Modified Tunneling Radiation of Fermions from a Spherically Symmetric Spacetime with Dark Matter

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Abstract

In the paper, we use the generalized Dirac equation to study the Hawking temperature and entropy of a spherically symmetric spacetime with the dark matter. The results show that the dark matter can influence the thermodynamic properties of the black hole. Meanwhile, we find the GUP corrected temperature and entropy are not only determined by the nature of black but also related to the properties of tunneling particles. Besides, the GUP can slow down the increase of Hawking temperature and causes the remnants.

Keywords

Generalized Uncertainty Principle, Spherically Symmetric Spacetime, Dark Matter

1. Introduction

Based on the quantum effect, people found that the black holes can radiate particles [1]. The radiation of black hole got researchers attention, many new methods were presented to study it [2]-[4]. Parikh and Wilzcek put forward the quantum tunneling method, which was an effective method to discuss the black hole's Hawking radiation. With the help of the quantum tunneling method, they have calculated the massless scalar particle's tunneling rate and Hawking temperature of the spherically symmetrical spacetime [5]. Then, Kerner and Mann developed the quantum tunneling method and studied the fermions tunneling from Schwarzschild (SC) spacetime [6]. Later, the tunneling behavior of particles with 0 spin, 1/2 spin, 1 spin and 3/2 spin from black holes were investigated via Hamilton-Jacobi ansatz, which is another quantum tunneling method. However, the results obtained in previous work showed that the standard Hawking temperature of black holes was inverse to their mass. As Hawking announced, the black hole would emit all their mass as the temperature increase, this process causes the black holes evaporate over. In other words, black holes would loss all their information, which was called as information paradox of black hole [7] [8].

In order to solve the information paradox problem, people proposed many kinds of correction theories. Recently, people studied the physical properties of the black hole via the generalized uncertainty principle (GUP) [9]-[11]. In quantum gravity theory and string theory, people believe the existence of the minimal observable

length, which can be described by the GUP [12] [13]. In [14], based on the associative Heisenberg algebra $[x_i, p_j] = i\hbar \delta_{ij}[1 + \beta p^2]$, where p_j is the momentum operator and x_i is the position operator. One kind of GUP is expressed as

$$\Delta x \Delta p \geq \hbar \left[1 + \beta (\Delta p)^2 + \beta \langle p \rangle \right] / 2, \qquad (1)$$

with $\beta = \beta_0 M_p^{-2}$ is a small value, which represents the effects of quantum gravity, M_p being the Planck mass [15]. Adopted GUP, Banerjee and Ghosh investigated the thermodynamics of SC spacetime, their results showed that the black hole have remnant mass at the end of evolution [16]. Soon, combining with quantum tunneling method and GPU, the Hawking radiation of Schwarzschild black hole was studied by Nozari and Saghafi [17].

On other hand, astronomers predict 27% of the universe is dark matter. It is natural to think how can the dark matter influence the properties of black hole and whether different corrected methods deduce different results? In this paper, we will apply the GUP to investigate the Hawking temperature and entropy of spherically symmetric black hole with dark matter.

The remainder of the paper goes as follows. In Section 2, we overview of the spherically symmetric black hole with dark matter is provided. In Section 3, we corrected the Hawking temperature and entropy via GUP. The last section is the discussion and concludes.

2. The Spherically Symmetric Spacetime with Dark Matter

In curved space-time, the line element of spherically symmetric black hole with dark matter can be expressed as [18] [19]

$$ds^2 = -e^{\mu(r)}dt^2 + e^{-\nu(r)}dr^2 + r^2 d\Omega^2, \qquad (2)$$

where

$$e^{\mu(r)} = \frac{f(r)}{4r} \left[-C_2 + 2\xi c \int r^{\frac{3}{2}} \Big/ f(r)^{\frac{3}{2}} \, dr + 2\alpha c \int r^{\frac{5}{2}} \Big/ f(r)^{\frac{3}{2}} \, dr \right], \qquad (3)$$

