Journal of Applied Mathematics and Physics, 2015, 3, 725-729
Published Online July 2015 in SciRes. http://www.scirp.org/journal/jamp
http://dx.doi.org/10.4236/jamp.2015.37086

Category of Attractor and Its Application

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Received 6 February 2015; accepted 23 June 2015; published 30 June 2015

Abstract

In this paper, we provide a new approach to study the geometry of attractor. By applying category, we investigate the relationship between attractor and its attraction basin. In a complete metric space, we prove that the categories of attractor and its attraction basin are always equal. Then we apply this result to both autonomous and non-autonomous systems, and obtain a number of corresponding results.

Keywords

Ljusternik-Schnirelmann Category, Attractor, Attraction Basin

1. Introduction

Attractors of a given system are of crucial importance, this is because that much of longtime dynamics is represented by the dynamics on and near the attractors. It is well known that the global attractors of dynamical systems can be very complicated. The geometry can be very pathological, even in the finite dimensional situation. To have a better understanding on the dynamics of a system, it is quite necessary for us to study the topology and geometry of the attractors. In the past few decades, there appeared many studies. In [1], Kapitanski and Rodnianski studied the shape of attractors of continuous semi-dynamical systems on general metric spaces. They proved that the global attractor has the same shape as the state space. Moreover, using the results on the shape of attractors, they developed an elementary Morse theory for an attractor. Lately, the author of [2] studied the Morse theory of attractors for semiflows on complete metric spaces by constructing continuous Lyapunov functions, and he introduced the concept of critical groups for Morse sets and established Morse inequalities and Morse equations for attractors. To study the geometry of the attractors, some concepts such as Lyapunov exponents, the Hausdorff dimension and the fractal dimension were also proposed, see [3] [4] etc. Recently, in [5] author studied the geometrical property of the global attractor for a class of symmetric p-Laplacian equations by means of Z_2 index, obtained some lower estimates for the fractal dimension of the global attractor.

In this paper, by using Ljusternik-Schnirelmann category (category for short), we try to provide a new approach to studying the geometry of the global attractor. Category is a topological invariant, which often be used in the estimate of the lower bound of the number of critical points, see [6]. Here we investigate the relationship between attractor and attraction basin in the sense of category. In a complete metric space, for asymptotic compact semiflow, we obtain that the categories of attractor and attraction basin are always equal. This result match with the result in [1]. Now we can directly describe this result by category. The result will be of most interest when we choose X be some special metric space. Finally, we have to point out that it is generally not very

How to cite this paper: Wei, J.Y., Li, Y.J. and Li, M.S. (2015) Category of Attractor and Its Application. *Journal of Applied Mathematics and Physics*, **3**, 725-729. http://dx.doi.org/10.4236/jamp.2015.37086

easy to compute the category of a given space. However, we can see there are more and more new results and methods about calculation of category, see [7] [8] etc.

We will prove the main results in Section 3 and give some applications in Section 4. Before that we provide some preliminaries and results in Section 2.

2. Preliminaries

We recall some basic definitions and facts in the theory of dynamical systems for semiflows on complete metric spaces. Let X be a complete metric space with metric $d(\cdot,\cdot)$.

Definition 2.1 A semiflow (semidynamical system) on X is a continuous mapping $S:\mathbb{R}^+\times X\to X$ that satisfies

$$S(0,x)=x,\quad S(t+s,x)=S(t,S(s,x))\ \text{ for all }\ x\in X\ \text{ and }\ t,s\geq 0.$$

We usually write $S(t,x)$ as $S(t)x$. Therefore a semiflow S can be viewed as a family of operators $\{S(t)\}_{t\geqslant 0}$ satisfying:

$$S(0)=id_X,\quad S(t+s)=S(t)S(s)\quad (\forall t,s\geqslant 0).$$

From now on, we will always assume that there has been given a semidynamical system $S(t)$ on X; Moreover, we assume $S(t)$ is asymptotically compact, that is, $S(t)$ satisfies the following assumption:

$(\mathbf{A})$: For any bounded sequence $x_n\in X$ and $t_n\to+\infty$, if the sequence $S(t_n)x_n$ is bounded, then it has a convergent subsequence.

The asymptotic compactness property (A) is fulfilled by a large number of infinite dimensional semiflows generated by PDEs in application [4].

