

Some Properties of CR -Submanifolds of a Nearly Trans-Sasakian Manifold with a Semi Symmetric Non-Metric Connection

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Abstract

This paper deals with the study of CR -submanifolds of a nearly trans-Sasakian manifold with a semi symmetric non-metric connection. Nijenhuis tensor, integrability conditions for some distributions on CR -submanifolds of a nearly trans-Sasakian manifold with a semi symmetric non-metric connection are discussed.

Keywords

CR -Submanifolds, Nearly Trans-Sasakian Manifold, Semi Symmetric Non-Metric Connection, Distribution

1. Introduction

A. Bejancu defined and studied CR -submanifolds of a Kaehler manifold [1]. Later on, CR -submanifolds of a Sasakian manifold were studied by M. Kobayashi [2], K. Yano and M. Kon [3]. J. A. Oubina introduced a new class of almost contact metric manifold known as trans-Sasakian manifold [4]. This class contains α -Sasakian and β -Kenmotsu manifold [5]. CR -submanifolds of a Kenmotsu manifold were studied by A. Bejancu and N. Papaghuic [6]. Geometry of CR -submanifolds of a trans-Sasakian manifold have been studied by M. H. Shahid in [7] [8]. CR -submanifolds of a nearly trans-Sasakian manifold were studied by Falleh R. Al-Solamy [9]. CR -submanifolds of an LP -Sasakian manifold with a semi-symmetric metric connection were studied by M. Ahmad *et al.* [10]. Motivated by the studies in [11]-[13], in this paper we study CR -submanifolds of a nearly trans-Sasakian manifold endowed with a semi symmetric non-metric connection.

Let ∇ be a linear connection in an n -dimensional differentiable manifold \bar{M} . The torsion tensor T of

∇ is given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

The connection ∇ is symmetric if torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

In [14], S. Golab introduced the idea of a semi-symmetric and quarter symmetric linear connections. A linear connection ∇ is said to be semi-symmetric if its torsion tensor T is of the form

$$T(X, Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form and ϕ is a tensor field of the type (1,1).

We consider integrabilities of horizontal and vertical distributions of CR -submanifolds with a semi symmetric non-metric connection. We also consider parallel horizontal distributions of CR -submanifolds.

The paper is organized as follows: In Section 2, we recall some necessary details of nearly trans-Sasakian manifold. In Section 3, we study CR -submanifolds of a nearly trans-Sasakian manifold. In Section 4, some useful lemmas are proved. In Section 5, some basic results on parallel distribution are investigated. In Section 6, we calculated Nijenhuis tensor and studied integrability conditions of the distributions on CR -submanifolds of a nearly trans-Sasakian manifold with a semi symmetric non-metric connection.

2. Nearly Trans-Sasakian Manifold

Let \bar{M} be an n -dimensional almost contact metric manifold [15] with an almost contact metric structure (ϕ, ξ, η, g) , that is, ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi \cdot \xi = 0, \quad \eta \cdot \phi = 0 \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X) \tag{3}$$

for all vector fields $X, Y \in T\bar{M}$. There are two well known classes of almost contact metric manifolds, namely Sasakian and Kenmotsu manifolds. Sasakian manifolds are characterized by the tensorial relation

$$(\bar{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

while Kenmotsu manifolds are given by the tensor equation

$$(\bar{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X.$$

An almost contact metric structure (ϕ, ξ, η, g) on \bar{M} is called a trans-Sasakian structure [4] if $(\bar{M} \times R, J, G)$ belongs to the class W_4 of Gray-Hervella classification of almost Hermitian manifolds [16], where J is the almost complex structure on $\bar{M} \times R$ defined by

$$J(X, ad/dt) = (\phi X - a\xi, \eta(X)d/dt)$$

for all vector fields X on M and smooth function a on $\bar{M} \times R$. This may be expressed by the condition [17]

$$(\bar{\nabla}_X \phi)Y = \alpha [g(X, Y)\xi - \eta(Y)X] + \beta [g(\phi X, Y)\xi - \eta(Y)\phi X] \tag{4}$$

for some smooth functions α and β on \bar{M} and we say that the trans-Sasakian structure is of type (α, β) .

In 2000, C. Gherghe [18] introduced a nearly trans-Sasakian structure of the type (α, β) . An almost contact metric structure (ϕ, ξ, η, g) on \bar{M} is called a nearly trans-Sasakian structure [18] if

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = \alpha [2g(X, Y)\xi - \eta(Y)X - \eta(X)Y] - \beta [\eta(Y)\phi X + \eta(X)\phi Y]. \tag{5}$$

A trans-Sasakian structure is always a nearly trans-Sasakian structure. Moreover, a nearly trans-Sasakian structure of type (α, β) is nearly Sasakian [19].

