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# Existence and Multiple of Positive Solution for Nonlinear Fractional Difference Equations with Parameter

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## Abstract

**Let $1 < \nu \le 2$, $\lambda > 0$. We study the existence and multiple positive solutions of $\nu$-th nonlinear discrete fractional boundary value problem of the form $\begin{array}{l} -\Delta^{\nu} u(t) = \lambda f(t+\nu-1, u(t+\nu-1)), \\ u(\nu-2) = 0 = u(\nu+b+1). \end{array}$ By using a fixed-point theorem on cone, the parameter intervals of problem is established.**

## Keywords

**Fractional Difference Equations, Parameter Intervals, Positive Solution, Fixed-Point Theorem**

## 1. Introduction

There have been of great interest recently on fractional difference equations. It is caused by the development of the theory of fractional calculus and discrete fractional calculus, also by its applications, see [1]-[7]. We noted that most papers on discrete fractional difference equation are devoted to solvability of linear initial fractional difference equations [8] [9]. Recently, there are some papers dealing with the existence of solutions of nonlinear boundary value problems, we also refer the readers to [10] [11]. However, there are few papers consider parameter intervals of fractional difference boundary value problems. In the present work, our purpose is to the parameter intervals of the following fractional difference boundary value problem

$$-\Delta^{\nu} u(t) = \lambda f(t+\nu-1, u(t+\nu-1)), \qquad t \in [1, b+1]_N \tag{1.1}$$

$$u(\nu-2) = 0 = u(\nu+b+1). \tag{1.2}$$

where $\nu \in (1,2]$, $b \ge 2$ is an integer, $f : [\nu, \nu+b]_N \times R \to R$ is continuous, $f(t,u) > 0$ for $t \in [\nu, \nu+b]_N$ and $u > 0$. For $a, b \in R$, define $[a,b]_N = \{a, a+1, \cdots, b-1, b\}$.

F. M. Atici and P. W. E. [10] studied fractional difference boundary value problem

$$-\Delta^{\nu} u(t) = f(t+\nu-1, u(t+\nu-1)), t \in [1, b+1]_N, \tag{1.3}$$

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with the boundary value condition (1.2). By using Krasnosel'skii fixed point theorem under condition

(H1) $f(t,\xi)\ge 0$, $(t,\xi)\in[\nu,\nu+b]_N\times[0,\infty)$;

(H2) $f(t,u)=h(t)g(u)$, where $h$ is a positive function, $g$ is a non-negative function and $\lim\limits_{y\to 0^+}\frac{g(u)}{u}=0, \lim\limits_{y\to\infty}\frac{g(u)}{u}=\infty;$

(H3) $f(t,u)=h(t)g(u)$, where $h$ is a positive function, $g$ is a non-negative function and $\lim\limits_{y\to 0^+}\frac{g(u)}{u}=\infty, \lim\limits_{y\to\infty}\frac{g(u)}{u}=0.$

They get the following.

**Theorem 1.1**[10] Assume that conditions (H1) and (H2) are satisfied, then problem (1.1) and (1.2) has at least one solution. Assume that conditions (H1) and (H3) are satisfied, then problem (1.1) and (1.2) has at least one solution.

The following conditions will be used in the paper

(A1) $f(t,u)=h(t)g(u)$, where $h$ is a positive function, $g:[0,\infty)\to R$ is continuous, and there exist $t_n\to 0$ such that $g(t_n)>0, n=1,2,\cdots$;

(A2) $\sup_{r>0}\min_{r/4\le t\le r} g(t)>0$.

## 2. Preliminaries

Recall the factorial polynomial $t^{(n)}=\prod_{j=0}^{n-1}(t-j)=\frac{\Gamma(t+1)}{\Gamma(t+1-n)}$, where $\Gamma$ denotes the special Gamma function and if $t+1-j=0$ for some $j$, we assume the product is zero. We shall employ the convention that division at a pole yields zero. For arbitrary $\nu$, define $t^{(\nu)}=\frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$. We also appeal to the convention that $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{(\nu)}=0$. Let $\nu>0$, $\sigma(s)=s+1$ and $f$ defined on $\{a,a+1,\cdots\}$, Miller and Ross [12] have defined the $\nu$-th fractional sum of $f$ by

$$\Delta^{-\nu}f(t,a)=\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-1)}f(s). \tag{2.1}$$

where $t\in\{a,a+1,\cdots\}$, also define the $\nu$-th fractional difference

$$\Delta^{\nu}f(t):=\Delta^N\Delta^{-(N-\nu)}f(t),$$

where $\nu>0$ and $0\le N-1<\nu\le N$ with $N\in N$, $t\in\{a+N-\nu,a+N-\nu+1,\cdots\}$.