$$e^{\nu(r)} = \left(1 - \alpha^2 r^2 - 2\alpha \xi r - \xi^2 - C_1/r \right)^{-1}, \qquad (4)$$

$$f(r) = 1 - \alpha^2 r^2 - 2\alpha \xi r - \xi^2 - C_1/r. \qquad (5)$$

In Equation (3)-Equation (5), the α, c, C_1, C_2 are parameters of the spacetime, note that ξ is a parameter of cold dark matter. In order to discuss the tunneling of the black hole, here we let parameter $c = 0$, $C_1 = 2M$, $C_2 = 4r$, Equation (3) and Equation (4) become to

$$e^{\mu(r)} = e^{-\nu(r)} = 1 - 2M/r - \alpha^2 r^2 - 2\alpha \xi r - \xi^2. \qquad (6)$$

Note that Equation (5) and Equation (6) have the same expression. Thus, one can use $f(r)$ to replace $e^{\mu(r)}$ and $e^{-\nu(r)}$. Now, the metric of spherically symmetric black hole with dark matter is given by

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \qquad (7)$$

The horizons of this black hole are determined by null super-surface equation. Now, assuming $f(r) = 0$, by a simple calculate, the roots can be expressed as r_c, r_H and r_0. According to the three roots, $f(r)$ can be written as

$$f(r) = -\alpha^2 (r - r_0)(r - r_H)(r - r_c)/r, \qquad (8)$$

where r_c is the cosmological horizon, r_H is the outer event horizon and r_0 is the inner event horizon. The three roots satisfy $r_0 < r_H < r_c$.

3. The GUP Corrected Hawking Temperature and Entropy of Spherically Symmetric Black Hole with Dark Matter

According to the GUP, Chen *et al.* have rewritten the original Dirac equation into generalized form. The massless generalized Dirac equation in curved spacetime is [20]

$$\left\{ i\gamma^0 \partial_0 + i\gamma^i \partial_i + i\hbar^2 \left(\partial_j \partial^j \right) \gamma^i \partial_i + i\overline{e}A_\mu \left(\frac{i}{\hbar} \right) \gamma^\mu \left[1 + \beta\hbar^2 \left(\partial_j \partial^j \right) \right] + i\gamma^\mu \Omega_\mu \left[1 + \beta\hbar^2 \left(\partial_j \partial^j \right) \right] \right\} \Psi = 0. \tag{9}$$

In order to discussing the fermions tunneling from black hole, people assume fermions with 1/2 spin have two states: spin up state and spin down state. Here we only consider the spin up state. The wave function of the spin up state is

$$\Psi_\uparrow = \begin{pmatrix} A \\ 0 \\ B \\ 0 \end{pmatrix} \exp\left[\frac{i}{\hbar} I(t,r,\theta,\phi) \right], \tag{10}$$

where I is the action of up spin state of fermions, which can be expanded in powers of $\hbar$. A, B are functions of coordinates t, r, θ, ϕ. In order to solve Equation (9), one needs to find a tetrad, which can construct a gamma matrix. The tetrad is

$$e_\mu^a = \mathrm{diag}\left(\sqrt{f(r)}, 1/\sqrt{f(r)}, r, r\sin\theta \right). \tag{11}$$

Then, the γ^μ matrices are

$$\gamma^t = f(r)^{-\frac{1}{2}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \gamma^r = \sqrt{f(r)} \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \gamma^\theta = \sqrt{g^{\theta\theta}} \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \gamma^\phi = \sqrt{g^{\phi\phi}} \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}. \tag{12}$$