Let A be a subset of X. We say that A attracts $B\subset X$, if for any $\varepsilon>0$ there exists a $T>0$ such that

$$S(t)B\subset B(A,\varepsilon),\quad \forall t>T.$$

The attraction basin of A, denote by $\Omega(A)$, is defined as:

$$\Omega(A)=\{x\,|\,\lim_{t\to\infty}d(S(t)x,A)=0\}.$$

The set A is said to be positively invariant (resp. invariant), if

$$S(t)A\subset A(resp.\ S(t)A=A),\quad \forall\ t\geqslant 0.$$

Definition 2.2 A compact set $\mathcal{A}\subset X$ is said to be an attractor of $S(t)$, if it is invariant and attracts a neighborhood of U itself. An attractor $\mathcal{A}$ is said to be the global attractor of $S(t)$, if it attracts each bounded subset of X.

Let U be an open subset of X, and K be a closed subset of U with $\overline{K}\subset int\,U$.

Definition 2.3 A function $\alpha(x)\in C(X)$ is said to be coercive with K, if for any $\varepsilon>0$,

$$\alpha(x)\geq\delta>0,\quad \forall x\in U\setminus B(K,\varepsilon).$$

In order to prove our result, we need following theorem (see Theorem 3.5 in [2]). Let there be given an attractor $\mathcal{A}$ with attraction basin $\Omega=\Omega(\mathcal{A})$.

Theorem 2.4 ([2]) The attractor $\mathcal{A}$ has radially unbounded Lyapunov function $V(x)$ on Ω such that

$$V(x)\equiv 0\ \ (on\mathcal{A}),\quad D^+V(X)\leqslant -v(x),\ \forall\ x\in\Omega,$$

where $v\in C(\Omega)$ is a nonnegative function satisfying

$$v(x)>0,\ (x\in\Omega\setminus\mathcal{A}),\ \ v(x)=0\ (x\notin\Omega\setminus\mathcal{A}.)$$

Remark 2.5 We emphasize that the $V(x)$ is coercive with $\mathcal{A}$ on Ω.This point is not contained in the statement of Theorem 2.4, but we can obtain this result from the proof of the Theorem 3.5 in [2] easily.

In the following, we recall some basic results on the Ljusternik-Schnirelmann category (category for short).

Definition 2.6 Let M be a topological space, $A\subset M$ be a closed subset. Set

$$cat_M(A)=\inf\{m\in\mathbb{N}\bigcup\{+\infty\}\,|\,\exists\ m\ contractible\ closed\ subsets\ of\ M:$$
$$F_1,F_2,\ldots,F_m\ such\ that\ A\subset\bigcup_{i=1}^n F_i\}.$$

A set F is called contractible (in M), if $\exists \eta:[0,1]\times M \to M$ such that $\eta(0,\cdot)=id_M$ and $\eta(1,F)=$ one point set.

The category defined above has properties as follows.

Lemma 2.7 Properties for the category:

1) $cat_M(A)=0 \Leftrightarrow A=\varnothing$;

2) (Monotonicity) $A\subset B \Rightarrow cat_M(A)\le cat_M(B)$;

3) (Subadditivity) $cat_M(A\bigcup B)\le cat_M(A)+cat_M(B)$;

4) (Deformation nondecreasing) If $\eta:[0,1]\times M \to M$ is continuous such that $\eta(0,\cdot)=id_M$, then

$$cat_M(A)\le cat_M(\overline{\eta(1,A)});$$

5) (Continuity) If A is compact, then there is a closed neighborhood N of A such that

$$A\subset int\,N \quad \text{and} \quad cat_M(A)=cat_M(N);$$

6) (Normality) $cat_M(\{p\})=1, \forall p\in M$.

For the proof of this lemma, we refer readers to [6].

Remark 2.8 By (2) and (5), we can easily obtain that if A is compact, then there exists a ε-neighborhood $B(A,\varepsilon)$ of A, such that $cat_M(A)=cat_M(B(A,\varepsilon))$.

Just by the definition of category, we can prove the following lemma:

Lemma 2.9 Let X,Σ are topology spaces, and $\mathbb{X}=X\times\Sigma$. F is a subset of X. If $cat_X F=1$, then $cat_{\mathbb{X}}(F\times\Sigma)=cat_\Sigma\Sigma.$

3. Category of Attractor

The main results can be stated as follows:

Theorem 3.1 Let X be a complete metric space and $S(t)$ is a semiflow on X, which is asymptotically compact. Let $\mathcal{A}$ be an attractor of $S(t)$ on X with attraction basin $\Omega=\Omega(\mathcal{A})$. Then $cat_\Omega(\Omega)=cat_\Omega(\mathcal{A})$.

