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Curvature Motion on Dual Hyperbolic Unit Sphere $\mathbb{H}_0^2$

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Abstract

In this paper, we define dual curvature motion on the dual hyperbolic unit sphere $\mathbb{H}_0^2$ in the dual Lorentzian space $\mathbb{D}_1^3$ with dual signature $(+,+,-)$. We carry the obtained results to the Lorentzian line space $\mathbb{R}_1^3$ by means of Study mapping. Then we make an analysis of the orbits during the dual hyperbolic spherical curvature motion. Finally, we find some line congruences, the family of ruled surfaces and ruled surfaces in $\mathbb{R}_1^3$.

Keywords

Dual Curvature Motion, Dual Lorentzian Space, Study Mapping

1. Introduction

Dual numbers had been introduced by W.K. Clifford (1845-1849) as a tool for his geometrical investigations. After him, E. Study (1860-1930) used dual numbers and dual vectors in his research on the geometry of lines and kinematics [1]. He devoted special attention to the representation of directed lines by dual unit vectors and defined the mapping that is known by his name. There exists one-to-one correspondence between the vectors of dual unit sphere S^2 and the directed lines of space of lines $\mathbb{R}^3$. Hence, a differentiable curve on the sphere S^2 corresponds to a ruled surface in the line space $\mathbb{R}^3$ [2]-[4]. Ruled surfaces have been widely applied in surface design, manufacturing technology and simulation of rigid bodies [5].

E. Study's mapping plays a fundamental role in the real and dual Lorentzian spaces [6]. By this mapping, a curve on a dual hyperbolic unit sphere $\mathbb{H}_0^2$ corresponds to a timelike ruled surface in the Lorentzian line space

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$\mathbb{R}_1^3$, in other words, there exists a one-to-one correspondence between the geometry of curves on $\mathbb{H}_0^2$ and the geometry of timelike ruled surfaces in $\mathbb{R}_1^3$. Similarly, a timelike (spacelike) curve on a dual Lorentzian unit sphere S_1^2 corresponds to a spacelike (timelike) ruled surface in the Lorentzian line space $\mathbb{R}_1^3$, this means that, there exists a one-to-one correspondence between the geometry of timelike (spacelike) curves on S_1^2 and the geometry of spacelike (timelike) ruled surfaces in $\mathbb{R}_1^3$ [7]. Since the dual Lorentzian metric is indefinite, the angle concept in this space is very interesting. For instance, the dual hyperbolic angle $\phi = \varphi + \varepsilon\varphi^*$ between two dual timelike unit vectors is a dual value formed with the (real) hyperbolic angle φ between corresponding two directed timelike lines in the Lorentzian line space $\mathbb{R}_1^3$ and the shortest Lorentzian distance φ^* between these directed timelike lines.

Real spherical curvature motion had been introduced by A. Karger and J. Novak [8]. Also, a dual spherical curvature motion has been defined by Z. Yapar [9]. In recent years, study about the real spherical motion has been generalized to the Lorentz spherical motion [6] [7] [10] [11]. In this work, we consider the curvature motion on the dual hyperbolic unit sphere $\mathbb{H}_0^2$ of the dual Lorentzian space $\mathbb{D}_1^3$ and the results are carried over to the Lorentzian line space by the E. Study's mapping.

2. Preliminaries and Definitions

In this section, we give a brief summary of the theory of dual numbers, dual Lorentzian vectors and Study's mapping.

Let $\mathbb{R}_1^3$ be the 3-dimensional Minkowski space over the field of real numbers $\mathbb{R}$ with the Lorentzian inner product $\langle , \rangle$ given by

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1 b_1 + a_2 b_2 - a_3 b_3$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3) \in \mathbb{R}_1^3$.

A vector $\boldsymbol{a} = (a_1, a_2, a_3)$ of $\mathbb{R}_1^3$ is said to be timelikeif $\langle \boldsymbol{a}, \boldsymbol{a} \rangle < 0$, spacelike if $\langle \boldsymbol{a}, \boldsymbol{a} \rangle > 0$ or $\boldsymbol{a} = 0$, and lighlike (null) if $\langle \boldsymbol{a}, \boldsymbol{a} \rangle = 0$ and $\boldsymbol{a} \neq 0$.

The norm of a vector $\boldsymbol{a}$ is defined by $|\boldsymbol{a}| = \sqrt{|\langle \boldsymbol{a}, \boldsymbol{a} \rangle|}$. Let $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ be two vectors in $\mathbb{R}_1^3$, then the Lorentzian cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by

$$\boldsymbol{a} \times \boldsymbol{b} = (a_3 b_2 - a_2 b_3, a_1 b_3 - a_3 b_1, a_1 b_2 - a_2 b_1).$$

If a and a^* are real numbers and $\varepsilon^2 = 0$, the combination $A = a + \varepsilon a^*$ is called a dual number, where ε is a dual unit.

