Journal of Applied Mathematics and Physics, 2014, 2, 1139-1148
Published Online December 2014 in SciRes. http://www.scirp.org/journal/jamp
http://dx.doi.org/10.4236/jamp.2014.213133

Scientific Research

Multifractal Analysis of the Asympyotically Additive Potentials

Lan Xu[1], Long Yang[2]

[1]Department of Mathematics and Physics, Suzhou Vocational University, Suzhou, China
[2]Department of Mathematics, Soochow University, Suzhou, China
Email: jssvcxulan@gmail.com, soochowyl@gmail.com

Received 14 September 2014; revised 15 October 2014; accepted 21 October 2014

Abstract

Multifractal analysis studies level sets of asymptotically defined quantities in dynamical systems. In this paper, we consider the u-dimension spectra on such level sets and establish a conditional variational principle for general asymptotically additive potentials by requiring only existence and uniqueness of equilibrium states for a dense subspace of potential functions.

Keywords

Multifractal Analysis, u-Dimension Spectra, Asymptotically Additive

1. Introduction

The theory of multifractal analysis is a subfield of the dimension theory in dynamical systems. A general framework for multifractal analysis of dynamical systems was laid out in [1] [2]. It studies a global dimensional quantity that assigns to each level set a "size" or "complexity", such as its topological entropy or Hausdorff dimension. Broadly speaking, let $f: X \to X$ be a continuous transformation of a compact metric space; let $\Phi = (\varphi_n)$, $\Psi = (\psi_n)$ be potential functions defined on X with value in $\mathbb{R}$. Given $\alpha \in \mathbb{R}$, we consider the level set:

$$K_\alpha = \left\{ x \in X : \lim_{n\to\infty} \frac{\varphi_n(x)}{\psi_n(x)} = \alpha \right\}$$

The dimension spectrum $\mathcal{D}: \mathbb{R} \to \mathbb{R}$ (of potential Φ/Ψ) is defined by $\mathcal{D}(\alpha) = \dim_H K_\alpha$ which has been extensively studied for Hólder continuous potentials for $C^{1+\alpha}$ conformal repellers in [3]-[5].

In [6], Barreira, Saussol, and Schmeling extended their work to higher-dimensional multifractal spectra, moreover, for which they consider the more general u-dimension in place of the topological entropy. Precisely,

How to cite this paper: Xu, L. and Yang, L. (2014) Multifractal Analysis of the Asympyotically Additive Potentials. *Journal of Applied Mathematics and Physics*, **2**, 1139-1148. http://dx.doi.org/10.4236/jamp.2014.213133

they consider functions $\Phi=\left(\varphi^1,\varphi^2,\cdots,\varphi^d\right)$, $\Psi=\left(\psi^1,\psi^2,\cdots,\psi^d\right)\in C(X)^d$ with $\psi^i>0$ $(i=1,2,\cdots,d)$ and examine the level sets

$$K_\alpha=\left\{x\in X\,\middle|\,\lim_{n\to\infty}\frac{\varphi^i(x)+\varphi^i(f(x))+\cdots+\phi^i(f^n(x))}{\psi^i(x)+\psi^i(f(x))+\cdots+\psi^i(f^n(x))}=\alpha_i,\ i=1,2,\cdots,d\right\}$$

for $\alpha=(\alpha_1,\alpha_1,\cdots,\alpha_d)\in\mathbb{R}^d$. We denote by $\mathcal{M}^f(x)$ the family of f-invariant Borel probability measures on X, and define a continuous function $\mathcal{P}:\mathcal{M}^f(x)\to\mathbb{R}^d$:

$$\mathcal{P}(\mu)=\left(\frac{\int_X\varphi^1\mathrm{d}\mu}{\int_X\psi^1\mathrm{d}\mu},\frac{\int_X\varphi^2\mathrm{d}\mu}{\int_X\psi_n^2\mathrm{d}\mu},\cdots,\frac{\int_X\varphi^d\mathrm{d}\mu}{\int_X\psi^d\mathrm{d}\mu}\right)$$

Given a positive function $u\in C(X)$ we denote by $\dim_u Z$ the u-dimension of the set $Z\subset X$ (see Section 2 for the definition). Let $D(X)\subset C(X)$ be the family of continuous functions with a unique equilibrium measure, they obtain the following result:

Theorem 1. Assume that *the metric entropy of* f *is upper semi-continuous, and that* $\operatorname{span}\left\{\phi^1,\psi^1,\phi^2,\psi^2,\cdots,\phi^d,\psi^d,u\right\}\subset D(v)$.

If $\alpha\notin\mathcal{P}\left(\mathcal{M}^f(X)\right)$, $K_\alpha=\varnothing$. *Otherwise, if* $\alpha\in\operatorname{int}\mathcal{P}\left(\mathcal{M}^f(X)\right)$, $K_\alpha\neq\varnothing$, *and the following properties hold*:

(I) $\mathcal{F}_u(\alpha)$ *satisfies the variational principle*:

$$\mathcal{F}_u(\alpha)=\max\left\{\frac{h_\mu(f)}{\int_X u\mathrm{d}\mu}:\mu\in\mathcal{M}^f(X),\mathcal{P}(\mu)=\alpha\right\}$$

(II) $\mathcal{F}_u(\alpha)=\min\left\{\mathcal{T}_u(\alpha,q):q\in\mathbb{R}^d\right\}$, *where* $\mathcal{T}_u(\alpha,q)$ *is the unique real number satisfying*:

$$P\left(\langle q,A-\alpha*B\rangle-\mathcal{T}_u(\alpha,q)\mathcal{U}\right)=0$$

