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Classifying Exact Traveling Wave Solutions to the Coupled-Higgs Equation

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Abstract

By the complete discrimination system for polynomials, we classify exact traveling wave solutions to the Coupled-Higgs Equation.

Keywords

Traveling Wave Solution, Complete Discrimination System for Polynomials, The Coupled-Higgs Equation

1. Introduction

There are many methods to study the exact traveling wave solutions of the nonlinear differential Equations, such as the inverse scattering method [1], Jacobi elliptic function expansion method [2], homogeneous balance method [3], (G'/G)-expansion method [4], and so on. At the same time, Liu [5] introduced the complete discrimination system method to give the classification of exact traveling wave solutions to some nonlinear equations, the method is simple and efficient. Using this method, some new traveling wave solutions were obtained to the Zhiber-Shabat Equation [6].

In this paper, we focus on the Coupled-Higgs Equation to classify its traveling wave solutions. A. Jabbari *et al*. [4] have got some traveling wave solutions to the Coupled-Higgs Equation. By Liu's method, we'll classify exact traveling wave solutions to the Coupled-Higgs Equation.

2. The Traveling Wave Solutions to the Coupled-Higgs Equation

The Coupled-Higgs Equation reads as

$$u_{tt} - u_{xx} + |u|^2 u - 2uv = 0, \tag{1}$$

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$$v_{tt}+v_{xx}-\left(|u|^2\right)_{xx}=0. \quad (2)$$

We introduce transformation as follows

$$u=e^{i\theta}U(\eta),\quad v=V(\eta),\quad \eta=\lambda(x-kt),\quad \theta=kx+t. \quad (3)$$

Substituting Equation (3) into Equation (1) and Equation (2) yields nonlinear ordinary differential equation as follows

$$\left(\lambda^2k^2+\lambda^2\right)U''-\left(k^2+1\right)U+U^3-2UV=0, \quad (4)$$

$$\left(k^2+1\right)V''-2\left(U'^2+UU''\right)=0. \quad (5)$$

Integrating Equation (5) twice with respect to η, and setting the integration constant to zero yields

$$V=\frac{U^2}{k^2+1}. \quad (6)$$

Substituting Equation (6) into Equation (4) yields the following nonlinear ordinary difference equation

$$U''=\frac{1-k^2}{\lambda^2\left(1+k^2\right)^2}U^3+\frac{1}{\lambda^2}U. \quad (7)$$

Integrating Equation (7) once with respect to η yields

$$\left(U'\right)^2=a_4U^4+a_2U^2+a_0, \quad (8)$$

where

$$a_4=\frac{1-k^2}{2\lambda^2\left(1+k^2\right)^2},\quad a_2=\frac{1}{\lambda^2}, \quad (9)$$

and a_0 is an arbitrary constant.

In order to find the traveling wave solutions to Equation (1) and Equation (2), let us solve Equation (8). In this article, there are two cases to discuss the exact solutions of Equation (8) according to the coefficient a_4.

Case 2.1. When $a_4>0$, we take the transformation as follows

$$w=\left(a_4\right)^{\frac{1}{4}}U,\quad \eta_1=\left(a_4\right)^{\frac{1}{4}}\eta. \quad (10)$$

Substituting (10) into (8) yields

$$w_{\eta_1}^2=w^4+pw^2+q, \quad (11)$$

where

$$p=\frac{1+k^2}{|\lambda|}\cdot\sqrt{\frac{2}{1-k^2}},\quad q=a_0. \quad (12)$$

In order to obtain the solutions to Equation (11), we let

$$w^2=\psi. \quad (13)$$

Substituting (13) into (11) yields

$$\psi_{\eta_1}^2=4\psi\left(\psi^2+p\psi+q\right). \quad (14)$$

Furthermore, integrating Equation (14), we have

$$\int \frac{\mathrm{d}\psi}{\sqrt{\psi F(\psi)}} = \pm 2(\eta_1 - \eta_0), \tag{15}$$

where

$$F(\psi) = \psi^2 + p\psi + q, \tag{16}$$

and η_0 is an integrating constant. Let $\Delta = p^2 - 4q$ be discriminant of second order polynomial $F(\psi)$, there are four cases for the solutions of Equation (15) according to the cases of roots of $F(\psi)$.

