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Oscillation of Second Order Nonlinear Neutral Differential Equations with Mixed Neutral Term

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Abstract

In this paper, we obtained some sufficient conditions for the oscillation of all solutions of the second order neutral differential equation of the form $\left(r(t)z'(t)\right)' + q(t) f\left(x\left(\sigma(t)\right)\right) = 0,\ t \ge t_0 \ge 0$ where $z(t) = x(t) + a(t)x(t-\tau) + b(t)x(t+\delta)$, and $\int_{t_0}^{\infty} \frac{1}{r(t)} \mathrm{d}t < \infty$. Examples are provided to illustrate the main results.

Keywords

Second Order, Nonlinear Differential Equation, Mixed Neutral Term, Oscillation

1. Introduction

In this paper, we are concerned with the oscillatory behavior of solutions of the second order nonlinear neutral differential equation of the form

$$\left(r(t)z'(t)\right)' + q(t) f\left(x\left(\sigma(t)\right)\right) = 0,\ t \ge t_0 \ge 0 \tag{1}$$

where $z(t) = x(t) + a(t)x(t-\tau) + b(t)x(t+\delta)$, subject to the following conditions:

(C_1) $a, b, q \in C\left([t_0, \infty), \mathbb{R}\right), 0 \le a(t) \le a < \infty, 0 \le b(t) \le b < \infty$, and $q(t) > 0$ for all $t \ge t_0$;

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(C_2) $r \in C([t_0,\infty),\mathbb{R}), r(t)>0$, and $\int_{t_0}^{\infty}\frac{1}{r(t)}\mathrm{d}t<\infty$;

(C_3) τ,δ are nonnegative constants, $\sigma \in C([t_0,\infty),\mathbb{R}), \sigma'(t)>0$, $\sigma(t)\le t, \lim_{t\to\infty}\sigma(t)=\infty$, and $\sigma(t\pm\alpha)=\sigma(t)\pm\alpha$ for any $\alpha>0$;

(C_4) $f \in C(\mathbb{R},\mathbb{R}), \frac{f(u)}{u}\ge k>0$ for $u\ne$, k is a constant.

By a solution of Equation (1), we mean a continuous function x defined on an interval $[t_x,\infty)$ such that $r(t)z'(t)$ is continuously differentiable and x satisfies Equation (1) for all $t\in[t_x,\infty)$. We consider only solutions satisfying condition $\sup\{|x(t)|: t\ge T\ge t_x\}>0$, and tacitly assume that Equation (1) possess such solutions. As usual, a solution of Equation (1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise we call it nonosicllatory.

From the literature, it is known that second order neutral functional differential equations have applications in problems dealing with vibrating masses attached to an elastic bar and in some variational problems. For further applications and questions regarding existence and uniqueness of solutions of neutral functional differential equations, see [1]-[3].

In recent years, there has been an increasing interest in establishing conditions for the oscillation or nonoscillation of solution of neutral functional differential equations, see [4]-[20] for example, and the references cited therein.

In [21], Xu and Meng obtained some sufficient conditions which guarantees that every solution x of equation (1) when $b(t)\equiv 0$, oscillates or $\lim_{t\to\infty}x(t)=0$.

Ye and Xu [22] studied equation when $b(t)\equiv 0$, and established some new oscillation criteria for Equation (1).

In [23], Han *et al*. considered Equation (1) with $b(t)\equiv 0$ and $0\le a(t)\le 1$, and obtained some sufficient conditions which ensure that every solution of Equation (1) is oscillatory.

In [24], the present authors established some sufficient conditions for the oscillation of all solutions of Equation (1) when $\int_{t_0}^{\infty}\frac{1}{r(t)}\mathrm{d}t=\infty$. Therefore in this paper we try to obtain some new oscillation criteria for Equation (1). In Section 2, we use Riccati transformation technique to obtain some sufficient conditions for the oscillation of all solutions of Equation (1). Examples are provided in Section 3 to illustrate the main results.

2. Oscillation Results

In this section, we obtain some new oscillation criteria for the Equation (1). We begin with the following theorem.