(III) *There exists ergodic measure* $\mu_\alpha\in\mathcal{M}^f(X)$ *with* $\mathcal{P}(\mu_\alpha)=\alpha$ *and* $\mu_\alpha(K_\alpha)=1$ *such that*

$$\dim_u\mu_\alpha=\frac{h_{\mu_\alpha}(f)}{\int_x u\mathrm{d}\mu}=\mathcal{F}_u(\alpha)$$

In [7], Barreira and Doutor study the spectrum of the u-dimension for the class of almost additive sequences with a unique equilibrium measure and establish a conditional variational principle for the dimension spectra in the context of the nonadditive thermodynamic formalism. We recall that a sequence of functions $\Phi=(\varphi_n)_n$ is said to be almost additive (with respect to a transformation f) if there is a constant $C>0$ such that for every $n,\ m\in\mathbb{N}$, we have:

$$-C+\varphi_n+\varphi_m\circ f^n\leq\varphi_{n+m}\leq C+\varphi_n+\varphi_m\circ f^n$$

In [8] Climenhaga proved a generalisation of Theorem 1 provided that there is a dense subspace of $C(X)$ comprising potentials with unique equilibrium states, *i.e.*, the result applies to all continuous functions, not just those whose span lies inside the collection of potentials with unique equilibrium states.

This paper is devoted to the study of higher-dimensional multifractal analysis for the class of asymptotically additive potentials. We consider the multifractal behavior of u-dimension spectrum of level sets and establish the conditional variational principle under the assumption proposed by Climenhaga.

Section 2 gives definitions and notions, and Section 3 gives precise formulations of the result and proofs.

2. Preliminaries

We recall in this section some notions and results from the thermodynamic formalism.

2.1. Nonadditive Topological Pressure

We first introduce the notion of nonadditive topological pressure. We also refer the reader to [2] and [7] for further references.

Let $f: X \to X$ be a continuous transformation of a compact metric space. We denote by $C(X)$ the space of continuous functions on X and $\mathcal{M}^f(X)$ the set of all f-invariant measures. Given a finite open cover $\mathcal{V}$ of X, we denote by $\mathcal{W}_n(V)$ the collection of vectors $V = (V_0, \cdots, V_n)$ with $V_0, \cdots, V_n \in V$. For each $V \in \mathcal{W}_n(V)$, we write $m(V) = n$, and we consider the open set

$$X(V) = \bigcap_{k=0}^{n} f^{-k} V_k = \left\{ x \in X \middle| f^k(x) \in V_k, k = 0, 1, \cdots, n \right\}$$

Now let Φ be a sequence of continuous functions $\varphi_n : X \to \mathbb{R}$. For each $n \in \mathbb{N}$ we define:

$$\gamma_n(\Phi, \mathcal{V}) = \sup\left\{ \left|\varphi_n(x) - \varphi_n(x)\right| : x, y \in X(V), V \in \mathcal{W}_n(V) \right\}$$

We always assume that

$$\limsup_{\operatorname{diam}\mathcal{V} \to 0} \limsup_{n \to \infty} \frac{\gamma_n(\Phi, \mathcal{V})}{n} = 0 \tag{1}$$

For each $V \in \mathcal{W}_n(V)$ we write:

$$\varphi(V) = \begin{cases} \sup\limits_{X(V)} \varphi_n & X(V) \neq \phi \\ -\infty, & X(V) = \phi \end{cases} \tag{2}$$

Given a set $Z \subset X$ and $\alpha \in \mathbb{R}$, we define the function:

$$M(Z, \alpha, \Phi, \mathcal{V}) = \lim_{n \to \infty} \inf_{\Gamma} \sum_{V \in \Gamma} \exp\left(-\alpha m(V) + \varphi(V)\right)$$

where the infimum is taken over all finite or countable collections $\Gamma \subset \bigcup_{k \geq n} \mathcal{W}_k(\mathcal{V})$, such that $\bigcup_{V \in \Gamma} X(V) \supset Z$. We also define

$$P_Z(\Phi, \mathcal{V}) = \inf\left\{ \alpha \in \mathbb{R} : M(Z, \alpha, \Phi, \mathcal{V}) = 0 \right\}$$

It was shown in [9] that the limit

$$P_Z(\Phi) = \lim_{\operatorname{diam}\mathcal{V} \to 0} P_Z(\Phi, \mathcal{V})$$

exists. The number $P_Z(\Phi)$ is called the nonadditive topological pressure of Φ in the set Z (with respect to f). In particular, if $\Phi = 0$, we get the topological entropy $h_{\text{top}}\left(f|_Z\right) = P_Z(0)$. We also write $P(\Phi) = P_X(\Phi)$.

The following proposition was established in [2].