The set of all dual numbers forms a commutative ring over the real numbers field and is denoted by $\mathbb{D}$. Then the set

$$\mathbb{D}^3 = \left\{ \tilde{\boldsymbol{a}} = (a_1, a_2, a_3) \middle| a_i \in \mathbb{D}, 1 \le i \le 3 \right\}$$

is a module over the ring $\mathbb{D}$ which is called a $\mathbb{D}$-module or dual space. The elements of $\mathbb{D}^3$ are called dual vectors. Thus a dual vector $\tilde{\boldsymbol{a}}$ can be written as

$$\tilde{\boldsymbol{a}} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*$$

where $\boldsymbol{a}$ and $\boldsymbol{a}^*$ are real vectors at $\mathbb{R}^3$.

If $A = a + \varepsilon a^*$ and $B = b + \varepsilon b^* \in \mathbb{D}$ with $B \neq 0$ then the division is given by

$$\frac{A}{B} = \frac{a + \varepsilon a^*}{b + \varepsilon b^*} = \frac{a}{b} + \varepsilon \left(\frac{a^*}{b} - \frac{ab^*}{b^2} \right).$$

Let f be a differentiable function with dual variable $X = x + \varepsilon x^*$. Then the Maclaurin series generated by f is

$$f(X) = f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x)$$

where $f'(x)$ is the derivative of $f(x)$. Then it is easy to see that

$$sh(x + \varepsilon x^*) = shx + \varepsilon x^* chx,$$

$$ch\left(x+\varepsilon x^*\right)=chx+\varepsilon x^* shx,$$

$$\sqrt{x+\varepsilon x^*}=\sqrt{x}+\varepsilon\frac{x^*}{2\sqrt{x}},\ \left(x>0\right).$$

The norm $|X|$ of a dual number $X=x+\varepsilon x^*$ is defined by

$$|X|=\left|x+\varepsilon x^*\right|=\sqrt{X^2}=\sqrt{x^2+2\varepsilon xx^*}.$$

Then we can write

$$|X|=|x|+\varepsilon x^*\frac{x}{|x|},\ \left(x\neq 0\right).$$

The Lorentzian inner product of two dual vectors $\tilde{\boldsymbol{a}}=\boldsymbol{a}+\varepsilon\boldsymbol{a}^*$, $\tilde{\boldsymbol{b}}=\boldsymbol{b}+\varepsilon\boldsymbol{b}^*\in\mathbb{D}^3$ is defined by

$$\left\langle\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}}\right\rangle=\left\langle\boldsymbol{a},\boldsymbol{b}\right\rangle+\varepsilon\left(\left\langle\boldsymbol{a},\boldsymbol{b}^*\right\rangle+\left\langle\boldsymbol{a}^*,\boldsymbol{b}\right\rangle\right),$$

where $\left\langle\boldsymbol{a},\boldsymbol{b}\right\rangle$ is the Lorentzian inner product of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in the Minkowski 3-space $\mathbb{R}_1^3$.

A dual vector $\tilde{\boldsymbol{a}}$ is said to be timelike if $\left\langle\tilde{\boldsymbol{a}},\tilde{\boldsymbol{a}}\right\rangle<0$, spacelike if $\left\langle\tilde{\boldsymbol{a}},\tilde{\boldsymbol{a}}\right\rangle>0$ and lightlike (or null) if

$$\left\langle\tilde{\boldsymbol{a}},\tilde{\boldsymbol{a}}\right\rangle=0,$$

where $\langle,\rangle$ is a Lorentzian inner product with signature $(+,+,-)$.

The set of all dual Lorentzian vectors is called dual Lorentzian space and it is denoted by

$$\mathbb{D}_1^3=\left\{\tilde{\boldsymbol{a}}=\boldsymbol{a}+\varepsilon\boldsymbol{a}^*:\boldsymbol{a},\boldsymbol{a}^*\in\mathbb{R}_1^3\right\}.$$

The Lorentzian cross product of dual vectors $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}}\in\mathbb{D}_1^3$ is defined by

$$\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}}=\boldsymbol{a}\times\boldsymbol{b}+\varepsilon\left(\boldsymbol{a}^*\times\boldsymbol{b}+\boldsymbol{a}\times\boldsymbol{b}^*\right),\ \left\langle\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}},\tilde{\boldsymbol{a}}\right\rangle=0$$

where $\boldsymbol{a}\times\boldsymbol{b}$ is the Lorentzian cross product in $\mathbb{R}_1^3$.