Theorem 2.1 *If*

$$\int_{t_0}^{\infty}Q(t)\mathrm{d}t=\infty \tag{2}$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[kQ(s)\delta^n(s)-\frac{n^2(1+a+b)}{4r(s)\delta^{2-n}(s)}\right]\mathrm{d}s=\infty \tag{3}$$

where $n\ge 1$, $Q(t)=\min\{q(t),q(t-\tau),q(t+\delta)\}$, and $\delta(t)=\int_t^{\infty}\frac{1}{r(s)}\mathrm{d}s$ *then every solution of Equation* (1) *is oscillatory*.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of Equation (1). Without loss of generality, we may assume that there exists $t_1\ge t_0$ such that. $x(t)>0, x(t-\tau)>0, x(t+\delta)>0$ and $x(\sigma(t))>0$ for all $t\ge t_1$. From the definition of $z(t)$, we have $z(t)>0$, and from Equation (1), $r(t)z'(t)$ is nonincreasing eventually. Hence, it is easy to conclude that there exist two possible cases of the sign of $z'(t)$, that is, $z'(t)>0$ or $z'(t)<0$ for all $t\ge t_2\ge t_1$.

First assume that $z'(t) > 0$ for all $t \geq t_2$. From the Equation (1), we have

$$\left(r(t)z'(t)\right)' + kq(t)x\left(\sigma(t)\right) + a\left(r(t-\tau)z'(t-\tau)\right)' + akq(t-\tau)x\left(\sigma(t-\tau)\right)$$
$$+b\left(r(t+\delta)z'(t+\delta)\right)' + bkq(t+\delta)x\left(\sigma(t+\delta)\right) \leq 0,$$

or

$$\left(r(t)z'(t)\right)' + a\left(r(t-\tau)z'(t-\tau)\right)' + b\left(r(t+\delta)z'(t+\delta)\right)' + kQ(t)z\left(\sigma(t)\right) \leq 0,\ t \geq t_2. \tag{4}$$

Integrating (4) from t_2 to t and using the fact $z(t) \geq c > 0$ for $t \geq t_2$, we obtain

$$\int_{t_2}^{t} Q(s)\mathrm{d}s < \infty$$

a contradiction to (2.1).

If $z'(t) < 0$, then we define the function w by

$$w(t) = \frac{r(t)z'(t)}{z(t)}, t \geq t_2. \tag{5}$$

Clearly $w(t) < 0$. Nothing that $r(t)z'(t)$ is nonincreasing, we obtain

$$r(s)z'(s) \leq r'(t)z'(t), s \geq t \geq t_2.$$

Dividing the last inequality by $r(s)$ and integrating it from t to ℓ, we obtain

$$z(\ell) \leq z(t) + r(t)z'(t)\int_t^{\ell} \frac{\mathrm{d}s}{r(s)}, \ell \geq t \geq t_2.$$

Letting $\ell \to \infty$ in the last inequality, we see that

$$0 \leq z(t) + r(t)z'(t)\delta(t),\ t \geq t_2.$$

Therefore,

$$\frac{r(t)z'(t)}{z(t)}\delta(t) \geq -1,\ t \geq t_2. \tag{6}$$

From (5), we have

$$-1 \leq w(t)\delta(t) \leq 0, t \geq t_2. \tag{7}$$

Next, we introduce another function u by

$$u(t) = \frac{r(t-\tau)z'(t-\tau)}{z(t)}, t \geq t_2. \tag{8}$$

Clearly $u(t) < 0$. Noting that $r(t)z'(t)$ is nonincreasing, we have $r(t-\tau)z'(t-\tau) \geq r(t)z'(t)$. Then, $u(t) \geq w(t)$. From (6), we obtain

$$-1 \leq u(t)\delta(t) \leq 0,\ t \geq t_2. \tag{9}$$

Similarly, we introduce another function v by

$$v(t) = \frac{r(t+\delta)z'(t+\delta)}{z(t)},\ t \geq t_2. \tag{10}$$

Clearly $v(t) < 0$. Since $r(t)z'(t)$ is nonincreasing, we have

$$r(s)z'(s) \leq r(t+\delta)z'(t+\delta), s \geq t+\delta \geq t_2.$$

Dividing the last inequality by $r(s)$ and integrating it from t to ℓ, we obtain

$$z(\ell) \le z(t) + r(t+\delta) z'(t+\delta) \int_t^{\ell} \frac{1}{r(s)} \mathrm{d}s, \ell \ge t+\delta \ge t_2.$$