Case 2.1.1. $\Delta = 0$, for $\psi > 0$. If $p > 0$, then the explicit solution of Equation (15) is

$$\psi = \frac{p}{2}\tan^2\left[\sqrt{\frac{p}{2}}(\eta_1 - \eta_0)\right]. \tag{17}$$

If $p = 0$, then the explicit solution of Equation (15) is

$$\psi = \frac{1}{(\eta_1 - \eta_0)^2}. \tag{18}$$

Case 2.1.2. $\Delta > 0$, $q = 0$, for $\psi > -p$. If $p > 0$, then the explicit solution of Equation (15) is

$$\psi = p\coth^2\left[\sqrt{p}(\eta_1 - \eta_0)\right] - p. \tag{19}$$

Case 2.1.3. $\Delta > 0$, $q \neq 0$. Suppose that $\alpha < \beta < \gamma$, one of α and β and γ is zero, and others are two roots of $F(\psi)$. As $\alpha < \psi < \beta$, the explicit solution of Equation (15) is

$$\psi = \alpha + (\beta - \alpha)\mathrm{sn}^2\left(\sqrt{\gamma - \alpha}(\eta_1 - \eta_0), m\right). \tag{20}$$

When $\psi > \gamma$, the explicit solution of Equation (15) is

$$\psi = \frac{-\beta\,\mathrm{sn}\left(\sqrt{\gamma - \alpha}(\eta_1 - \eta_0), m\right) + \gamma}{\mathrm{cn}\left(\sqrt{\gamma - \alpha}(\eta_1 - \eta_0), m\right)}, \tag{21}$$

where $m^2 = \dfrac{\beta - \alpha}{\gamma - \alpha}$.

Case 2.1.4. $\Delta < 0$, as $\psi > 0$, the explicit solution of Equation (15) is

$$\psi = \frac{2\sqrt{q}}{1 + \mathrm{cn}\left(2q^{\frac{1}{4}}(\eta_1 - \eta_0), m\right)} - \sqrt{q}, \tag{22}$$

where $m^2 = \dfrac{1}{2}\left(1 - \dfrac{p}{2\sqrt{q}}\right)$.

Case 2.2. When $a_4 < 0$, we take the transformation as follows

$$w = (-a_4)^{\frac{1}{4}}U, \quad \eta_1 = (-a_4)^{\frac{1}{4}}\eta. \tag{23}$$

Substituting (23) into (8) yields

$$w_{\eta_1}^2 = -\left(w^4 - pw^2 - q\right), \tag{24}$$

where

$$p = \frac{1 + k^2}{|\lambda|}\cdot\sqrt{\frac{2}{k^2 - 1}}, \quad q = a_0. \tag{25}$$

In order to obtain the solutions to Equation (24), we let

$$w^2 = \psi . \tag{26}$$

Substituting (26) into (24) yields

$$\psi_{\eta_1}^2 = 4\psi\left(-\psi^2 + p\psi + q\right). \tag{27}$$

Furthermore, integrating Equation (27), we have

$$\int \frac{\mathrm{d}\psi}{\sqrt{\psi F(\psi)}} = \pm 2(\eta_1 - \eta_0), \tag{28}$$

where

$$F(\psi) = -\psi^2 + p\psi + q, \tag{29}$$

And η_0 is an integrating constant. Let $\Delta = p^2 + 4q$ be discriminant of second order polynomial $F(\psi)$, there are two cases for the solutions of Equation (28) according to the cases of roots of $F(\psi)$.

Case 2.2.1. $\Delta > 0$, $q = 0$, for $0 < \psi < p$, then the explicit solutions of Equation (28) is

$$\psi = p - p\tanh^2\left[\sqrt{p}(\eta_1 - \eta_0)\right]. \tag{30}$$

Case 2.2.2. $\Delta > 0$, $q \neq 0$, this case is completely similar to Case 2.1.3. So the Equation (20) and (21) are the explicit solution of Equation (28).

From the above we know that Equations (17)-(22) and (30) are all possible solutions of Equations (15) and (28). According to Equations (13), (10), (6), (3) and (26), (23), (6), (3), we can give the classification of all single traveling wave solutions to the Coupled-Higgs Equation with respective parameter conditions as follows:

$$u_1 = \pm\frac{1+k^2}{\sqrt{1-k^2}}\exp\left[i(kx+t)\right]\tan\left[\sqrt{\frac{1+k^2}{|\lambda|}\sqrt{\frac{1}{2(1-k^2)}}}\left(\sqrt[4]{\frac{1-k^2}{2\lambda^2(1+k^2)^2}}\eta - \eta_0\right)\right],$$

$$v_1 = \frac{1+k^2}{1-k^2}\tan^2\left[\sqrt{\frac{1+k^2}{|\lambda|}\sqrt{\frac{1}{2(1-k^2)}}}\left(\sqrt[4]{\frac{1-k^2}{2\lambda^2(1+k^2)^2}}\eta - \eta_0\right)\right],$$

$$u_2 = \pm\left(\frac{1-k^2}{2\lambda^2(1+k^2)^2}\right)^{-\frac{1}{4}}\cdot\frac{1}{\sqrt[4]{\frac{1-k^2}{2\lambda^2(1+k^2)^2}}\eta - \eta_0}\cdot\exp\left[i(kx+t)\right],$$

$$v_2 = |\lambda|\sqrt{\frac{2}{1-k^2}}\cdot\frac{1}{\left(\sqrt[4]{\frac{1-k^2}{2\lambda^2(1+k^2)^2}}\eta - \eta_0\right)^2},$$