Proposition 1. *For any* $Z \subset X$, *we have*

$$P_Z(\Phi) \geq \sup\left\{ h_\mu(f) + \lim_{n \to \infty} \frac{1}{n} \int_X \varphi_n \mathrm{d}\mu \middle| \mu \in \mathcal{M}^f(X), \mu(Z) = 1 \right\}$$

2.2. u-Dimension

We recall here a notion introduced by Barreira and Schmeling in [10]. Let $u: X \to \mathbb{R}$ be a strictly positive continuous function. Likewise, we define

$$N(Z, \alpha, \Phi, \mathcal{V}) = \lim_{n \to \infty} \inf_{\Gamma} \sum_{V \in \Gamma} \exp\left(-\alpha u(V)\right)$$

where $u(V)$ is defined as in (2) and where the infimum is taken over all finite or countable collections $\Gamma \subset \bigcup_{k \geq n} \mathcal{W}_k(\mathcal{V})$ such that $\bigcup_{V \in \Gamma} X(V) \supset Z$. We also define

$$\dim_{u,\mathcal{V}} Z = \inf\left\{\alpha \in \mathbb{R} : M\left(Z,\alpha,\Phi,\mathcal{V}\right) = 0\right\}$$

Theorem 2. ([10]) *The following limits exist*:

$$\dim_u Z \overset{\text{def}}{=} \lim_{\operatorname{diam}\mathcal{V}\to 0} \dim_{u,\mathcal{V}} Z$$

We call $\dim_u Z$ the u-dimension of Z. If $u \equiv 1$, then the number $\dim_u Z$ coincides with the topological entropy of f on Z. The following result is an easy consequence of the definitions.

Proposition 2. *The number* $\dim_u Z = \alpha$ *is the unique root of the equation* $P_Z\left(-\alpha\mathcal{U}\right) = 0$, *where* $\mathcal{U} = \left(u_n\right)_n$ *with* $u_n = \sum_{k=0}^{n-1} u \circ f^k$ *for each* $n \in \mathbb{N}$.

Furthermore, given a probability measure μ in X, we set:

$$\dim_{u,\mathcal{V}} \mu = \inf\left\{\dim_{u,\mathcal{V}} Z : \mu\left(Z\right) = 1\right\}$$

We can show that the limit $\dim_u \mu = \lim_{\operatorname{diam}\mathcal{V}\to 0} \dim_{u,\mathcal{V}} \mu$ exists, and we call it the u-dimension of μ. When $\mu \in \mathcal{M}^f\left(X\right)$ is ergodic, one can show that (see [10])

$$\dim_u \mu = \frac{h_\mu\left(f\right)}{\int_X u \mathrm{d}\mu} \tag{3}$$

2.3. Asymptotically Additive Sequences

This kind of potential was introduced by Feng and Huang ([11]).

Definition 1. *A sequence* $\Phi = \left(\varphi_n\right)_n$ *of functions on* X *is said to be asymptotically additive if for any* $\varepsilon > 0$, *there exists* $\overline{\varphi} \in C\left(X\right)$ *such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} \left|\varphi_n\left(x\right) - \sum_{k=0}^{n-1} \overline{\varphi}\left(f^k\left(x\right)\right)\right| \le \varepsilon$$

We denote by $AA\left(X\right)$ the family of asymptotically additive sequences of continuous functions (satisfying (1)). Now we give two propositions whose proof can be found in [11].

Proposition 3. *If* $f : X \to X$ *is a continuous transformation of a compact metric space,* $\Phi = \left(\varphi_n\right)_n \in AA\left(X\right)$ *is an asymptotically additive sequence, and* $\mu \in \mathcal{M}^f\left(X\right)$, *then*

(I) *The limit* $\varphi\left(x\right) = \lim_{n\to\infty} \frac{\varphi_n\left(x\right)}{n}$ *exists for* $\mu - a.e.\ x \in X$;

(II) *The limit* $\lim_{n\to\infty} \int_X \frac{\varphi_n\left(x\right)}{n} \mathrm{d}\mu = \int_X \varphi\left(x\right) \mathrm{d}\mu$ *exists*;

(III) *If* μ *is ergodic, then for* $\mu - a.e.$ x,

$$\lim_{n\to\infty} \int_X \frac{\varphi_n\left(x\right)}{n} \mathrm{d}\mu = \lim_{n\to\infty} \frac{\varphi_n\left(x\right)}{n} \tag{4}$$

(IV) *The function* $\mu \mapsto \lim_{n\to\infty} \int_X \frac{\varphi_n\left(x\right)}{n} \mathrm{d}\mu$ *is continuous with the weak* topology in* $\mathcal{M}^f\left(X\right)$.

Proposition 4. *If* $f : X \to X$ *is a continuous transformation of a compact metric space,* $\Phi = \left(\varphi_n\right)_n \in AA\left(X\right)$ *is an asymptotically additive sequence, then the topological pressure* $P\left(\Phi\right)$ *satisfies the following variational principle*:

$$P\left(\Phi\right) = \sup\left\{h_\mu\left(f\right) + \lim_{n\to\infty} \int_X \frac{\varphi_n\left(x\right)}{n} \mathrm{d}\mu \,\middle|\, \mu \in \mathcal{M}^f\left(X\right)\right\}$$

We call $\mu \in \mathcal{M}^f\left(X\right)$ an equilibrium measure for the potential Φ if

$$P\left(\Phi\right) = h_\mu\left(f\right) + \lim_{n\to\infty} \int_X \frac{\varphi_n\left(x\right)}{n} \mathrm{d}\mu$$

Note that if the function $\mu \mapsto h_\mu(f)$ is upper semicontinuous, then every sequence in $AA(X)$ has an equilibrium measure.

3. Main Result

Let $d \in \mathbb{N}$ and take $(A,B) \in AA(X)^d \times AA(X)^d$. We write $A = (\Phi^1, \Phi^2, \cdots, \Phi^d)$ and $B = (\Psi^1, \Psi^2, \cdots, \Psi^d)$, and also $\Phi^i = (\varphi_n^i)_n$, $\Psi^i = (\psi_n^i)_n$ $(i = 1, 2, \cdots, d)$.

We assume that

(1) There exists constant $\sigma > 0$ such that $\psi_n^i(x) > n\sigma > 0$ $(i = 1, 2, \cdots d)$ for any $n \in \mathbb{N}$ and any $x \in X$.