**Lemma 2.1** [10] Let $1<\nu\le 2$, $h:[1,\infty]_N\to R$, the unique solution problem

$$\begin{aligned}&-\Delta^{\nu}u(t)=h(t+\nu-1), t\in[1,b+1]_N,\\&u(\nu-2)=0=u(\nu+b+1).\end{aligned} \tag{2.2}$$

is $u(t)=\sum_{s=1}^{b+1}G(t,s)h(s+\nu-1), t\in[1,b+1]_N$, where

$$G(t,s)=\frac{1}{\Gamma(\nu)}\begin{cases}\frac{t^{(\nu-1)}(\nu+b+1-\sigma(s))^{(\nu-1)}}{(\nu+b+1)^{(\nu-1)}}-(t-\sigma(s))^{(\nu-1)}, & s<t-\nu+1\le b+1,\\ \frac{t^{(\nu-1)}(\nu+b+1-\sigma(s))^{(\nu-1)}}{(\nu+b+1)^{(\nu-1)}}, & t-\nu+1\le s\le b+1,\end{cases} \tag{2.3}$$

**Lemma 2.2** [10] The Green's function $G(t,s)$ in Lemma 2.1 satisfies the following conditions:

(i) $G(t,s)>0$ for $t\in[\nu-1,\nu+b]_N$ and $s\in[1,b+1]_N$;

(ii) $\max_{t\in[\nu-1,\nu+b]}G(t,s)=G(s+\nu-1,s)$ for $s\in[1,b+1]_N$;

(iii) There exists a positive number $\rho\in(0,1)$ such that for $s\in[1,b+1]_N$

$$\min_{\frac{b+\nu}{4}\le t\le\frac{3(b+\nu)}{4}} G(t,s) \ge \rho \max_{t\in[\nu-1,\nu+b]_N} G(t,s) = \rho G(s+\nu-1,s). \tag{2.4}$$

where

$$\rho = \min\left\{ \frac{1}{\left(\frac{3(b+\nu)}{3}\right)^{(\nu-1)}} \left[ \left(\frac{3(b+\nu)}{4}\right)^{(\nu-1)} - \frac{\left(\frac{3(b+\nu)}{4}-\sigma(1)\right)^{(\nu-1)}(\nu+b+1)^{(\nu-1)}}{(\nu+b+1-\sigma(1))^{(\nu-1)}} \right], \frac{\left(\frac{b+\nu}{4}\right)^{(\nu-1)}}{(b+\nu)^{(\nu-1)}} \right\}. \tag{2.5}$$

In the rest of the paper, we will use the fixed point index theory in cones to deal with (1.1) and (1.2).

**Lemma2.3** [12] Let $E$ be a Banach space, $P\subseteq E$ be a cone, and suppose that $\Omega_1,\Omega_2$ are bounded open balls of $E$ centered at the origin with $\bar{\Omega}_1\subset\Omega_2$. Suppose further that $A: P\cap(\bar{\Omega}_2\setminus\Omega_1)\to P$ is a completely continuous operator such that either

(i) $\|Au\|\le\|u\|$, $u\in P\cap\partial\Omega_1$ and $\|Au\|\ge\|u\|$, $u\in P\cap\partial\Omega_2$, or

(ii) $\|Au\|\ge\|u\|$, $u\in P\cap\partial\Omega_1$ and $\|Au\|\le\|u\|$, $u\in P\cap\partial\Omega_2$

holds, then $A$ has a fixed point in $P\cap(\bar{\Omega}_2\setminus\Omega_1)$.

We will need the following notations. Let

$$X := \{u \mid u:[\nu-2,\nu+b+1]_N \to R, u(\nu-2)=u(\nu+b+1)=0\}.$$

Then $X$ is a Banach space with the norm $\|u\|=\max\{|u(t)|: t\in[\nu-2,\nu+b+1]_N\}$.

So, $u$ is a solution of (1.1) and (1.2) if, and only if $u$ is a fixed point of the operator $T: X\to X$ defined by

$$Tu(t) := \lambda\sum_{s=1}^{b+1} G(t,s) f(s+\nu-1,u(s+\nu-1)), t\in[\nu-2,\nu+b+1]_N.$$

Note $Y=[\frac{\nu+b}{4},\frac{3(\nu+b)}{4}]_N$, let $\rho$ be defined by (2.5) and define cones $P$ in $X$ by $P=\{u\in X: \min_{t\in Y} u(t)\ge\rho\|u\|\}$. For some $r>0$, $P_r=\{u\in P:\|u\|\le r\}$. Since $X$ is finite dimensional, we have the $T: X\to X$ is compact. Obviously, $T(P)\subseteq P$.