$$u_3 = \pm\left(1+k^2\right)\sqrt{\frac{2}{1-k^2}}\exp\left[i(kx+t)\right]\cdot\sqrt{\coth^2\left[\sqrt{\frac{1+k^2}{|\lambda|}\sqrt{\frac{2}{1-k^2}}}\left(\sqrt[4]{\frac{1-k^2}{2\lambda^2(1+k^2)^2}}\eta - \eta_0\right)\right] - 1},$$

$$v_3=\frac{2\left(1+k^2\right)}{1-k^2}\cdot\left\{\coth^2\left[\sqrt{\frac{1+k^2}{|\lambda|}\sqrt{\frac{2}{1-k^2}}}\left(\sqrt[4]{\frac{1-k^2}{2\lambda^2\left(1+k^2\right)^2}}\eta-\eta_0\right)\right]-1\right\},$$

$$u_4=\pm\left(\frac{1-k^2}{2\lambda^2\left(1+k^2\right)^2}\right)^{-\frac{1}{4}}\exp\left[i\left(kx+t\right)\right]\cdot\sqrt{\alpha+\left(\beta-\alpha\right)\operatorname{sn}^2\left[\sqrt{\gamma-\alpha}\left(\sqrt[4]{\frac{1-k^2}{2\lambda^2\left(1+k^2\right)^2}}\eta-\eta_0\right),m\right]},$$

$$v_4=|\lambda|\sqrt{\frac{2}{1-k^2}}\left\{\alpha+\left(\beta-\alpha\right)\operatorname{sn}^2\left[\sqrt{\gamma-\alpha}\left(\sqrt[4]{\frac{1-k^2}{2\lambda^2\left(1+k^2\right)^2}}\eta-\eta_0\right),m\right]\right\},$$

$$u_5=\pm\left(\frac{1-k^2}{2\lambda^2\left(1+k^2\right)^2}\right)^{-\frac{1}{4}}\exp\left[i\left(kx+t\right)\right]\cdot\sqrt{\frac{-\beta\operatorname{sn}\left[\sqrt{\gamma-\alpha}\left(\sqrt[4]{\frac{1-k^2}{2\lambda^2\left(1+k^2\right)^2}}\eta-\eta_0\right),m\right]+\gamma}{\operatorname{cn}\left[\sqrt{\gamma-\alpha}\left(\sqrt[4]{\frac{1-k^2}{2\lambda^2\left(1+k^2\right)^2}}\eta-\eta_0\right),m\right]}},$$

$$v_5=|\lambda|\sqrt{\frac{2}{1-k^2}}\cdot\frac{-\beta\operatorname{sn}\left[\sqrt{\gamma-\alpha}\left(\sqrt[4]{\frac{1-k^2}{2\lambda^2\left(1+k^2\right)^2}}\eta-\eta_0\right),m\right]+\gamma}{\operatorname{cn}\left[\sqrt{\gamma-\alpha}\left(\sqrt[4]{\frac{1-k^2}{2\lambda^2\left(1+k^2\right)^2}}\eta-\eta_0\right),m\right]},$$

$$u_6=\pm\left(\frac{1-k^2}{2\lambda^2q\left(1+k^2\right)^2}\right)^{-\frac{1}{4}}\exp\left[i\left(kx+t\right)\right]\cdot\sqrt{\frac{2}{1+\operatorname{cn}\left[2q^{\frac{1}{4}}\left(\sqrt[4]{\frac{1-k^2}{2\lambda^2\left(1+k^2\right)^2}}\eta-\eta_0\right),m\right]}-1},$$

$$v_6=|\lambda|\sqrt{\frac{2q}{1-k^2}}\left[\frac{2}{1+\operatorname{cn}\left[2q^{\frac{1}{4}}\left(\sqrt[4]{\frac{1-k^2}{2\lambda^2\left(1+k^2\right)^2}}\eta-\eta_0\right),m\right]}-1\right],$$

$$u_7=\pm\left(1+k^2\right)\sqrt{\frac{2}{k^2-1}}\exp\left[i\left(kx+t\right)\right]\cdot\sqrt{1-\tanh^2\left[\sqrt{\frac{1+k^2}{|\lambda|}\sqrt{\frac{2}{k^2-1}}}\left(\sqrt[4]{\frac{k^2-1}{2\lambda^2\left(1+k^2\right)^2}}\eta-\eta_0\right)\right]},$$

$$v_7=\frac{2\left(1+k^2\right)}{k^2-1}\cdot\left\{1-\tanh^2\left[\sqrt[4]{\frac{2\left(1+k^2\right)^2}{\lambda^2\left(k^2-1\right)}}\left(\sqrt[4]{\frac{k^2-1}{2\lambda^2\left(1+k^2\right)^2}}\eta-\eta_0\right)\right]\right\}.$$

3. Conclusion

By the complete discrimination system for polynomial method, we have obtained the classification of traveling wave solutions to the Coupled-Higgs Equation. These solutions include triangle periodic solutions, rational function solution, Jacobi elliptic function periodic solutions, and so on. This method is simple and efficient.

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