In Equation (12), $\sqrt{g^{\theta\theta}} = r^{-1}$ and $\sqrt{g^{\phi\phi}} = (r\sin\theta)^{-1}$, σ^i are Pauli matrices. Substituting Equation (10) and Equation (12) into Equation (9), then using WKB approximation and ignoring the higher order of $\mathcal{O}(\hbar)$, one gets four Hamilton-Jacobi equations

$$-Aif(r)^{-\frac{1}{2}} \partial_t I - B\sqrt{f(r)} \partial_r I + KB\beta\sqrt{f(r)} \partial_r I = 0, \tag{13}$$

$$Bif(r)^{-\frac{1}{2}} \partial_t I - A\sqrt{f(r)} \partial_r I + KA\beta\sqrt{f(r)} \partial_r I = 0, \tag{14}$$

$$A\left\{ -\sqrt{g^{\theta\theta}} \partial_\theta I + K\beta\sqrt{g^{\theta\theta}} \partial_\theta I - i\sqrt{g^{\phi\phi}} \partial_\phi I + i\beta\sqrt{g^{\phi\phi}} \partial_\phi I - \sqrt{g^{\theta\theta}} \partial_\theta I + iK\beta\sqrt{g^{\theta\theta}} \partial_\theta I \right\} = 0, \tag{15}$$

$$B\left\{ -\sqrt{g^{\theta\theta}} \partial_\theta I + K\beta\sqrt{g^{\theta\theta}} \partial_\theta I - i\sqrt{g^{\phi\phi}} \partial_\phi I + i\beta\sqrt{g^{\phi\phi}} \partial_\phi I - \sqrt{g^{\theta\theta}} \partial_\theta I + iK\beta\sqrt{g^{\theta\theta}} \partial_\theta I \right\} = 0, \tag{16}$$

where $\mathrm{K} = g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\phi\phi}(\partial_\phi I)^2$. For solving the four equations above, one needs to carry out separation of variables of I. Adopting the following ansatz for the separation of variables

$$I = -\omega t + W(r) + \Theta(\theta,\phi), \tag{17}$$

where ω being the energy of this tunneling particle. One finds that Equation (15) and Equation (16) can be decoupled into the purely angular equation

$$\left(\sqrt{g^{\theta\theta}} \partial_\theta \Theta + i\sqrt{g^{\phi\phi}} \partial_\phi \Theta \right) \left\{ \beta\left[g^{rr}(\partial_r W)^2 + g^{\theta\theta}(\partial_\theta W)^2 + g^{\phi\phi}(\partial_\phi W)^2 \right] \right\} = 0. \tag{18}$$

In Equation (18), the β is a small coefficient associated with quantum gravity effective, it cannot be zero. Thus, we get an important relation

$$\sqrt{g^{\theta\theta}} \partial_\theta \Theta + i\sqrt{g^{\phi\phi}} \partial_\phi \Theta = 0. \tag{19}$$

Now, substituting Equation (17) and Equation (19) into Equation (15) and Equation (16), keep the first order term of β, yields

$$P_4\left(\partial_r W\right)^4 + P_2\left(\partial_r W\right)^2 + P_0 = 0, \tag{20}$$

where $P_4 = -2\beta f(r)^3$, $P_2 = f(r)^2$, $P_0 = -\omega^2$. Neglecting the higher order term of β and solving Equation (20) at the outer event horizon. The result of Equation (20) is

$$
\begin{aligned}
W(r) &= \pm\int \frac{1}{f(r)}\sqrt{\omega^2}\left[1+\beta\frac{\omega^2}{f(r)}\right]dr \\
&= \pm i\pi \frac{\omega r_H}{\alpha^2\left(r_0-r_H\right)\left(r_H-r_c\right)}\left[1-\frac{\beta}{2}\frac{4\omega^2\left(r_0 r_c - r_H\right)^2}{\alpha^2\left(r_0-r_H\right)^2\left(r_H-r_c\right)^2}\right] + \mathcal{O}(RealPart).
\end{aligned}
\tag{21}
$$

