin{aligned}(\Phi y)(x)&=\int_0^1 G(x,s)a(s)h(y(s))\mathrm{d}s\le H\int_0^1\beta(x)\gamma(s)a(s)\mathrm{d}s\\&\le H\|\beta\|\int_0^1\gamma(s)a(s)\mathrm{d}s\le M_2\Phi y=y,\end{aligned}$$

i.e. $\|\Phi y\| \le \|y\|$.

2) If $h(y)$ is unbounded on $y \in [0,+\infty)$, Let $M_2 > \max\{2M_1, N\}$, such that $h(y) \le h(M_2)$, $\forall\ 0 < y \le M_2$. Since $h(y)$ is unbounded, then $\forall\ y \in K \cap \partial\Omega_2$, we have

$$(\Phi y)(x) = \int_0^1 G(x,s)a(s)h(y(s))\mathrm{d}s \le \|\beta\| \int_0^1 \gamma(s)a(s)h(M_2)\mathrm{d}s$$
$$\le \mu_2 M_2 \|\beta\| \int_0^1 \gamma(s)a(s)\mathrm{d}s \le M_2,$$

i.e. $\|\Phi y\| \le \|y\|$.

In conclusion, according to the theorems on compression and expansion of cones, Φ has at least one fixed point $y \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. This showed that $\Phi y = y$, and y satisfied integral equation

$$y(x) = \int_0^1 G(x,s)a(s)h(y(s))\mathrm{d}s = \int_0^1 G(x,s) f(s, y(s))\mathrm{d}s,$$

So, $y(x)$ is the positive of BVP (1), where $f(x,y) = a(x)h(y)$.

Theorem 2 If condition (H_1)-(H_4) holds, then the BVP (1) has a uniqueness positive solution.

Proof If $y_1(x)$, $y_2(x)$ are the positive solution of BVP (1), Let $y(x) = y_1(x) - y_2(x)$, where $y(x)$ satisfied boundary value problem

$$\begin{cases} (-1)^m y^{(2m)}(x) = f(x, y_1) - f(x, y_2), & \\ y^{(i)}(0) = y^{(i)}(1) = 0, & 0 \le i \le m-1 \end{cases},$$

Notice that $(-1)^m y(x) y^{(2m)}(x) \le 0 \quad \forall x \in [0,1]$, integral the left from 0 to 1, notice that

$$\begin{aligned} &(-1)^m \int_0^1 y(x) y^{(2m)}(x)\mathrm{d}x \\ &= (-1)^m \Big[y(x) y^{(2m-1)}(x)\Big|_0^1 - y'(x) y^{(2m-2)}(x)\Big|_0^1 + (-1)^2 y''(x) y^{(2m-3)}(x)\Big|_0^1 \\ &\quad + \cdots + (-1)^{m-1} y^{(m-1)}(x) y^{(m)}(x)\Big|_0^1 + (-1)^m \int_0^1 \big[y^{(m)}(x)\big]^2 \mathrm{d}x \Big] \\ &= (-1)^{2m} \int_0^1 \big[y^{(m)}(x)\big]^2 \mathrm{d}x = \int_0^1 \big[y^{(m)}(x)\big]^2 \mathrm{d}x, \end{aligned}$$

So we obtain

$$0 \le \int_0^1 \big[y^{(m)}(x)\big]^2 \mathrm{d}x \le 0,$$

Thus $\int_0^1 \big[y^{(m)}(x)\big]^2 \mathrm{d}x = 0$, *i.e.* $y^{(m)}(x) = 0$, $\forall\ x \in [0,1]$. And since $y^{(m-1)}(x) = c$, we have $y^{(m-1)}(0) = 0$, $c = 0$, *i.e.* $y^{(m-1)}(x) = 0$. Repeat above process, and conditions $y^{(i)}(0) = 0$, $0 \le i \le m-1$,

In the last, we have $y(x) = 0$, $\forall\ x \in [0,1]$. It is obvious that $y_1(x) = y_2(x)$, $\forall\ x \in [0,1]$. The uniqueness has been proved.

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