Lemma 2.1. Let $\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}},\tilde{\boldsymbol{c}},\tilde{\boldsymbol{d}}\in\mathbb{D}_1^3$. Then [12]

1) $\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}}=-\tilde{\boldsymbol{b}}\times\tilde{\boldsymbol{a}},$

2) $\left\langle\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}},\tilde{\boldsymbol{a}}\right\rangle=0$; and $\left\langle\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}},\tilde{\boldsymbol{b}}\right\rangle=0$,

3) $\left\langle\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}},\tilde{\boldsymbol{c}}\right\rangle=-\det\left(\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}},\tilde{\boldsymbol{c}}\right)$,

4) $\left\langle\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}},\tilde{\boldsymbol{c}}\times\tilde{\boldsymbol{d}}\right\rangle=-\left\langle\tilde{\boldsymbol{a}},\tilde{\boldsymbol{c}}\right\rangle\left\langle\tilde{\boldsymbol{b}},\tilde{\boldsymbol{d}}\right\rangle+\left\langle\tilde{\boldsymbol{a}},\tilde{\boldsymbol{d}}\right\rangle\left\langle\tilde{\boldsymbol{b}},\tilde{\boldsymbol{c}}\right\rangle$.

Let $\tilde{\boldsymbol{a}}=\boldsymbol{a}+\varepsilon\boldsymbol{a}^*\in\mathbb{D}_1^3$. Then $\tilde{\boldsymbol{a}}$ is said to be dual timelike unit vector (resp., dual spacelike unit vector) if the vectors $\boldsymbol{a}$ and $\boldsymbol{a}^*$ satisfy the following properties:

$$\left\langle\boldsymbol{a},\boldsymbol{a}\right\rangle=-1,\ \text{(resp., } \left\langle\boldsymbol{a},\boldsymbol{a}\right\rangle=1\text{)},\ \left\langle\boldsymbol{a},\boldsymbol{a}^*\right\rangle=0.$$

The set of all dual timelike unit vectors (resp., all dual spacelike unit vectors) is called the dual hyperbolic unit sphere (resp., dual Lorentzian unit sphere) and is denoted by $\mathbb{H}_0^2$ (resp., S_1^2) [6]. (See [13]-[16] for Lorentzian basic concepts.)

Theorem 2.2. (E. Study Map) [6] There exists one-to-one correspondence between directed timelike (resp., spacelike) lines of $\mathbb{R}_1^3$ and an ordered pair of vectors $\left(\boldsymbol{a},\boldsymbol{a}^*\right)$ such that $\left\langle\boldsymbol{a},\boldsymbol{a}\right\rangle=-1$ (resp., $\left\langle\boldsymbol{a},\boldsymbol{a}\right\rangle=1$) and $\left\langle\boldsymbol{a},\boldsymbol{a}^*\right\rangle=0$.

Definition 2.1. A directed timelike line in $\mathbb{R}_1^3$ may be given by two points on it, p and q. If λ is any non-zero constant, the parametric equation of the line is $\boldsymbol{q}=\boldsymbol{p}+\lambda\boldsymbol{y}$. In this case, the vector given by

$$\boldsymbol{y}^*=\boldsymbol{p}\times\boldsymbol{y}=\boldsymbol{q}\times\boldsymbol{y}$$

is called the moment of the vector $\boldsymbol{y}$ with respect to the origin 0.

This means that the direction vector $\boldsymbol{y}$ of the timelike line and its moment vector $\boldsymbol{y}^*$ are independent of the choice of the points $p,q,r,\cdots$ on the line. However the vector $\boldsymbol{y}$ and $\boldsymbol{y}^*$ are not independent of one another. Also, they satisfy the following properties:

$$\langle \boldsymbol{y}, \boldsymbol{y} \rangle = 1, \ \langle \boldsymbol{y}, \boldsymbol{y}^* \rangle = 0.$$

Let $\mathbb{H}_0^2$, 0 and $\{0, \tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3 \text{ (timelike)}\}$ denote the dual hyperbolic unit sphere, the center of $\mathbb{H}_0^2$ and the dual orthonormal system at 0, respectively where we have

$$\tilde{\boldsymbol{e}}_i = \boldsymbol{e}_i + \varepsilon \boldsymbol{e}_i^*, \ 1 \le i \le 3$$

$$\tilde{\boldsymbol{e}}_1 \times \tilde{\boldsymbol{e}}_2 = \tilde{\boldsymbol{e}}_3, \quad \tilde{\boldsymbol{e}}_2 \times \tilde{\boldsymbol{e}}_3 = -\tilde{\boldsymbol{e}}_1, \ \tilde{\boldsymbol{e}}_3 \times \tilde{\boldsymbol{e}}_1 = -\tilde{\boldsymbol{e}}_2,$$

and

$$\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3, \quad \boldsymbol{e}_2 \times \boldsymbol{e}_3 = -\boldsymbol{e}_1, \quad \boldsymbol{e}_3 \times \boldsymbol{e}_1 = -\boldsymbol{e}_2$$

[7]. In this case the orthonormal system $\{0, \tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3\}$ is the system of the space of lines $\mathbb{R}_1^3$.

A ruled surface is a surface generated by the motion of a straight line in $\mathbb{R}^3$. This line the generator of the surface. This follows the following definition.

Definition 2.2. A ruled surface is said to be timelike if the normal of surface at every point is spacelike, and spacelike if the normal of surface at every point is timelike [7].