(2) For every $\mu \in \mathcal{M}^f(X)$, $\lim_{n\to\infty} \frac{\psi_n^i(x)}{n} = \psi^i(x) > 0$ for $\mu - a.e.\, x \in X$ and every $i = 1, \cdots, d$, where the limit exists by proposition 3.

Given $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbb{R}^d$, we define:

$$K_\alpha = \bigcap_{i=1}^{d} \left\{ x \in X : \lim_{n\to\infty} \frac{\varphi_n^i(x)}{\psi_n^i(x)} = \alpha_i \right\}$$

and function $\mathcal{F}_u : \mathbb{R}^d \to \mathbb{R}$ by $\mathcal{F}_u(\alpha) = \dim_u K_\alpha$.

We also consider the function $\mathcal{P} : \mathcal{M}^f(X) \to \mathbb{R}^d$ defined by:

$$\mathcal{P}(\mu) = \left(\frac{\lim_{n\to\infty} \frac{1}{n}\int_X \varphi_n^1 \mathrm{d}\mu}{\lim_{n\to\infty} \frac{1}{n}\int_X \psi_n^1 \mathrm{d}\mu}, \frac{\lim_{n\to\infty} \frac{1}{n}\int_X \varphi_n^2 \mathrm{d}\mu}{\lim_{n\to\infty} \frac{1}{n}\int_X \psi_n^2 \mathrm{d}\mu}, \cdots, \frac{\lim_{n\to\infty} \frac{1}{n}\int_X \varphi_n^d \mathrm{d}\mu}{\lim_{n\to\infty} \frac{1}{n}\int_X \psi_n^d \mathrm{d}\mu} \right) = \lim_{n\to\infty} \left(\frac{\int_X \varphi_n^1 \mathrm{d}\mu}{\int_X \psi_n^1 \mathrm{d}\mu}, \frac{\int_X \varphi_n^2 \mathrm{d}\mu}{\int_X \psi_n^2 \mathrm{d}\mu}, \cdots, \frac{\int_X \varphi_n^d \mathrm{d}\mu}{\int_X \psi_n^d \mathrm{d}\mu} \right)$$

Given vectors $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbb{R}^d$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_d) \in \mathbb{R}^d$ we use the notations:

$$\langle \alpha, \beta \rangle = \sum_{i=1}^{d} \alpha_i \beta_i$$

$$\alpha * \beta = (\alpha_1\beta_1, \alpha_2\beta_2, \cdots, \alpha_d\beta_d)$$

and

$$\|\alpha\| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_d|$$

We also consider the positive sequence of functions $\mathcal{U} = (u_n)_n$ with $u_n = \sum_{k=0}^{n-1} u \circ f^k$.

Our main result is the following theorem.

Theorem 3. *Let f be a continuous transformation of a compact metric space X such that the entropy map $\mu \mapsto h_\mu(f)$ is upper semicontinuous, and assume that there exists a dense subset $D \subset AA(X)$ such that every $\Phi \in D$ has a unique equilibrium measure.*

If $\alpha \notin \mathcal{P}(\mathcal{M}^f(X))$, then $K_\alpha = \varnothing$. Otherwise, if $\alpha \in \operatorname{int}\mathcal{P}(\mathcal{M}^f(X))$, then $K_\alpha \neq \varnothing$, and the following properties hold:

(I) $\mathcal{F}_u(\alpha)$ *satisfies the variational principle*:

$$\mathcal{F}_u(\alpha) = \max\left\{ \frac{h_\mu(f)}{\int_X u \mathrm{d}\mu} : \mu \in \mathcal{M}^f(X), \mathcal{P}(\mu) = \alpha \right\}$$

(II) $\mathcal{F}_u(\alpha) = \min\{\mathcal{T}_u(\alpha, q) : q \in \mathbb{R}^d\}$, *where $\mathcal{T}_u(\alpha, q)$ is the unique real number satisfying*:

$$P(\langle q, A - \alpha * B \rangle - \mathcal{T}_u(\alpha, q)\mathcal{U}) = 0$$

(III) *There exists ergodic measure $\mu_\alpha \in \mathcal{M}^f(X)$ with $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$, and*

$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(f)}{\int_X u \mathrm{d}\mu_\alpha}$$

which is arbitrarily close to $\mathcal{F}_u(\alpha)$.

Proof. We first establish several auxiliary results.

Lemma 1. *For* $\Psi = (\psi_n)_n \in AA(X)$ *there exists constant* $C > 0$ *such that for every* $m \in \mathbb{N}$ *we have*

$$\|\psi_m\|_\infty \le m\left(\|\overline{\psi}\|_\infty + C\right) \tag{5}$$

where $\|\cdot\|_\infty$ *denotes the supremum norm.*

Proof. For any $\varepsilon > 0$, since the sequence $(\psi_n)_n$ is asymptotically additive, there exists $\overline{\psi} \in C(X)$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} \left| \psi_n(x) - \sum_{k=0}^{n-1} \overline{\psi}\left(f^k(x)\right) \right| \le \varepsilon$$

Therefore, there exists $C > 0$, such that for every $m \in \mathbb{N}$ and $x \in X$, we have

$$\left| \psi_m(x) - \sum_{k=0}^{m-1} \overline{\psi}\left(f^k(x)\right) \right| \le mC$$

and thus

$$\|\psi_m\|_\infty \le m\left(\|\overline{\psi}\|_\infty + C\right)$$ □

Lemma 2. *If* $\alpha \in \mathcal{P}\left(\mathcal{M}^f(X)\right)$, *then* $\inf_{q\in\mathbb{R}^d} P\left(\langle q, A - \alpha * B\rangle - \mathcal{F}_u(\alpha)\mathcal{U}\right) \ge 0$.