Letting $\ell \to \infty$, we see that

$$-1 \le v(t)\delta(t) \le 0, t+\delta \ge t_2. \tag{11}$$

Differentiating (5), we obtain

$$w'(t) \le \frac{\left(r(t) z'(t)\right)'}{z(t)} - \frac{w^2(t)}{r(t)}, t \ge t_3 \ge t_2 + \delta. \tag{12}$$

Differentiating (8), we have

$$u'(t) \le \frac{\left(r(t-\tau) z'(t-\tau)\right)'}{z(t)} - \frac{u^2(t)}{r(t)}, t \ge t_3. \tag{13}$$

Differentiating (10), we have

$$v'(t) \le \frac{\left(r(t+\delta) z'(t+\delta)\right)'}{z(t)} - \frac{v^2(t)}{r(t)}, t \ge t_3. \tag{14}$$

Inview of (12), (13) and (14), we can obtain

$$\begin{aligned} w'(t) + au'(t) + bv'(t) \le{}& \frac{\left(r(t) z'(t)\right)'}{z(t)} + a\frac{\left(r(t-\tau) z'(t-\tau)\right)'}{z(t)} + b\frac{\left(r(t+\delta) z'(t+\delta)\right)'}{z(t)} \\ & - \frac{w^2(t)}{r(t)} - a\frac{u^2(t)}{r(t)} - b\frac{v^2(t)}{r(t)}, t \ge t_3. \end{aligned} \tag{15}$$

From (4) and (15), we obtain

$$w'(t) + au'(t) + bv'(t) \le -kQ(t) - \frac{w^2(t)}{r(t)} - a\frac{u^2(t)}{r(t)} - b\frac{v^2(t)}{r(t)}, t \ge t_3. \tag{16}$$

Multiplying (16) by $\delta^n(t)$ and integrating from t_3 to t, we have

$$\begin{aligned} &\delta^n(t) w(t) - \delta^n(t_3) w(t_3) + n\int_{t_3}^t \frac{w(s)\delta^{n-1}(s)}{r(s)} \mathrm{d}s + n\int_{t_3}^t \frac{w^2(s)\delta^n(s)}{r(s)} \mathrm{d}s + a\delta^n(t) u(t) \\ &-a\delta^n(t_3) u(t_3) + an\int_{t_3}^t \frac{u(s)\delta^{n-1}(s)}{r(s)} \mathrm{d}s + an\int_{t_3}^t \frac{u^2(s)\delta^n(s)}{r(s)} \mathrm{d}s + b\delta^n(t) v(t) \\ &-b\delta^n(t_3) v(t_3) + bn\int_{t_3}^t \frac{v(s)\delta^{n-1}(s)}{r(s)} \mathrm{d}s + b\int_{t_3}^t \frac{v^2(s)\delta^n(s)}{r(s)} \mathrm{d}s + k\int_{t_3}^t \delta^n(s) Q(s) \mathrm{d}s \le 0. \end{aligned}$$

From the above inequality, we obtain

$$\begin{aligned} &\delta^n(t) w(t) - \delta^n(t_3) w(t_3) + a\delta^n(t) u(t) - a\delta^n(t_3) u(t_3) + b\delta^n(t) v(t) - b\delta^n(t_3) v(t_3) \\ &+k\int_{t_3}^t \delta^n(s) Q(s) \mathrm{d}s - n^2 \frac{(1+a+b)}{4} \int_{t_3}^t \frac{\mathrm{d}s}{r(s)\delta^{n-2}(s)} \le 0. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} &\delta^n(t) w(t) + a\delta^n(t) u(t) + b\delta^n(t) v(t) + \int_{t_3}^t \left[k\delta^n(s) Q(s) - n^2 \frac{1+a+b}{4r(s)\delta^{2-n}(s)} \right] \mathrm{d}s \\ &\le \delta^n(t_3) w(t_3) + a\delta^n(t_3) u(t_3) + b\delta^n(t_3) v(t_3). \end{aligned}$$