Let M be an m -dimensional isometrically immersed submanifold of a nearly trans-Sasakian manifold \bar{M} and denote by the same g the Riemannian metric tensor field induced on M from that of \bar{M} .

3. CR-Submanifolds of Nearly Trans-Sasakian Manifolds

Definition 3.1 An m -dimensional Riemannian submanifold M of a nearly trans-Sasakian manifold \bar{M} is called a CR-submanifold if ξ is tangent to M and there exists on M a differentiable distribution $D : x \rightarrow D_x \subset T_x(M)$ such that

(i) the distribution D_x is invariant under ϕ , i.e., $\phi D_x \subset D_x$ for each $x \in M$;

(ii) The orthogonal complementary distribution $D^\perp : x \rightarrow D_x^\perp \subset T_x(M)$ of the distribution D on M is anti-invariant under ϕ , i.e., $\phi D_x^\perp(M) \subset T_x^\perp(M)$ for all $x \in M$, where $T_x(M)$ and $T_x^\perp(M)$ are tangent space and normal space of M at $x \in M$ respectively.

If $\dim D_x^\perp = 0$ (resp., $\dim D_x = 0$), then CR-submanifold is called an invariant (resp., anti-invariant). The distribution D (resp., D^\perp) is called the horizontal (resp., vertical) distribution. The pair (D, D^\perp) is called ξ -horizontal (resp., ξ -invariant) if $\xi_x \in D_x$ (resp., $\xi_x \in D_x^\perp$) for $x \in M$.

For any vector field X tangent to M , we put

$$X = PX + QX, \tag{6}$$

where PX and QX belong to the distribution D and D^\perp respectively.

For any vector field N normal to M , we put

$$\phi N = BN + CN, \tag{7}$$

where BN (resp., CN) denotes the tangential (resp., normal) component of ϕN .

Now, we remark that owing to the existence of the 1-form η , we can define a semi symmetric non-metric connection $\bar{\nabla}$ in any almost contact metric manifold by

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \eta(Y)X \tag{8}$$

such that $(\bar{\nabla}_X g)(Y, Z) = -\eta(Y)g(X, Z) - \eta(Z)g(X, Y)$ for any $X, Y \in TM$, where $\bar{\bar{\nabla}}$ is the induced connection with respect to g on M .

By using (4) and (8), we get

$$(\bar{\nabla}_X \phi)Y = \alpha [g(X, Y)\xi - \eta(Y)X] + \beta [g(\phi X, Y)\xi - \eta(Y)\phi X] - \eta(Y)\phi X. \tag{9}$$

Similarly, we have

$$(\bar{\nabla}_Y \phi)X = \alpha [g(Y, X)\xi - \eta(X)Y] + \beta [g(\phi Y, X)\xi - \eta(X)\phi Y] - \eta(X)\phi Y.$$

On adding above equations, we obtain

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = \alpha [2g(X, Y)\xi - \eta(X)Y - \eta(Y)X] - (\beta + 1)[\eta(Y)\phi X - \eta(X)\phi Y]. \tag{10}$$

This is the condition for $\bar{M}(\phi, \xi, \eta, g)$ with a semi symmetric non-metric connection to be nearly trans-Sasakian manifold.

We denote by g the metric tensor of \bar{M} as well as that induced on M . Let $\bar{\nabla}$ be the semi symmetric non-metric connection on \bar{M} and ∇ be the induced connection on M with respect to the unit normal N . Then we have:

Theorem 3.2 (i) If M is ξ -horizontal, $X, Y \in D$ and D is parallel with respect to ∇ , then the connection induced on a CR-submanifold of a nearly trans-Sasakian manifold with a semi symmetric non-metric connection is also a semi symmetric non-metric connection.

(ii) If M is ξ -vertical, $X, Y \in D^\perp$ and D^\perp is parallel with respect to ∇ , then the connection induced on a CR-submanifold of a nearly trans-Sasakian with a semi symmetric non-metric connection is also a semi symmetric non-metric connection.