Proof. Using (5), a slight modification of the proof of Lemma 2 in [7] yields this statement, and thus we omit it. □

Lemma 3. *If* $\alpha \notin \mathcal{P}\left(\mathcal{M}^f(X)\right)$, *then* $K_\alpha = \varnothing$. *Otherwise, if* $\alpha \in \mathcal{P}\left(\mathcal{M}^f(X)\right)$, *then* $K_\alpha \ne \varnothing$.

Proof. Take $\alpha \in \mathbb{R}^d$ with $K_\alpha \ne \varnothing$ and let $x \in K_\alpha$. Then $\lim_{n\to\infty} \frac{\varphi_n^i(x)}{\varphi_n^i(x)} = \alpha_i$ for $i = 1, 2, \cdots, d$. We consider the sequence $(\mu_n)_n$ of probability measures in X defined by $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x}$. Let μ be a limitpoint of μ_n, clearly $\mu \in \mathcal{M}^f(X)$. We always assume μ is ergodic, or else taking an ergodic decomposition of μ. The desired statements are thus immediate consequences of (4). □

Now proceed with the proof of (1) in theorem 3. We use analogous arguments to those in the proof of lemma 3 in [7]. First show that

$$\mathcal{F}_u(\alpha) \le \sup\left\{ \frac{h_{\mu_\alpha}(f)}{\int_X u \mathrm{d}\mu_\alpha} \middle| \mu \in \mathcal{M}^f(X), \mathcal{P}(\mu) = \alpha \right\}$$

Let $r > 0$ be the distance of α to $\mathbb{R}^d \setminus \mathcal{P}\left(\mathcal{M}^f(X)\right)$. Take $q \in \mathbb{R}^d$ and define:

$$F(q) = P\left(\langle q, A - \alpha * B\rangle - \mathcal{F}_u(\alpha)\mathcal{U}\right)$$

Given $\beta = (\beta_1, \cdots, \beta_d) \in \mathbb{R}^d$ with $\beta_i = \alpha_i + r \cdot \mathrm{sgn}\frac{q_i}{2d}$ for each i, we have:

$$\|\beta - \alpha\| = \sum_{i=1}^d |\beta_i - \alpha_i| = \sum_{i=1}^d \left| \frac{1}{2d} r \cdot \mathrm{sgn} q_i \right| = \frac{r}{2} < r$$

and hence $\beta \in \mathcal{P}\left(\mathcal{M}^f(X)\right)$. Therefore, there exists $\mu \in \mathcal{M}^f(X)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \int_X A_n \mathrm{d}\mu = \lim_{n\to\infty} \frac{1}{n} \int_X \beta * B_n \mathrm{d}\mu$$

where $A_n = \left(\varphi_n^1, \varphi_n^2, \cdots, \varphi_n^d\right)$, $B_n = \left(\psi_n^1, \psi_n^2, \cdots, \psi_n^d\right)$. Moreover,

$$\begin{aligned} F(q) &\geq h_\mu(f) + \lim_{n\to\infty}\frac{1}{n}\int_X \left[\langle q, A_n - \alpha * B_n\rangle - \mathcal{F}_u(\alpha) u_n\right] \mathrm{d}\mu \\ &= h_\mu(f) + \lim_{n\to\infty}\frac{1}{n}\left\langle q, \int_X (A_n - \alpha * B_n)\mathrm{d}\mu\right\rangle - \mathcal{F}_u(\alpha)\int_X u \mathrm{d}\mu \end{aligned}$$

Since

$$\left\langle q, \int_X (\beta - \alpha) * B_n \mathrm{d}\mu\right\rangle = \sum_{i=1}^d q_i \int_X (\beta_i - \alpha_i)\psi_n^i \mathrm{d}\mu = \sum_{i=1}^d \frac{1}{2d} r q_i \cdot \mathrm{sgn} q_i \int_X \psi_n^i \mathrm{d}\mu = \frac{1}{2d} r \sum_{i=1}^d |q_i| \int_X \psi_n^i \mathrm{d}\mu$$

we obtain:

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{n}\left\langle q, \int_X (A_n - \alpha * B_n)\mathrm{d}\mu\right\rangle &= \lim_{n\to\infty}\left(\frac{1}{2d} r \sum_{i=1}^d |q_i| \frac{1}{n}\int_X \psi_n^i \mathrm{d}\mu + \frac{1}{n}\left\langle q, \int_X (A_n - \beta * B_n)\mathrm{d}\mu\right\rangle\right) \\ &= \frac{1}{2d} r \sum_{i=1}^d |q_i| \lim_{n\to\infty}\frac{1}{n}\int_X \psi_n^i \mathrm{d}\mu = \frac{r\|q\|}{2d}\lim_{n\to\infty}\frac{1}{n}\int_X \psi_n^i \mathrm{d}\mu \geq \frac{r\|q\|}{2d}\min_i \lim_{n\to\infty}\frac{1}{n}\int_X \psi_n^i \mathrm{d}\mu \end{aligned}$$

Since $h_\mu(f) \geq 0$, it follows that

$$F(q) \geq \frac{r\|q\|}{2d}\min_i \lim_{n\to\infty}\frac{1}{n}\int_X \psi_n^i \mathrm{d}\mu - \mathcal{F}_u(\alpha)\int_X u \mathrm{d}\mu$$

It implies that $F(q)$ takes arbitrarily large values for $\|q\|$ sufficiently large, and hence there exists $R \in \mathbb{R}$ such that $F(q) \geq F(0)$ for every $q \in \mathbb{R}^d$ with $\|q\| \geq R$. The continuity of F implies that it attains a minimum at some point q_0 with $\|q_0\| \leq R$.