By (7), (9) and (11), we obtain that

$$\int_{t_3}^{t}\left[k\delta^n(s)Q(s)-n^2\frac{(1+a+b)}{4r(s)\delta^{2-n}(s)}\right]\mathrm{d}s\le\delta^{n-1}(t_3)(1+a+b)+\delta^n(t_3)\big(w(t_3)+au(t_3)+bv(t_3)\big)$$

which contradicts (3). The proof is now complete. □

Corollary 2.1. *Assume that* $\sigma(t)=t-\tau$ *with* $\sigma\ge\tau$ *for* $t\ge t_0$. *Further assume that* (2.1) *and* (3) *hold. Then every solution of* Equation (1) *is oscillatory.*

Proof. The proof follows from Theorem 2.1. □

Theorem 2.2. *Assume that* $\sigma(t)\le t-\tau$ *for* $t\ge t_0$. *If condition* (2.1) *holds and*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\delta^2(s)Q(s)\mathrm{d}s=\infty \tag{17}$$

then every solution of Equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (1). Without loss of generality, we may assume that there exists $t_1\ge t_0$ such that $x(t)>0, x(t-\tau)>0, x(t+\delta)>0$ and $x(\sigma(t))>0$ for all $t\ge t_1$. By equation (1), $r(t)z'(t)$ is nonincreasing eventually. Hence, it is easy to conclude that there exist two possible cases of the sign of $z'(t)$, that is, $z'(t)>0$ or $z'(t)<0$ for all $t\ge t_2\ge t_1$. If $z'(t)>0$, then we are back to the case of Theorem 2.1, and we can obtain a contradiction to (2.1). If $z'(t)<0$, then we define w,u and v as in Theorem 2.1. Then proceed as in the proof of Theorem 2.1, we obtain (7), (9), (11) and (16) for $t\ge t_3\ge t_2$. Multiplying (16) by $\delta^2(t)$ and integrating from t_3 to t yields

$$\begin{aligned}&\delta^2(t)w(t)-\delta^2(t_3)w(t_3)+2\int_{t_3}^{t}\frac{w(s)\delta(s)}{r(s)}\mathrm{d}s+\int_{t_3}^{t}\frac{w^2(s)\delta^2(s)}{r(s)}ds+a\delta^2(t)u(t)\\&-a\delta^2(t_3)u(t_3)+2a\int_{t_3}^{t}\frac{u(s)\delta(s)}{r(s)}\mathrm{d}s+a\int_{t_3}^{t}\frac{u^2(s)\delta^2(s)}{r(s)}\mathrm{d}s+b\delta^2(t)v(t)\\&-b\delta^2(t_3)v(t_3)+2b\int_{t_3}^{t}\frac{v(s)\delta(s)}{r(s)}ds+b\int_{t_3}^{t}\frac{v^2(s)\delta^2(s)}{r(s)}ds+k\int_{t_3}^{t}\delta^2(s)Q(s)\mathrm{d}s\le 0.\end{aligned} \tag{18}$$

It follows from (C_2) and (7) that

$$\left|\int_{t_3}^{\infty}\frac{w(s)\delta(s)}{r(s)}ds\right|\le\int_{t_3}^{\infty}\frac{|w(s)\delta(s)|}{r(s)}\mathrm{d}s\le\int_{t_3}^{\infty}\frac{1}{r(s)}\mathrm{d}s<\infty,$$

$$\int_{t_3}^{t}\frac{w^2(s)\delta^2(s)}{r(s)}\mathrm{d}s\le\int_{t_3}^{\infty}\frac{1}{r(s)}\mathrm{d}s<\infty.$$

Inview of (9), we have

$$\left|\int_{t_3}^{\infty}\frac{u(s)\delta(s)}{r(s)}\mathrm{d}s\right|\le\int_{t_3}^{\infty}\frac{|u(s)\delta(s)|}{r(s)}\mathrm{d}s\le\int_{t_3}^{\infty}\frac{1}{r(s)}\mathrm{d}s<\infty,$$

$$\int_{t_3}^{t}\frac{u^2(s)\delta^2(s)}{r(s)}\mathrm{d}s\le\int_{t_3}^{\infty}\frac{1}{r(s)}\mathrm{d}s<\infty.$$