**Lemma 2.4** Suppose that conditions (A1) hold, and there exist two different positive numbers $a$ and $b$ such that

$$\max_{0\le t\le a} g(t)\le\frac{a}{\lambda A}, \min_{b/4\le t\le b} g(t)\ge\frac{b}{\lambda B}, \quad \text{where } A=\max_{t\in[1,b+1]_N}\sum_{s=1}^{b+1}G(t,s)h(s), B=\min_{t\in Y}\sum_{s=1}^{b+1}G(t,s)h(s).$$

Then, problem (1.1), (1.2) has at least one positive solution $u^*\in P$ such that $\min\{a,b\}\le\|u^*\|\le\max\{a,b\}$.

**Proof.** We can suppose that $a<b$. For $u\in\partial P_a$, $t\in[1,b+1]_N$, there is $g(u(t))\le\frac{a}{\lambda A}$, then

$$(Tu)(t)=\lambda\sum_{s=1}^{b+1}G(t,s)f(s+\nu-1,u(s+\nu-1))=\lambda\sum_{s=1}^{b+1}G(t,s)h(s)g(u(s))\le\lambda\sum_{s=1}^{b+1}G(t,s)h(s)\frac{a}{\lambda A}=\lambda A\cdot\frac{a}{\lambda A}=a,$$

these mains that for $u\in\partial P_a$, there is $\|Tu\|\le\|u\|$. For $u\in\partial P_b$, $t\in Y$, there is $g(u(t))\ge\frac{b}{\lambda B}$, then

$$(Tu)(t)=\lambda\sum_{s=1}^{b+1}G(t,s)h(s)g(u(s))\ge\lambda\sum_{s\in Y}G(t,s)h(s)g(u(s))\ge\lambda\sum_{s\in Y}G(t,s)h(s)\frac{b}{\lambda B}=\lambda B\cdot\frac{b}{\lambda B}=b,$$

these mains that for $u\in\partial P_b$, there is $\|Tu\|\ge\|u\|$. By using Lemma 2.3, there exist $u^*\in\bar{\Omega}_b\setminus\Omega_a$ such that $Tu^*=u^*$. This means that, $u^*$ is a solution of problems (1.1), (1.2) and $a\le\|u^*\|\le b$. Also, because $\|u^*\|\ge a>0$, so $u^*(t)>0$ for $t\in Y$, taking into account that conditions(A1) and (A2) hold and $Tu^*=u^*$, we have that

$u^*(t)>0$ for $t\in[1,b+1]_N$, *i.e.* $u^*$ is a positive solution of (1.1), (1.2).

## 3. Main Results

For some $r>0$, denote $\lambda^*=\frac{1}{A}\sup_{r>0}\frac{r}{\max_{0\le t\le r} g(t)}, \lambda^{**}=\frac{1}{B}\inf_{r>0}\frac{r}{\max_{r/4\le t\le r} g(t)}$.

By using Lemma 2.4, we get

**Theorem 3.1** Assume that (A1) hold, and $\lim_{t\to 0}\frac{g(t)}{t}=+\infty$ and $\lim_{t\to+\infty}\frac{g(t)}{t}=+\infty$, then, there exist $\lambda^*>0$, for every $\lambda\in(0,\lambda^*)$, problem (1.1) and (1.2) has at least two positive solutions.

**Theorem 3.2** Assume that (A1) hold, and $\lim_{t\to 0}\frac{g(t)}{t}=+\infty$ or $\lim_{t\to\infty}\frac{g(t)}{t}=+\infty$, then, for every $\lambda\in(0,\lambda^*)$, problem (1.1) and (1.2) has at least one positive solutions.

**Theorem 3.3** Assume that (A1) hold, and $\lim_{t\to 0}\frac{g(t)}{t}=0$ and $\lim_{t\to\infty}\frac{g(t)}{t}=0$, then, for every $\lambda\in(\lambda^{**},\infty)$, problem (1.1) and (1.2) has at least two positive solutions.

**Theorem 3.4** Assume that (A1) and (A2) hold, and $\lim_{t\to 0}\frac{g(t)}{t}=0$ or $\lim_{t\to+\infty}\frac{g(t)}{t}=0$, then, for every $\lambda\in(\lambda^{**},\infty)$, problem (1.1) and (1.2) has at least one positive solutions.

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