The plus (minus) means the outgoing (ingoing) wave. Here we only keep the imaginary part of Equation (24), because the real part $\mathcal{O}(RealPart)$ do not contribution to calculate the tunneling rate. The tunneling rate of fermions is

$$\Gamma = \frac{\Gamma_{out}}{\Gamma_{in}} = \frac{\exp\left(-2\operatorname{Im}I_+\right)}{\exp\left(-2\operatorname{Im}I_-\right)} = 4\pi \frac{\omega r_H}{\alpha^2\left(r_0-r_H\right)\left(r_H-r_c\right)}\left[1-\frac{\beta}{2}\frac{4\omega^2\left(r_0 r_c - r_H\right)^2}{\alpha^2\left(r_0-r_H\right)^2\left(r_H-r_c\right)^2}\right]. \tag{22}$$

Comparing with the Boltzman factor expression, the GUP corrected Hawking temperature of the black hole with dark matter is

$$T_H = T_0\left[1-\frac{\beta}{2}\frac{4\omega^2\left(r_0 r_c - r_H\right)^2}{\alpha^2\left(r_0-r_H\right)^2\left(r_H-r_c\right)^2}\right]^{-1}. \tag{23}$$

where $T_0 = \alpha^2\left(r_H-r_0\right)\left(r_H-r_c\right)/4\pi r_H$ is the semi-classical Hawking temperature of the black hole with dark matter. However, the temperature of an object should not depend on what it is emitting. Therefore, using the saturated form of the uncertainty principle $\omega = \hbar/2\Delta x$, one has

$$T_H = T_0\left[1-\frac{\beta}{2}\frac{\hbar^2\left(r_0 r_c - r_H\right)^2}{\alpha^2\Delta x^2\left(r_0-r_H\right)^2\left(r_H-r_c\right)^2}\right]^{-1}. \tag{24}$$

Then, near the event horizon of the black hole, the position uncertainty of a particle can be expressed [21] [22]

$$\Delta x = \varepsilon r_H, \tag{25}$$

where ε is a calibration factor and r_H is the event horizon of the black hole [23]. With the help of Equation (25), The temperature can be written as

$$T_H = T_0\left[1-\frac{\beta}{2}\frac{\left(r_0 r_c - r_H\right)^2}{\alpha^2 r_H^2\left(r_0-r_H\right)^2\left(r_H-r_c\right)^2}\right]^{-1} = T_0\left[1+\frac{\beta}{2}\frac{\left(r_0 r_c - r_H\right)^2}{\alpha^2 r_H^2\left(r_0-r_H\right)^2\left(r_H-r_c\right)^2}\right]. \tag{26}$$

In above equation, we set $\hbar = \varepsilon = 1$. Obviously, the GUP corrected temperature is lower than the semi-classical case. Besides, it is not only determined by parameters of α, M and ζ the β, which belongs to the effects of quantum gravity.

4. Conclusions

In the previous work, people found that the GUP can cause the remnants of black holes. For calculating the remnants, we need neglect the parameters of α, ξ. The metric of SC black hole is recovered, Equation (26) becomes to $T_H = \frac{1}{8\pi M}\left(1-\frac{\beta}{8M^2}\right)$. When considering $\left(M-dM\right)\left(1+\beta\omega^2\right) \simeq M$ with $dM = \omega$ and $\beta = \beta_0/M_p^2$, the temperature of black hole will stop increasing. The remnants are

$$M_{res} \simeq \frac{M_p^2}{\beta_0 \omega} \geq \frac{\sqrt{2} M_p}{\sqrt{\beta_0}}, T_{res} \leq \frac{\sqrt{2} \beta_0}{16 \pi M_p}, \tag{27}$$

where $\beta_0 > 10^{34}$. Equation (27) implies that the remnant mass and temperature are relate to the Planck mass M_p and the dimensionless parameter β_0.

In this work, with the help of GUP, we corrected the thermodynamic properties of spherically symmetric black hole with dark matter. We find that the GUP corrected temperatures is related to the properties of black hole (mass of black hole, α and dark matter ξ), and β, which represents the effects of quantum gravity. In addition, we can obtain the same result when consider the down spin state of fermions. It implies that the GUP may solve the information paradox of the black hole.

Acknowledgements

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