Let $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{y}}$ denote two different points at $\mathbb{H}_0^2$ and ϕ denote the dual hyperbolic angle $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$. The hyperbolic angle ϕ has a value $\phi = \varphi + \varepsilon \varphi^*$ which is a dual number, where φ and φ^* are the hyperbolic angle and the minimal Lorentzian distance between directed lines $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{y}}$, respectively.

3. Dual Curvature Motion on the Dual Hyperbolic Unit Sphere $\mathbb{H}_0^2$

Let us consider a fixed dual orthonormal frame $R = \{\tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2, \tilde{\boldsymbol{u}}_3 \text{ (timelike)}\}$ and represent this frame by the dual hyperbolic unit sphere H'. Consider the dual hyperbolic spherical motion of a hyperbolic spherical segment $\widehat{AB}$ of constant such that its endpoints move along circles, one of them being a great circle $\mathbb{H}_2^1$ which lying on the plane $(\tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_3)$ on H'. Let a circle $\mathbb{H}_1^1$ with radius $\frac{\sqrt{2}}{2}$ which is perpendicular to the great circle $\mathbb{H}_2^1$ be given in a plane which is parallel to the plane $(\tilde{\boldsymbol{u}}_2, \tilde{\boldsymbol{u}}_3)$. Its center is on the vector $\tilde{\boldsymbol{u}}_1$ and with distance $\frac{\sqrt{2}}{2}$ from the plane $(\tilde{\boldsymbol{u}}_2, \tilde{\boldsymbol{u}}_3)$. The segment $\widehat{AB}$ moves so that $A \in \mathbb{H}_1^1, B \in \mathbb{H}_2^1$. The position vectors of the endpoints of segment $\widehat{AB}$ are chosen as the vectors $\tilde{\boldsymbol{v}}_3$ (timelike) and $\tilde{\boldsymbol{v}}_2$ (spacelike) of the moving frame $\bar{R}$. The vector $\tilde{\boldsymbol{v}}_1$ is then defined by the relation $\tilde{\boldsymbol{v}}_1 = -(\tilde{\boldsymbol{v}}_2 \times \tilde{\boldsymbol{v}}_3)$. As the parameter of motion we choose the dual hyperbolic angle ϕ of the timelike vectors $\tilde{\boldsymbol{u}}_3$ and $\tilde{\boldsymbol{v}}_3$. Let us denote the dual hyperbolic angle of the vectors $\tilde{\boldsymbol{v}}_2$ and $\tilde{\boldsymbol{u}}_2$ by α. Then

$$\tilde{\boldsymbol{v}}_2 = \frac{\sqrt{2}}{2} \tilde{\boldsymbol{u}}_1 + \frac{\sqrt{2}}{2} \tilde{\boldsymbol{u}}_2 ch\alpha + \frac{\sqrt{2}}{2} \tilde{\boldsymbol{u}}_3 sh\alpha,$$

where the vector $\tilde{\boldsymbol{v}}_2 = \boldsymbol{OA}$ is spacelike. Further, we have

$$\tilde{\boldsymbol{v}}_3 = \tilde{\boldsymbol{u}}_1 sh\phi + \tilde{\boldsymbol{u}}_3 ch\phi \ \ (\tilde{\boldsymbol{u}}_3 \ \text{ timelike}).$$

where $\tilde{\boldsymbol{v}}_3$ is timelike. It must be

$$\langle \tilde{\boldsymbol{v}}_2, \tilde{\boldsymbol{v}}_3 \rangle = 0, \ i.e., \ \frac{\sqrt{2}}{2} sh\phi - \frac{\sqrt{2}}{2} sh\alpha ch\phi = 0,$$

i.e. $sh\phi = sh\alpha ch\phi$ or $sh\alpha = \tanh\phi$, where $\phi = \varphi + \varepsilon\varphi^*$. Then $ch\alpha = (1 + \tanh^2\phi)^{1/2}$. Thus we obtain

$$\begin{aligned} \tilde{\boldsymbol{v}}_1 &= -(\tilde{\boldsymbol{v}}_2 \times \tilde{\boldsymbol{v}}_3) = -\frac{\sqrt{2}}{2} ch\phi (1 + \tanh^2\phi)^{1/2} \tilde{\boldsymbol{u}}_1 + \left(\frac{\sqrt{2}}{2} ch\phi - \frac{\sqrt{2}}{2} sh\phi \tanh\phi \right) \tilde{\boldsymbol{u}}_2 - \frac{\sqrt{2}}{2} sh\phi (1 + \tanh^2\phi)^{1/2} \tilde{\boldsymbol{u}}_3 \\ \tilde{\boldsymbol{v}}_2 &= \frac{\sqrt{2}}{2} \tilde{\boldsymbol{u}}_1 + \frac{\sqrt{2}}{2} (1 + \tanh^2\phi)^{1/2} \tilde{\boldsymbol{u}}_2 + \frac{\sqrt{2}}{2} \tanh\phi \tilde{\boldsymbol{u}}_3 \\ \tilde{\boldsymbol{v}}_3 &= sh\phi \tilde{\boldsymbol{u}}_1 + ch\phi \tilde{\boldsymbol{u}}_3. \end{aligned} \tag{1}$$

Thus, we have the orthonormal dual frame $\{0,\tilde{\boldsymbol{v}}_1,\tilde{\boldsymbol{v}}_2,\tilde{\boldsymbol{v}}_3\}$. Let this system be represented by moving hyperbolic sphere H. Then, a dual hyperbolic curvature motion H/H' takes place. This motion will be called a dual hyperbolic curvature motion. Let $\boldsymbol{X}$ be a fixed point on the arc $\widehat{v_2v_3}$. During the dual hyperbolic curvature motion, the point $\boldsymbol{X}$ draws an orbit on the fixed hyperbolic sphere H'.