From (11), we obtain

$$\left|\int_{t_3}^{\infty}\frac{v(s)\delta(s)}{r(s)}\mathrm{d}s\right|\le\int_{t_3}^{\infty}\frac{|v(s)\delta(s)|}{r(s)}\mathrm{d}s\le\int_{t_3}^{\infty}\frac{1}{r(s)}\mathrm{d}s<\infty,$$

$$\int_{t_3}^{t}\frac{v^2(s)\delta^2(s)}{r(s)}\mathrm{d}s\le\int_{t_3}^{\infty}\frac{1}{r(s)}\mathrm{d}s<\infty.$$

Therefore from (18), we obtain

$$\limsup_{t\to\infty}\int_{t_0}^{t}\delta^2(s)Q(s)\mathrm{d}s<\infty,$$

which is a contradiction with (17). The proof is now complete. □

Corollary 2.2. *Assume that* $\sigma(t) \le t-\tau$ *for* $t \ge t_0$. *In condition* (2.1) *and* (17) *hold, then every solution of* Equation (1) *is oscillatory.*

Proof. The proof follows from Theorem 2.2. □

To prove our next theorem, we need a class of function γ and the operator *T* defined as follows:

Following [16], we say that a function $\phi = \phi(t,s,\ell)$ belongs to the function class Y, denoted by $\phi \in Y$ if $\phi \in C(E,\mathbb{R})$, where $E = \{(t,s,\ell): t_0 \le \ell \le s \le t < \infty\}$, which satisfies $\phi(t,t,\ell)=0, \phi(t,\ell,\ell)=0$ and $\phi(t,s,\ell)>0$ for $\ell < s < t$, and has the partial derivative $\frac{\partial \phi}{\partial s}$ on E such that $\frac{\partial \phi}{\partial s}$ is locally integrable with respect to s in E.

Define the operator T by

$$T[g;\ell,t] = \int_\ell^t \phi(t,s,\ell) g(s)\,\mathrm{d}s, \tag{19}$$

for $t \ge s \ge \ell \ge t_0$ and $g \in C'[t_0,\infty)$. The function $\psi = \psi(t,s,\ell)$ is defined by

$$\frac{\partial \psi}{\partial s}(t,s,\ell) = \psi(t,s,\ell)\phi(t,s,\ell) \tag{20}$$

then, it is easy to see that T is a linear operator and

$$T[g';\ell,t] = -T[g\psi;\ell,t] \quad \text{for } g \in C'[t_0,\infty). \tag{21}$$

Theorem 2.3. *Assume that* $\sigma(t) \le t-\tau$, *and there exist functions* $\phi \in Y$ *and* $\rho \in C'([t_0,\infty),\mathbb{R}^+)$ *such that*

$$\limsup_{t\to\infty} T\left[k\rho(s)Q(s) - \frac{(1+a+b)\left(\psi + \frac{\rho'(s)}{\rho(s)}\right)^2}{4\sigma'(s)} r(\sigma(s))\rho(s);\ell,t \right] > 0 \tag{22}$$

and

$$\limsup_{t\to\infty} T\left[kQ(s) - \frac{(1+a+b)r(s)\psi^2}{4};\ell,t \right] > 0 \tag{23}$$

where $Q(t)$ is defined as in Theorem 2.1, the operator T defined by (19), and $\psi = \psi(t,s,\ell)$ is defined by (20). Then every solution of Equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (1). Then there exists a $t_1 \ge t_0$ such that $x(t) \ne 0$ for all $t \ge t_1$. Without loss of generality, we may assume that $x(t)>0, x(t-\tau)>0, x(t+\delta)>0$ and $x(\sigma(t))>0$ for all $t \ge t_1$. Then proceeding as in the proof of Theorem 2.1 we have

$z(t)>0, z'(t)>0, (r(t)z'(t))' \le 0$ or $z(t)>0, z'(t)<0$ and $(r(t)z'(t))' \le 0$ for all $t \ge t_1$.