Note that $D \subset AA(X)$ is a dense subset such that every $\Phi \in D$ has a unique equilibrium measure, then for every $\delta > 0$ and $q \in \overline{B(0,R)}$ there exists $\tilde{A} \in D^d$, $\tilde{B} \in D^d$ and $\tilde{\mathcal{U}} \in D$ with the following properties:

(1) $\left\langle q, \tilde{A} - \alpha * \tilde{B}\right\rangle - \mathcal{F}_u(\alpha)\tilde{\mathcal{U}}$ has a unique equilibrium measure μ_q^δ which depends continuously on q (for fixed δ);

(2) $\left\langle q, \lim_{n\to\infty}\frac{1}{n}\int_X \left(\tilde{A}_n - \alpha * \tilde{B}_n\right)\mathrm{d}\mu_q^\delta\right\rangle - \mathcal{F}_u(\alpha)\int_X \tilde{u}\mathrm{d}\mu_q^\delta < \left\langle q, \lim_{n\to\infty}\frac{1}{n}\int_X (A_n - \alpha * B_n)\mathrm{d}\mu_q^\delta\right\rangle - \mathcal{F}_u(\alpha)\int_X u \mathrm{d}\mu_q^\delta + \frac{\delta}{2}$;

(3) $\tilde{F}(q) \overset{\Delta}{=} P\left(\left\langle q, \tilde{A} - \alpha * \tilde{B}\right\rangle - \mathcal{F}_u(\alpha)\tilde{\mathcal{U}}\right) > F(q) - \frac{\delta}{2}$.

Therefore,

$$\begin{aligned} \tilde{F}(q) &= h_{\mu_q^\delta}(f) + \left\langle q, \lim_{n\to\infty}\frac{1}{n}\int_X \left(\tilde{A}_n - \alpha * \tilde{B}_n\right)\mathrm{d}\mu_q^\delta\right\rangle - \mathcal{F}_u(\alpha)\int_X \tilde{u}\mathrm{d}\mu_q^\delta \\ &< h_{\mu_q^\delta}(f) + \left\langle q, \lim_{n\to\infty}\frac{1}{n}\int_X (A_n - \alpha * B_n)\mathrm{d}\mu_q^\delta\right\rangle - \mathcal{F}_u(\alpha)\int_X u \mathrm{d}\mu_q^\delta + \frac{\delta}{2} \end{aligned}$$

and thus

$$h_{\mu_q^\delta}(f) + \left\langle q, \lim_{n\to\infty}\frac{1}{n}\int_X (A_n - \alpha * B_n)\mathrm{d}\mu_q^\delta\right\rangle - \mathcal{F}_\alpha \int_X u \mathrm{d}\mu_q^\delta > F(q) - \delta$$

Denote μ_q a limit point of $\left\{\mu_q^\delta\right\}$ as $\delta \to 0$, then

$$h_{\mu_q}(f) + \left\langle q, \lim_{n\to\infty}\frac{1}{n}\int_X (A_n - \alpha * B_n)\mathrm{d}\mu_q^\delta\right\rangle - \mathcal{F}_\alpha \int_X u \mathrm{d}\mu_q \geq F(q) \tag{6}$$

For each vector $e \in \mathbb{R}^k$ with $\|e\| = 1$, let $q = q_0 + \varepsilon e$ and let μ_q be taken as in (6). We have

$$F(q)-F(q_0)\le h_{\mu_q}(f)+\left\langle q,\lim_{n\to\infty}\frac{1}{n}\int_X(A_n-\alpha*B_n)\mathrm{d}\mu_q\right\rangle-\mathcal{F}_u(\alpha)\int_X u\mathrm{d}\mu_q$$
$$-\left[h_{\mu_q}(f)+\left\langle q_0,\lim_{n\to\infty}\frac{1}{n}\int_X(A_n-\alpha*B_n)\mathrm{d}\mu_q\right\rangle-\mathcal{F}_u(\alpha)\int_X u\mathrm{d}\mu_q\right]$$
$$=\varepsilon\left\langle e,\lim_{n\to\infty}\frac{1}{n}\int_X(A_n-\alpha*B_n)\mathrm{d}\mu_q\right\rangle$$

If $\varepsilon>0$, then

$$\left\langle e,\lim_{n\to\infty}\frac{1}{n}\int_X(A_n-\alpha*B_n)\mathrm{d}\mu_q\right\rangle\ge 0$$

Now assume that $\mu_q\to\nu_e^+$ when $\varepsilon\to 0$ for some measure ν_e^+. The upper semicontinuity of the entropy implies that

$$F(q_0)=\lim_{\varepsilon\to 0}F(q)\le h_{\nu_e^+}(f)+\left\langle q_0,\lim_{n\to\infty}\frac{1}{n}\int_X(A_n-\alpha*B_n)\mathrm{d}\nu_e^+\right\rangle-\mathcal{F}_u(\alpha)\int_X\tilde{u}\mathrm{d}\nu_e^+$$

This shows that ν_e^+ is an equilibrium measure of

$$\Xi=\langle q_0,A-\alpha*B\rangle-\mathcal{F}_u(\alpha)\mathcal{U}$$

satisfying

$$\left\langle e,\lim_{n\to\infty}\frac{1}{n}\int_X(A_n-\alpha*B_n)\mathrm{d}\nu_e^+\right\rangle\ge 0$$