Denote the dual hyperbolic angles of $\widehat{v_2X},\widehat{Xv_3}$ by $\Theta_1=\theta_1+\varepsilon\theta_1^*$ and $\Theta_2=\theta_2+\varepsilon\theta_2^*$ respectively. Then it can be written

$$\boldsymbol{X}=\frac{\tilde{\boldsymbol{v}}_2 sh\Theta_1+\tilde{\boldsymbol{v}}_3 sh\Theta_2}{sh(\Theta_1+\Theta_2)}=\frac{\tilde{\boldsymbol{v}}_2 sh\Theta_1+\tilde{\boldsymbol{v}}_3 sh\Theta_2}{sh\Delta}, \tag{2}$$

where $\Delta=\Theta_1+\Theta_2=\sigma+\varepsilon\sigma^*$ [5]. From Equation (2), making the necessary calculations for $\boldsymbol{X}$, we have

$$\boldsymbol{x}=\left(\frac{\sqrt{2}}{2}\frac{sh\theta_1}{sh\sigma}+\frac{sh\theta_2 sh\varphi}{sh\sigma},\frac{\sqrt{2}}{2}\frac{\sqrt{ch2\varphi}sh\theta_1}{sh\sigma ch\varphi},\frac{\sqrt{2}}{2}\frac{sh\theta_1 sh\varphi}{sh\sigma ch\varphi}+\frac{sh\theta_2 ch\varphi}{sh\sigma}\right) \tag{3}$$

$$\begin{aligned}\boldsymbol{x}^*=\Bigg(&\frac{\sqrt{2}}{2}\frac{1}{sh^2\sigma}\left(\theta_1^* sh\sigma ch\theta_1-\sigma^* sh\theta_1 ch\sigma\right)+\frac{sh\varphi}{sh^2\sigma}\left(\theta_2^* ch\theta_2 sh\sigma-\sigma^* sh\theta_2 ch\sigma\right)\\&+\varphi^*\frac{sh\theta_2 ch\varphi}{sh\sigma},\frac{\sqrt{2}}{2}\varphi^*\frac{\sqrt{ch2\varphi}sh\theta_1 sh\varphi}{sh\sigma ch2\varphi ch^2\varphi}+\frac{\sqrt{2}}{2}\frac{\sqrt{ch2\varphi}}{sh^2\sigma ch\varphi}\left(\theta_1^* sh\sigma ch\theta_1-\sigma^* sh\theta_1 ch\sigma\right),\\&\frac{\sqrt{2}}{2}\varphi^*\frac{sh\theta_1}{sh\sigma ch^2\varphi}+\frac{\sqrt{2}}{2}\frac{sh\varphi}{sh^2\sigma ch\varphi}\left(\theta_1^* sh\sigma ch\theta_1-\sigma^* sh\theta_1 ch\sigma\right)\\&+\varphi^*\frac{sh\theta_2 sh\varphi}{sh\sigma}+\frac{ch\varphi}{sh^2\sigma}\left(\theta_2^* ch\theta_2 sh\sigma-\sigma^* sh\theta_2 ch\sigma\right)\Bigg),\end{aligned} \tag{4}$$

where $\boldsymbol{x}$ and $\boldsymbol{x}^*$ are the real and dual parts of $\boldsymbol{X}$, also. Since, Θ_1 and Θ_2 are constant (*i.e.* θ_1,θ_1^*, θ_2,θ_2^* are all constants), $A=\sigma+\varepsilon\sigma^*$ is constant. Equations (3) and (4) depend only two parameters φ and φ^*. Thus, Equations (3) and (4) represent a timelike congruence in $\mathbb{R}_1^3$ (for more details on congruences, see [10] [12]).