First assume that $z(t)>0, z'(t)>0,$ and $(r(t)z'(t))' \le 0$ for all $t \ge t_1$. Define

$$w(t) = \rho(t)\frac{r(t)z'(t)}{z(\sigma(t))},\ t \ge t_1. \tag{24}$$

Then $w(t)>0$, and

$$\begin{aligned} w'(t) &= \rho(t)\frac{(r(t)z'(t))'}{z(\sigma(t))} + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{w(t)}{z(\sigma(t))}z'(\sigma(t))\sigma'(t) \\ &\le \rho(t)\frac{(r(t)z'(t))'}{z(\sigma(t))} + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{w^2(t)}{\rho(t)r(\sigma(t))}\sigma'(t). \end{aligned} \tag{25}$$

Since $r(t)z'(t)$ is nonincreasing and $z(t)$ is increasing. Next, define

$$u(t)=\rho(t)\frac{\left(r(t-\tau)z'(t-\tau)\right)}{z(\sigma(t))},\ t\geq t_1. \tag{26}$$

Then $u(t)>0$, and

$$\begin{aligned} u'(t)&=\rho(t)\frac{\left(r(t-\tau)z'(t-\tau)\right)}{z(\sigma(t))}+\frac{\rho'(t)}{\rho(t)}u(t)-\frac{u(t)}{z(\sigma(t))}z'(\sigma(t))\sigma'(t)\\ &\leq\rho(t)\frac{\left(r(t-\tau)z'(t-\tau)\right)'}{z(\sigma(t))}+\frac{\rho'(t)}{\rho(t)}u(t)-\frac{u^2(t)\sigma'(t)}{\rho(t)r(\sigma(t))}. \end{aligned} \tag{27}$$

Since $r(t)z'(t)$ is nonincreasing, $z(t)$ is increasing and $\sigma(t)\leq t-\tau$. Again, define

$$v(t)=\rho(t)\frac{\left(r(t+\delta)z'(t+\delta)\right)}{z(\sigma(t))},\ t\geq t_1. \tag{28}$$

Then $v(t)>0$, and

$$\begin{aligned} v'(t)&=\rho(t)\frac{\left(r(t+\delta)z'(t+\delta)\right)}{z(\sigma(t))}+\frac{\rho'(t)}{\rho(t)}v(t)-\frac{v(t)}{z(\sigma(t))}z'(\sigma(t))\sigma'(t)\\ &\leq\rho(t)\frac{\left(r(t+\delta)z'(t+\delta)\right)'}{z(\sigma(t))}+\frac{\rho'(t)}{\rho(t)}v(t)-\frac{v^2(t)\sigma'(t)}{\rho(t)r(\sigma(t))}. \end{aligned} \tag{29}$$

Since $r(t)z'(t)$ is nonincreasing, $z(t)$ is increasing and $\sigma(t)\leq t+\delta$. Combining (25) and (29), and then using (4), we obtain

$$\begin{aligned} w'(t)+au'(t)+bv'(t)\leq{}&-k\rho(t)Q(t)+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\sigma'(t)}{r(\sigma(t))\rho(t)}w^2(t)+a\frac{\rho'(t)}{\rho(t)}u(t)\\ &-a\frac{\sigma'(t)}{r(\sigma(t))\rho(t)}u^2(t)+b\frac{\rho'(t)}{\rho(t)}v(t)-b\frac{\sigma'(t)}{r(\sigma(t))\rho(t)}v^2(t). \end{aligned} \tag{30}$$

Now applying the operator T to (30) and then using (21), we have

$$\begin{aligned} T\left[k\rho(s)Q(s);\ell,t\right]\leq T\Bigg[&\left(\psi+\frac{\rho'(s)}{\rho(s)}\right)w(s)-\frac{\sigma'(s)}{r(\sigma(s))\rho(s)}w^2(s)+a\left(\psi+\frac{\rho'(s)}{\rho(s)}\right)u(s)\\ &-a\frac{\sigma'(s)}{r(\sigma(s))\rho(s)}u^2(s)+b\left(\psi+\frac{\rho'(s)}{\rho(s)}\right)v(s)-b\frac{\sigma'(s)}{r(\sigma(s))\rho(s)}v^2(s);\ell,t\Bigg]. \end{aligned}$$

From the last inequality, we obtain

$$T\left[k\rho(s)Q(s);\ell,t\right]\leq T\left[\frac{(1+a+b)}{4\sigma'(s)}\left(\psi+\frac{\rho'(s)}{\rho(s)}\right)^2 r(\sigma(s))\rho(s);\ell,t\right]$$

or

$$T\left[k\rho(s)Q(s)-\frac{(1+a+b)}{4\sigma'(s)}\left(\psi+\frac{\rho'(s)}{\rho(s)}\right)^2 r(\sigma(s))k(s);\ell,t\right]\leq 0.$$

Taking the sup limit in the last inequality, we obtain a contradiction with (22).