Proof. Since $\mathcal{A}\subset\Omega$, by monotonicity,

$$cat_\Omega(\mathcal{A})\le cat_\Omega(\Omega). \tag{3.1}$$

Since $\mathcal{A}$ is compact, by continuity (Remark 2.8}), fixed $\varepsilon>0$ small enough, we have

$$cat_\Omega(\overline{B(\mathcal{A},\varepsilon)})=cat_\Omega(\mathcal{A}). \tag{3.2}$$

If we find a set $K\subset B(\mathcal{A},\varepsilon)$ such that

$$cat_\Omega\Omega\le cat_\Omega(K), \tag{3.3}$$

by using monotonicity again and (3.2}), we have

$$cat_\Omega(\Omega)\le cat_\Omega(\mathcal{A}). \tag{3.4}$$

Then combine (3.1}) and (3.4), we will obtain the result $cat_\Omega(\Omega)=cat_\Omega(\mathcal{A})$.

Now the rest of the work in this proof is in finding the appropriate set K, which is subset of $B(\mathcal{A},\varepsilon)$ and satisfies (3.3). In order to obtain the proper set K, the key tool here is the level set of Lyapunov function on attractor $\mathcal{A}$. Thanks to Theorem 2.4, we can construct a Lapunov function $V(x)$. For $a\in R$, we devote by V_a the level set of V in Ω,

$$V_a=\{x\in\Omega\,|V(x)\le a\}.$$

V_a is clearly positively invariant and satisfies $\mathcal{A}\subset V_a\subset\Omega$ as $a>0$.

By the Remark 2.5, $V(x)$ is coercive with $\mathcal{A}$, that is for the fixed ε above, there exists $\delta>0$ such that

$$V(x)\ge\delta>0,\quad \forall x\in\Omega\setminus B(\mathcal{A},\varepsilon).$$

Hence, let $0<a<\delta$, we have $V_a\subset B(\mathcal{A},\varepsilon)$.

We use the method in [2], Define a function $t(x)$ on Ω as

$$t(x)=\begin{cases}\sup\{t\ge 0\mid S([0,t))x\subset\Omega\setminus V_a\}, & x\in\Omega\setminus V_a;\\ 0, & x\in V_a.\end{cases}$$

Here $t(x)<+\infty$ and $t(x)$ is continuous on Ω. (See Theorem 5.1 and Lemma 5.2 in [2], in which Ω replaced by V_b.) Define

$$\eta(\sigma,x)=S(\sigma t(x))x,\ \ x\in\Omega,\sigma\in[0,1]$$

Then $\eta:[0,1]\times\Omega\to\Omega$ satisfies:

$$\eta(0,\cdot)=id_{\Omega},\quad \eta(1,\Omega)\subset V_a,$$

Since $t(x)$ is continuous on Ω, we see that η is a continuous mapping, by deformation nondecreasing and monotonicity, we have

$$cat_{\Omega}(\Omega)\le cat(\overline{\eta(1,\Omega)})\le cat_{\Omega}(V_a).$$

Now we just let $K=V_a$, which completes the proof.

Now to extend our result to non-autonomous case, we consider a skew-product system, which consists of a base semiflow, and a semiflow on the phase space that is in some sense driven by the base semiflow. More precisely, the base semiflow consists of the base space Σ, which we take to be a metric space with metric ρ, and a group of continuous transformations $\{\theta_t\}_{t\in R}$ from Σ into itself such that $\theta_0=id_{\Sigma}$; $\theta_t\theta_s=\theta_{t+s}$ for all $t,s\ge 0$.

The dynamics on the phase space (X,d) is given by a family of continuous mappings

$$R^+\times\Sigma\ni(t,\sigma)\to\varphi(t,\sigma)\in\mathcal{C}(X)$$

satisfy the cocycle property

1) $\varphi(0,\sigma)=id_X$ for all $\sigma\in\Sigma$;
2) $\varphi(t+s,\sigma)=\varphi(t,\theta_s\sigma)\varphi(s,\sigma)$ for all $t,s\ge 0$ and $\sigma\in\Sigma$;
3) $R^+\times\Sigma\ni(t,\sigma)\to\varphi(t,\sigma)x\in X$ is continuous.

Then we can define an autonomous semigroup $T(\cdot)$ on $\mathbb{X}=X\times\Sigma$ by setting

$$T(t)(x,\sigma)=(\varphi(t,\sigma)x,\theta_t\sigma),\quad t\ge 0.$$

If we assume that the autonomous semigroup $T(\cdot)$ is asymptotically compact on $\mathbb{X}$, and has an global attractor $\mathbb{A}$, then we can generalize Theorem 3.1 to the non-autonomous case as follows:

Corollary 3.2 Let $T(t)$ is a asymptotically compact semiflow on $\mathbb{X}$. If $\mathbb{A}$ is a global attractor of $T(t)$ on $\mathbb{X}$. Then $cat_{\mathbb{X}}\mathbb{X}=cat_{\mathbb{X}}\mathbb{A}$.