Let $\boldsymbol{p}$ denote the position vector of an arbitrary point $P(y_1,y_2,y_3)$ of a directed timelike line of this timelike line congruence in $\mathbb{R}_1^3$. Then we have

$$\boldsymbol{p}=\boldsymbol{x}(\varphi,\varphi^*)\times\boldsymbol{x}^*(\varphi,\varphi^*)+u\boldsymbol{x}(\varphi,\varphi^*). \tag{5}$$

Since (y_1,y_2,y_3) are the coordinates of P, making the necessary calculations, we obtain

$$\begin{aligned}y_1=&\frac{-\varphi^*\sqrt{ch2\varphi}sh\theta_1}{2sh^2\sigma ch\varphi ch2\varphi}\left(sh\theta_1+2\sqrt{2}sh\theta_2 sh^3\varphi\right)+\frac{\sqrt{2}}{2}\frac{\sqrt{ch2\varphi}sh\theta_2}{sh^3\sigma}\left(\theta_1^* sh\sigma ch\theta_1-\sigma_1^* sh\theta_1 ch\sigma\right)\\&-\frac{\sqrt{2}}{2}\frac{\sqrt{ch2\varphi}sh\theta_1}{sh^3\sigma}\left(\theta_2^* ch\theta_2 sh\sigma-\sigma^* sh\theta_2 ch\varphi\right)+\frac{\lambda}{sh\sigma}\left(\frac{\sqrt{2}}{2}sh\theta_1+sh\theta_2 sh\varphi\right)\end{aligned} \tag{6}$$

$$\begin{aligned}y_2=&\varphi^*\left(\frac{1}{2}\frac{sh^2\theta_1}{sh^2\sigma ch^2\varphi}+\frac{\sqrt{2}}{2}\frac{sh\theta_1 sh\theta_2 sh\varphi}{sh^2\sigma}+\frac{\sqrt{2}}{2}\frac{sh\theta_1 sh\theta_2 sh\varphi}{sh^2\sigma ch^2\varphi}+\frac{sh^2\theta_2 sh^2\varphi}{sh^2\sigma}\right.\\&\left.-\frac{\sqrt{2}}{2}\frac{sh\theta_1 sh\theta_2 sh\varphi}{sh^2\sigma}-\frac{sh^2\theta_2 ch^2\varphi}{sh^2\sigma}\right)+\frac{\sqrt{2}}{2}\frac{sh\theta_1 ch\varphi}{sh^3\sigma}\left(\theta_2^* ch\theta_2 sh\sigma-\sigma^* sh\theta_2 ch\sigma\right)\\&+\frac{\sqrt{2}}{2}\frac{sh\theta_2 sh^2\varphi}{sh^3\sigma ch\varphi}\left(\theta_1^* sh\sigma ch\theta_1-\sigma_1^* sh\theta_1 ch\sigma\right)+\frac{sh\theta_2 sh\varphi ch\varphi}{sh^3\sigma}\left(\theta_2^* ch\theta_2 sh\sigma-\sigma^* sh\theta_2 ch\sigma\right)\\&-\frac{\sqrt{2}}{2}\frac{sh\theta_1 sh^2\varphi}{sh^3\sigma ch\varphi}\left(\theta_2^* ch\theta_2 sh\sigma-\sigma^* sh\theta_2 ch\sigma\right)-\frac{\sqrt{2}}{2}\frac{sh\theta_2 ch\varphi}{sh^3\sigma}\left(\theta_1^* sh\sigma ch\theta_1-\sigma^* sh\theta_1 ch\sigma\right)\\&-\frac{sh\theta_2 sh\varphi ch\varphi}{sh^3\sigma}\left(\theta_2^* ch\theta_2 sh\sigma-\sigma^* sh\theta_2 ch\sigma\right)+\frac{\sqrt{2}}{2}\lambda\frac{\sqrt{ch2\varphi}sh\theta_1}{ch\varphi sh\sigma}\end{aligned} \tag{7}$$

and

$$\begin{aligned}y_3 &= \frac{\varphi^*}{2}\frac{\sqrt{ch2\varphi}sh\theta_1}{sh^2\sigma}\left(\frac{sh\theta_1 sh\varphi}{ch^2\varphi ch2\varphi}+\sqrt{2}\frac{sh\theta_2 sh^2\varphi}{ch^2\varphi ch2\varphi}-\sqrt{2}sh\theta_2\right)\\ &+\frac{\sqrt{2}}{2}\frac{\sqrt{ch2\varphi}sh\theta_2 sh\varphi}{sh^3\sigma ch\varphi}\left(\theta_1^* sh\sigma ch\theta_1-\sigma^* sh\theta_1 ch\sigma\right)\\ &-\frac{\sqrt{2}}{2}\frac{\sqrt{ch2\varphi}sh\theta_1 sh\varphi}{sh^3\sigma ch\varphi}\left(\theta_2^* ch\theta_2 sh\sigma-\sigma^* sh\theta_2 ch\sigma\right)+\frac{\lambda}{sh\sigma}\left(\frac{\sqrt{2}}{2}\frac{sh\theta_1 sh\varphi}{ch\varphi}+sh\theta_2 ch\varphi\right)\end{aligned} \tag{8}$$

If $\Theta_1=\theta_1+\varepsilon\theta_1^*=0$, (*i.e.* $\theta_1=\theta_1^*=0$) then $\Delta=\theta_2$, *i.e.* $\sigma=\theta_2, \sigma^*=\theta_2^*$. In this case, from Equation (2) we have $\boldsymbol{x}=\tilde{\boldsymbol{v}}_3$. Thus, from Equations (6)-(8) we obtain

$$\begin{aligned}y_1 &= \lambda sh\varphi\\ y_2 &= -\varphi^*\\ y_3 &= \lambda ch\varphi\end{aligned} \tag{9}$$

From Equation (9) we have

$$\begin{aligned}y_1^2+y_3^2 &= \lambda^2\\ y_2 &= -\varphi^*\end{aligned} \tag{10}$$

which represents a line congruence. Thus, we have the following theorem.