Next consider the case $z(t)>0, z'(t)<0$ and $\left(r(t)z'(t)\right)'\leq 0$ for all $t\geq t_2$. From the proof of Theorem 2.1, we have the inequality (16). Now apply the operator T to (16) and then using (21), we have

$$T\left[kQ(s);\ell,t\right] \le T\left[\psi w(s) - \frac{w^2(s)}{r(s)} + a\psi u(s) - a\frac{u^2(s)}{r(s)} + b\psi v(s) - b\frac{v^2(s)}{r(s)};\ell,t\right]$$

From the last inequality, we obtain

$$T\left[kQ(s);\ell,t\right] \le T\left[\frac{(1+a+b)r(s)\psi^2}{4};\ell,t\right]$$

or

$$T\left[kQ(s) - \frac{(1+a+b)}{4}\psi^2 r(s);\ell,t\right] \le 0.$$

Taking the sup limit in the last inequality, we obtain a contradiction with (23). The proof is now completed. □

Remark 2.1. *With different choices of functions* ρ *and* ϕ, *Theorem* 2.3 *can be stated with different conditions for oscillations of* Equation (1).

For example, if we take $\phi(t,s,\ell) = (t-s)^{\alpha}(s-\ell)^{\beta}$ for $\alpha > \frac{1}{2}, \beta > \frac{1}{2}$, then

$$\psi(t,s,\ell) = \frac{\beta t - (\alpha+\beta)s + \alpha\ell}{(t-s)(s-\ell)}.$$

From Theorem 2.3, we obtain the following oscillation criteria for Equation (1).

Corollary 2.3. *Assume that* $\sigma(t) \le t - \tau$, *and there exists a function* $\rho \in C'([t_0,\infty),\mathbb{R}^+)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[k(t-s)^{\alpha}(s-t_0)^{\beta}\rho(s)Q(s) - \frac{(1+a+b)\left(\psi + \frac{\rho'(s)}{\rho(s)}\right)^2}{4\sigma'(s)} r(\sigma(s))\rho(s) \right] ds > 0$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[k(t-s)^{\alpha}(s-t_0)^{\beta} Q(s) - \frac{(1+a+b)r(s)\psi^2}{4} \right] ds > 0$$

where $\alpha > \frac{1}{2}, \beta > \frac{1}{2}$ and $\psi = \frac{\beta t - (a+b)s + \alpha t_0}{(t-s)(s-t_0)}$. Then every solution of Equation (1) is oscillatory.

3. Examples

In this section, we provide three examples to illustrate the main results.

Example 3.1. *Consider the neutral differential equation*

$$\left(t^2\left(x(t) + 3x(t-\pi) + 2x(t+\pi)\right)'\right)' + 5tx(t-\pi) = 0,\ t \ge 1. \tag{31}$$

Here $r(t) = t^2, a(t) = 3, b(t) = 2, q(t) = 5t, \tau = \delta = \pi$, and $\sigma(t) = t - \pi$. By taking $n = 1$ and $\delta(t) = \frac{1}{t}$, it is easy to see that all conditions of Theorem 2.1 are satisfied and hence every solution of Equation (31) is oscillatory.

Example 3.2. *Consider the neutral differential equation*

$$\left(t^2\left(x(t) + 2x(t-2) + 2x(t+1)\right)'\right)' + \frac{100}{(t-1)^2}x(t-3) = 0,\ t \ge 4. \tag{32}$$

Here $r(t)=t^2, a(t)=2, b(t)=1, q(t)=\frac{100}{(t-1)^2}, \tau=2, \delta=1$, and $\sigma(t)=t-3$. By taking $\alpha=\beta=2$ and $\rho(t)=1$, it is easy to see that all conditions of Corollary 2.3 are satisfied and hence every solution of Equation (32) is oscillatory.

We conclude this paper with the following remark.

Remark 3.1. *The results presented in* [24] *are not applicable to* Equations (31) *and* (32) *since in these equations* $\int_{t_0}^{\infty}\frac{1}{r(t)}\mathrm{d}t<\infty$ *and the neutral term contains advanced arguments. Therefore, our results complement and generalize some of the known results in the literature.*

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