Similarly, one can consider $\varepsilon<0$ and find an invariant measure ν_e^- that is an equilibrium measure of Ξ satisfying

$$\left\langle e,\lim_{n\to\infty}\frac{1}{n}\int_X(A_n-\alpha*B_n)\mathrm{d}\nu_e^-\right\rangle\le 0$$

For each $a\in[0,1]$, let $\nu_e(a)=a\nu_e^+(1-a)\nu_e^-$. Then the function

$$p(a)=\left\langle e,\lim_{n\to\infty}\frac{1}{n}\int_X(A_n-\alpha*B_n)\mathrm{d}\nu_e(a)\right\rangle$$

is continuous. Moreover, $p(0)\le 0$ and $p(1)\ge 0$. Hence, there exists a_0 such that $p(a_0)=0$. Since ν_e^+ and ν_e^- are equilibrium measures of Ξ, this implies that $\nu_e(a)$ is also an equilibrium measure of Ξ. Therefore, for each unit vector $e\in\mathbb{R}^k$ there exists an equilibrium measure $\nu_e(a)$ of Ξ such that

$$\left\langle e,\lim_{n\to\infty}\frac{1}{n}\int_X(A_n-\alpha*B_n)\mathrm{d}\nu_e(a)\right\rangle=0 \tag{7}$$

We claim that there exists an equilibrium measure ν of Ξ such that

$$\lim_{n\to\infty}\frac{1}{n}\int_X(A_n-\alpha*B_n)\mathrm{d}\nu=0 \tag{8}$$

Let us assume that such a measure does not exist. We denote by $\mathcal{I}(\alpha)$ the set of all equilibrium measures of Ξ. Then

$$\mathcal{K}=\left\{\lim_{n\to\infty}\frac{1}{n}\int_X(A_n-\alpha*B_n)\mathrm{d}\mu:\mu\in\mathcal{I}(\alpha)\right\}$$

is a compact convex subset of $\mathbb{R}^k\setminus\{0\}$. Hence, there exist a unit vector $v\in\mathbb{R}^k$ and $c>0$ such that $\langle v,b\rangle<-c$ for $b\in\mathcal{K}$.

For every $\mu\in\mathcal{I}(\alpha)$, we have

$$\left\langle v,\left(\lim_{n\to\infty}\frac{1}{n}\int_X(A_n-\alpha*B_n)\mathrm{d}\mu\right)\right\rangle<-c<0$$

which contradicts (7). This completes the proof of claim. Observe that this claim implies $\mathcal{P}(\nu)=\alpha$.

By lemma 2, for the measure ν satisfying (8), we have

$$h_\nu(f)-\mathcal{F}_u(\alpha)\int_X u\mathrm{d}\nu = F(q_0)\ge 0$$

and hence

$$\mathcal{F}_u(\alpha)\le\frac{h_\nu(f)}{\int_X u\mathrm{d}\mu}\le\sup\left\{\frac{h_\mu(f)}{\int_X u\mathrm{d}\mu}\middle|\mu\in\mathcal{M}^f(X),\mathcal{P}(\mu)=\alpha\right\}$$

We now to prove the reverse inequality. We need the following lemma.

Lemma 4. ([8]) *Under the assumptions of theorem 3, for* $\alpha\in\operatorname{int}\mathcal{P}\left(\mathcal{M}^f(X)\right)$ *and* $t\in\mathbb{R}$, *we have*:

$$P_{K_\alpha}(tu)=\sup\left\{h_\mu(f)+t\int_X u\mathrm{d}\mu\middle|\mu\in\mathcal{M}^f(X),\mu(K_\alpha)=1\right\}=\inf_{q\in\mathbb{R}^d}P\left(\langle q,A-\alpha*B\rangle+t\mathcal{U}\right)$$

In fact, this is a particular case of Theorem C in [8].

For any $t<\sup\left\{\frac{h_{\mu_\alpha}(f)}{\int_X u\mathrm{d}\mu_\alpha}\middle|\mu\in\mathcal{M}^f(X),\mathcal{P}(\mu)=\alpha\right\}$, there exists $\mu\in\mathcal{M}^f(X)$ with $\mathcal{P}(\mu)=\alpha$ such that $t>\frac{h_\mu(f)}{\int_X u\mathrm{d}\mu}$. Therefore

$$P_{K_\alpha}(-tu)=\inf_{q\in\mathbb{R}^d}P\left(\langle q,A-\alpha*B\rangle-t\mathcal{U}\right)\ge h_\mu(f)-t\int_X u\mathrm{d}\mu>0$$

and hence by proposition 2 we have $t<\mathcal{F}_u(\alpha)$. The arbitrariness of t implies that

$$\mathcal{F}_u(\alpha)\ge\sup\left\{\frac{h_{\mu_\alpha}(f)}{\int_X u\mathrm{d}\mu_\alpha}\middle|\mu\in\mathcal{M}^f(X),\mathcal{P}(\mu)=\alpha\right\}$$

and thus

$$\mathcal{F}_u(\alpha)=\sup\left\{\frac{h_{\mu_\alpha}(f)}{\int_X u\mathrm{d}\mu_\alpha}\middle|\mu\in\mathcal{M}^f(X),\mathcal{P}(\mu)=\alpha\right\}\tag{9}$$

Furthermore, since the map $\mu\mapsto\frac{h_{\mu_\alpha}(f)}{\int_X u\mathrm{d}\mu}$ is upper semicontinuous on the compact set $\left\{\mu\in\mathcal{M}^f(X),\mathcal{P}(\mu)=\alpha\right\}$, then the supremum of (9) can be obtained, *i.e.*

$$\mathcal{F}_u(\alpha)=\max\left\{\frac{h_{\mu_\alpha}(f)}{\int_X u\mathrm{d}\mu_\alpha}\middle|\mu\in\mathcal{M}^f(X),\mathcal{P}(\mu)=\alpha\right\}$$

This completes (I) of theorem 3.