Theorem 3.1. During the dual hyperbolic spherical curvature motion H/H' in the case of $\Theta_1=0$ (hence $\boldsymbol{x}=\tilde{\boldsymbol{v}}_3$) in Equation (2), the Study map of the orbit which is drawn on the H' by $\boldsymbol{x}=\tilde{\boldsymbol{v}}_3$ is the congruence in $\mathbb{R}_1^3$

$$\begin{aligned}y_1^2+y_3^2 &= \lambda^2\\ y_2 &= -\varphi^*.\end{aligned} \tag{11}$$

If we take $\lambda=\varphi^*=-y_2$ in the Equation (9), then we have

$$y_1^2+y_3^2=y_2^2. \tag{12}$$

Thus, we have the following theorem.

Theorem 3.2. During the dual hyperbolic spherical curvature motion H/H' in the case of $\lambda=\varphi^*=-y_2$, the Study map in $\mathbb{R}_1^3$ of the orbit drawn on the H' by $\boldsymbol{x}=\tilde{\boldsymbol{v}}_3$ is the cone which is given by

$$y_1^2+y_3^2=y_2^2.$$

In addition, if we take $-\varphi^*=c\varphi$ (c = constant) then we have

$$y_2=c\tanh^{-1}\frac{y_1}{y_3},$$

which represents a right helicoid.

If $\Theta_2=\theta_2+\varepsilon\theta_2^*=0$, *i.e.* $\theta_2=\theta_2^*=0$, then $\Delta=\Theta_1$, *i.e.* $\sigma=\theta_1, \sigma^*=\theta_1^*$. In this case , from Equation (2) we have $\boldsymbol{x}=\tilde{\boldsymbol{v}}_2$. Thus, from Equations (6)-(8) we obtain

$$\begin{aligned}2y_1 &= -\varphi^*\frac{\sqrt{ch2\varphi}}{ch\varphi ch2\varphi}+\sqrt{2}\lambda,\\ 2y_2 &= \frac{\varphi^*}{ch^2\varphi}+\sqrt{2}\lambda\frac{\sqrt{ch2\varphi}}{ch\varphi},\\ 2y_3 &= \varphi^*\frac{\sqrt{ch2\varphi}}{ch2\varphi}\frac{sh\varphi}{ch^2\varphi}+\sqrt{2}\lambda\frac{sh\varphi}{ch\varphi}.\end{aligned} \tag{13}$$

If we put $\varphi^*=0, \varphi\neq 0, \lambda\neq 0$, from Equation (13) we have

$$2y_1 = \sqrt{2}\lambda,$$
$$2y_2 = \sqrt{2}\lambda \frac{\sqrt{ch2\varphi}}{ch\varphi}, \tag{14}$$
$$2y_3 = \sqrt{2}\lambda \frac{sh\varphi}{ch\varphi}.$$

From Equation (14) we have

$$y_1^2 + y_3^2 = y_2^2, \tag{15}$$

which represents a cone whose axis is the vector y_2. Thus, we have the following theorem.

Theorem 3.3. During the dual hyperbolic spherical curvature motion, the orbit drawn on H' by $\tilde{\boldsymbol{v}}_2$ (if $\varphi^* = 0, \varphi \neq 0, \lambda \neq 0$) represents a cone in the $\mathbb{R}_1^3$, whose axis is the vector y_2.

If we put $\varphi^* \neq 0, \varphi \neq 0, \lambda = 0$, from Equation (13) we have

$$2y_1 = -\varphi^* \frac{\sqrt{ch2\varphi}}{ch\varphi ch2\varphi},$$
$$2y_2 = \frac{\varphi^*}{ch^2\varphi}, \tag{16}$$
$$2y_3 = \varphi^* \frac{\sqrt{ch2\varphi}}{ch2\varphi} \frac{sh\varphi}{ch^2\varphi}.$$

From Equation (16) we have

$$y_1^2 + y_3^2 = y_2^2, \tag{17}$$

which represents a cone whose axis is the vector y_2. Thus, we have the following theorem.

Theorem 3.4. During the dual hyperbolic spherical curvature motion, the orbit drawn on H' by $\tilde{\boldsymbol{v}}_3$ (if $\varphi^* \neq 0, \varphi \neq 0, \lambda = 0$) represents a cone in the $\mathbb{R}_1^3$, whose axis is the vector y_2.