4. Applications

In this section, we further apply our results to some special metric space X, we will see some interesting results.

Example 1. Assume $X=S^n(or\,T^n)$. Let $S(t)$ is a asymptotically compact semiflow on X. If $\mathcal{A}$ is a global attractor of $S(t)$ on X. Then $\mathcal{A}=X$.

Proof. Suppose the contrary. Then there exist at least one point $s\in X$ such that $s\notin\mathcal{A}$. Then we deduce that $\mathcal{A}\subset X\backslash\{s\}\subset X$. By the monotonicity, we have $cat_X\mathcal{A}\le cat_X(X\backslash\{s\})\le cat_X X$.

Note that $X\backslash\{s\}=S^n\backslash\{s\}$ is a punctured n-dimensional sphere,

$$cat_X(X\backslash\{s\})=1,\quad cat_X X=cat_{S^n}S^n=2,$$

Thus, we have $cat_X\mathcal{A}<cat_X X$.

On the other hand, by virtue of Theorem, we have $cat_X\mathcal{A}=cat_X X$, which leads to a contradiction! Hence, the global attractor $\mathcal{A}$ must be phase space X itself.

Using similar arguments, one can prove the case of $X=T^n$.

Example 2. In skew-product flow case, we assume $\mathbb{X} = X \times \Sigma = S^m \times T^n$. Let $T(t)$ is a asymptotically compact semiflow on $\mathbb{X}$. If $\mathbb{A}$ is a global attractor of $T(t)$ on $\mathbb{X}$. Then $\mathbb{A} = \mathbb{X}$.

Proof. Suppose the contrary. Then there exist at least one point $s \in X$ such that $s \notin \mathbb{A}$. Then we deduce that $\mathbb{A} \subset (X \setminus \{s\}) \times \Sigma \subset \mathbb{X}$. By the monotonicity, we have

$$cat_{\mathbb{X}}\mathbb{A} \le cat_{\mathbb{X}}((X \setminus \{s\}) \times \Sigma) \le cat_{\mathbb{X}}\mathbb{X}.$$

Note that $X \setminus \{s\} = S^m \setminus \{s\}$ is a punctured m- dimensional ball, $cat_X(X \setminus \{s\}) = 1,$

By Lemma 2.9, $cat_{\mathbb{X}}(X \setminus \{s\}) \times T^n = cat_{T^n} T^n = n+1,$ while by Theorem 15 in [7], we have

$$cat_{\mathbb{X}}\mathbb{X} = cat_{\mathbb{X}} S^m \times T^n = n+2$$

Thus, we have $cat_{\mathbb{X}}\mathbb{A} < cat_{\mathbb{X}}\mathbb{X}.$

On the other hand, by Virtue of Theorem 3.1, we have $cat_{\mathbb{X}}\mathbb{A} = cat_{\mathbb{X}}\mathbb{X},$ which leads to a contradiction! Hence, we obtain $\mathcal{A} = \mathbb{X}$.

Remark 3.3 If $X = T^m$, since

$$cat_{T^m} T^m > cat_{T^m}(T^m \setminus \{s\}), \quad \text{and} \quad cat_{T^n \times T^m}(T^n \times T^m) = n + m + 1,$$

we can obtain the same result.

Remark 3.4 By Theorem 15.7 in [9], if $\mathbb{A}$ is a global attractor of $S(t)$ on $\mathbb{X}$. Then $A(\cdot)$ with $A(\sigma) = \pi_\sigma \mathbb{A}$ is the pullback attractor of the skew-product flow (ϕ, θ), where $\pi_\sigma \mathbb{A}$ is the section of $\mathbb{A}$ over $\sigma \in \Sigma$. Since corollary 3.2, we can show that the pull back attractor of the skew-product flow (ϕ, θ) must be $A(\cdot) = X$.

Acknowledgements

This work was supported by the National Nature Science Foundation of China (11261027) and Longyuan youth innovative talents support programs of 2014, and by the innovation Funds of principal (LZCU-XZ2014-05).

Support

This work was supported by the National Nature Science Foundation of China (11261027) and Longyuan youth innovative talents support programs of 2014, and by the innovation Funds of principal (LZCU-XZ2014-05).

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