We now proceed with the proof of (II) and (III). By lemma 2 we have

$$F(q)=P\left(\langle q,A-\alpha*B\rangle-\mathcal{F}_u(\alpha)\mathcal{U}\right)\ge 0$$

for every $q\in\mathbb{R}^d$. Therefore,

$$\mathcal{F}_u(\alpha)\le\inf\left\{\mathcal{T}_u(\alpha,q):q\in\mathbb{R}^d\right\}$$

On the other hand, for any $t<\inf\left\{\mathcal{T}_u(\alpha,q):q\in\mathbb{R}^d\right\}$ we have

$$P_{K_\alpha}(-tu)=\inf_{q\in\mathbb{R}^d}P\left(\langle q,A-\alpha*B\rangle-\mathcal{F}_u(\alpha)\mathcal{U}\right)>0$$

and hence $t<\mathcal{F}_u(\alpha)$. So we conclude that

$$\mathcal{F}_u(\alpha) = \inf\left\{\mathcal{T}_u(\alpha, q) : q \in \mathbb{R}^d\right\}$$

By ergodic decomposition we obtain

$$\mathcal{F}_u(\alpha) = \max\left\{\frac{h_{\mu_\alpha}(f)}{\int_X u\mathrm{d}\mu_\alpha} \middle| \mu \in \mathcal{M}^f(X) \text{ is ergdic}, \mathcal{P}(\mu) = \alpha\right\}$$

For any $\delta > 0$, there exists ergodic μ_δ with $\mathcal{P}(\mu_\delta) = \alpha$ such that

$$\frac{h_{\mu_\delta}(f)}{\int_X u\mathrm{d}\mu_\delta} \geq \mathcal{F}_u(\alpha) - \delta$$

Note that $\mu_\delta(K_\alpha) = 1$, then by (3)

$$\mathcal{F}_u(\alpha) - \delta \leq \frac{h_{\mu_\delta}(f)}{\int_X u\mathrm{d}\mu_\delta} = \dim_u \mu_\delta \leq \mathcal{F}_u(\alpha)$$

It follows that statement (III) in theorem 3 holds. □

Acknowledgements

The author wishes to thank Professor Cao Yongluo for his invaluable suggestions and encouragement.

References

[1] Barreira, L., Pesin, Y. and Schmeling, J. (1997) On a General Concept of Multifractality: Multifractal Spectra for Dimensions, Entropies, and Lyapunov Exponents. Multifractal Rigidity. *Chaos*, **7**, 27-38. http://dx.doi.org/10.1063/1.166232

[2] Pesin, Y. (1997) Dimension Theory in Dynamical Systems: Contemporary Views and Applications. Chicago Lectures in Mathematics, Chicago University Press, Chicago. http://dx.doi.org/10.7208/chicago/9780226662237.001.0001

[3] Barreira, L. and Saussol, B. (2001) Variational Principles and Mixed Multifractal Spectra. *Transactions of the American Mathematical Societyy*, **353**, 3919-3944. http://dx.doi.org/10.1090/S0002-9947-01-02844-6

[4] Pesin, Y. and Weiss, H. (1997) A Multifractal Analysis of Equilibrium Measures for Confromal Expanding Maps and Moran-Like Geometric Constructions. *Journal of Statistical Physics*, **86**, 233-275. http://dx.doi.org/10.1007/BF02180206

[5] Weiss, H. (1999) The Lyapunov Spectrum of Equilibrium Measures for Conformal Expanding Maps and Axiom-A Surface Diffeomorphisms. *Journal of Statistical Physics*, **95**, 615-632. http://dx.doi.org/10.1023/A:1004591209134

[6] Barreira, L., Saussol, B. and Schmeling, J. (2002) Higher-Dimensional Multifractal Analysis. *Journal de Mathématiques Pures et Appliquées*, **81**, 67-91. http://dx.doi.org/10.1016/S0021-7824(01)01228-4

[7] Barreira, L. and Doutor, P. (2009) Almost Additive Multifractal Analysis. *Journal de Mathématiques Pures et Appliquées*, **92**, 1-17. http://dx.doi.org/10.1016/j.matpur.2009.04.006

[8] Climenhaga, V. (2013) Topological Pressure of Simultaneous Level. *Nonlinearity*, **26**, 241-268. http://dx.doi.org/10.1088/0951-7715/26/1/241

[9] Barreira, L. (1996) A Nonadditive Thermodynamic Formalism and Applications to Dimension Theory of Hyperbolic Dynamical Systems. *Ergodic Theory and Dynamical Systems*, **16**, 871-927. http://dx.doi.org/10.1017/S0143385700010117

[10] Barreira, L. and Schmeling, J. (2000) Sets of "Non-Typical" Points Have Full Topological Entropy and Full Hausdorff Dimension. *Israel Journal of Mathematics*, **116**, 29-70. http://dx.doi.org/10.1007/BF02773211

[11] Feng, D. and Huang, W. (2010) Lyapunov Spectrum of Asymptotically Sub-Additive Potential. *Communications in Mathematical Physics*, **297**, 1-43.