4. Analysis of the Orbit of $\tilde{v}_1$ during the Dual Hyperbolic Spherical Curvature Motion

Seperating real and dual parts of $\tilde{\boldsymbol{v}}_1$, from Equation (1) we have

$$v_1 = \left(-\frac{\sqrt{2}}{2}\sqrt{ch2\varphi}, \frac{\sqrt{2}}{2}\frac{1}{ch\varphi}, -\frac{\sqrt{2}}{2}\sqrt{ch2\varphi}\frac{sh\varphi}{ch\varphi} \right), \tag{18}$$

$$v_1^* = \left(-\frac{\sqrt{2}}{2}\varphi^* \frac{\sqrt{ch2\varphi}sh\varphi}{ch\varphi}\left(1 + \frac{1}{ch2\varphi}\right), -\frac{\sqrt{2}}{2}\varphi^* \frac{sh\varphi}{ch^2\varphi}, -\frac{\sqrt{2}}{2}\varphi^* \left(\sqrt{ch2\varphi} + \frac{\sqrt{ch2\varphi}sh^2\varphi}{ch2\varphi ch^2\varphi} \right) \right) \tag{19}$$

Equations (18) and (19) have only two parameters φ and φ^*. Hence Equations (18) and (19) represent a line congruence in $\mathbb{R}_1^3$. Let $\boldsymbol{n}$ denote the position vector of an arbitrary point $N(y_1, y_2, y_3)$ of an oriented line of this congruence in $\mathbb{R}_1^3$, then considering Equation (5) we have

$$\boldsymbol{n} = v_1(\varphi, \varphi^*) \times v_1^*(\varphi, \varphi^*) + u v_1(\varphi, \varphi^*). \tag{20}$$

Since (y_1, y_2, y_3) are the coordinates of N, making the necessary calculations, we obtain

$$2y_1 = \varphi^* \frac{\sqrt{ch2\varphi}}{ch\varphi}\left(\frac{3sh^2\varphi + ch^2\varphi}{ch2\varphi} \right) - \sqrt{2}u\sqrt{ch2\varphi},$$
$$2y_2 = \varphi^* \frac{ch2\varphi}{ch^2\varphi} + \sqrt{2}\frac{u}{ch\varphi}, \tag{21}$$
$$2y_3 = \varphi^* \frac{\sqrt{ch2\varphi}sh\varphi}{ch^2\varphi ch2\varphi}\left(3ch^2\varphi + sh^2\varphi\right) - \sqrt{2}u\frac{\sqrt{ch2\varphi}sh\varphi}{ch\varphi}.$$

In the case of $\varphi^* = 0, \varphi \neq 0, u \neq 0$ from Equation (21) we have

$$\begin{aligned} 2y_1 &= -\sqrt{2}u\sqrt{ch2\varphi}, \\ 2y_2 &= \sqrt{2}\frac{u}{ch\varphi}, \\ 2y_3 &= -\sqrt{2}u\frac{\sqrt{ch2\varphi}sh\varphi}{ch\varphi}. \end{aligned} \tag{22}$$

From Equation (22) we obtain

$$y_1^2 = y_3^2 + y_2^2 ch2\varphi \tag{23}$$

which represents an one-parameter family of cone in $\mathbb{R}_1^3$.

If we put $\varphi = \ln\left(1+\sqrt{2}\right)$ in the Equation (23), then we have

$$y_1^2 = y_3^2 + 3y_2^2 \tag{24}$$

which represents an elliptic cone, whose axis is the vector y_1. Thus, we have the following theorem.

Theorem 4.1. During the dual hyperbolic spherical curvature motion, the orbit drawn on H' by $\tilde{\boldsymbol{v}}_1$ (if $\varphi^* = 0, \varphi = \ln\left(1+\sqrt{2}\right), u \neq 0$) represents an elliptic cone, whose axis is the vector y_1 in the $\mathbb{R}_1^3$.

In addition, putting various values of parameters in the Equations (21) or (22) we have different line congruences or ruled surfaces in $\mathbb{R}_1^3$.

5. Conclusion

This paper presents the curvature motion on the dual hyperbolic unit sphere $\mathbb{H}_0^2$. We define the curvature motion on the dual hyperbolic unit sphere $\mathbb{H}_0^2$ of the dual Lorentzian space $\mathbb{D}_1^3$ and the results are carried over to the Lorentzian line space by the E. Study mapping. The orbits drawn on the fixed dual hyperbolic unit sphere by unit dual vectors of an orthonormal base $\{\tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_2, \tilde{\boldsymbol{v}}_3\}$ are obtained. During this carrying, we do an analysis of orbits they drawn by the vectors $\tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_2, \tilde{\boldsymbol{v}}_3$ of dual hyperbolic unit sphere and then we get some line congruences, the families of ruled surfaces and ruled surfaces in according to variables of parameters. Moreover we find equations of these line congruences, the families of ruled surfaces and ruled surfaces. This motion and its results may give a way to define new motions and contribute to the study of surface design, manufacturing technology, robotic research and special and general theory of relativity, and many other areas in 3-dimensional Lorentzian space.

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