(iii) The Gauss formula with respect to the semi symmetric non-metric connection is of the form

$$\bar{\nabla}_x Y = \nabla_x Y + h(X, Y).$$

Proof. Let ∇ be the induced connection with respect to the unit normal N on a CR -submanifold of a nearly trans-Sasakian manifold from a semi symmetric non-metric connection $\bar{\nabla}$, then

$$\bar{\nabla}_x Y = \nabla_x Y + m(X, Y), \tag{11}$$

where m is a tensor field of the type (0,2) on CR -submanifold M . If $\nabla^{\hat{a}}$ be the induced connection on CR -submanifold from Riemannian connection $\bar{\bar{\nabla}}$, then

$$\bar{\bar{\nabla}}_x Y = \nabla_x^{\hat{a}} Y + h(X, Y), \tag{12}$$

where h is a second fundamental form. By the definition of the semi symmetric non-metric connection, we have

$$\bar{\nabla}_x Y = \bar{\bar{\nabla}}_x Y + \eta(Y)X.$$

Now, using (11) and (12) in above equation, we have

$$\nabla_x Y + m(X, Y) = \nabla_x^{\hat{a}} Y + h(X, Y) + \eta(Y)X.$$

Using (6), the above equation can be written as

$$P\nabla_x Y + Q\nabla_x Y + m(X, Y) = P\nabla_x^{\hat{a}} Y + Q\nabla_x^{\hat{a}} Y + h(X, Y) + \eta(Y)PX + \eta(Y)QX. \tag{13}$$

From (13), comparing the tangential and normal components from both the sides, we get

$$h(X, Y) = m(X, Y) \tag{14}$$

$$P\nabla_x Y - \eta(Y)PX = P\nabla_x^{\hat{a}} Y \tag{15}$$

$$Q\nabla_x Y - \eta(Y)QX = Q\nabla_x^{\hat{a}} Y. \tag{16}$$

Using (14), the Gauss formula for a CR -submanifold of a nearly trans-Sasakian manifold with a semi symmetric non-metric connection is

$$\bar{\nabla}_x Y = \nabla_x Y + h(X, Y). \tag{17}$$

This proves (iii). In view of (15), if M is ξ -horizontal, $X, Y \in D$ and D is parallel with respect to ∇ , then the connection induced on a CR -submanifold of a nearly trans-Sasakian manifold with a semi symmetric non-metric connection is also a semi symmetric non-metric connection.

Similarly, using (16), if M is ξ -vertical, $X, Y \in D^\perp$ and D^\perp is parallel with respect to ∇ , then the connection induced on a CR -submanifold of a nearly trans-Sasakian manifold with a semi symmetric non-metric connection is also a semi symmetric non-metric connection.

Weingarten formula is given by

$$\bar{\nabla}_x N = -A_N X + \nabla_x^\perp N + \eta(N)X \tag{18}$$

for $X, Y \in TM, N \in T^\perp M, h: TM \times TM \rightarrow TM^\perp$ (resp., $A_N: TM \rightarrow TM$) is the second fundamental form (resp., tensor) of M in \bar{M} and ∇^\perp denotes the operator of the normal connection. Moreover, we have

$$g(h(X, Y), N) = g(A_N X, Y). \tag{19}$$

4. Some Basic Lemmas

Lemma 4.1 Let M be a CR -submanifold of a nearly trans-Sasakian manifold \bar{M} with a semi symmetric non-metric connection. Then

$$\begin{aligned} &P(\nabla_x \phi PY) + P(\nabla_y \phi PX) - P(A_{\phi QY} X) - P(A_{\phi QX} Y) \\ &= \phi P\nabla_x Y + \phi P\nabla_y X + 2\alpha g(X, Y)P\xi - \alpha\eta(X)PY - \alpha\eta(Y)PX - (\beta + 1)[\eta(Y)\phi PX + \eta(X)\phi PY] \end{aligned} \tag{20}$$

$$\begin{aligned} & Q(\nabla_x \phi PY) + Q(\nabla_y \phi PX) - Q(A_{\phi QY} X) - Q(A_{\phi QX} Y) \\ & = 2\alpha g(X, Y)Q\xi - \alpha\eta(X)QY - \alpha\eta(Y)QX + 2Bh(X, Y) \end{aligned} \tag{21}$$

$$\begin{aligned} & h(X, \phi PY) + h(Y, \phi PX) + \nabla_x^\perp \phi QY + \nabla_y^\perp \phi QX \\ & = \phi(Q\nabla_x Y) + \phi(Q\nabla_y X) + 2Ch(X, Y) - (\beta + 1)[\eta(Y)\phi QX + \eta(X)\phi QY] \end{aligned} \tag{22}$$

for $X, Y \in TM$.

Proof. By direct covariant differentiation, we have

$$\bar{\nabla}_x \phi Y = (\bar{\nabla}_x \phi)Y + \phi(\bar{\nabla}_x Y).$$

By virtue of (6), (9), (17) and (18), we get

$$\begin{aligned} \nabla_x \phi PY + h(X, \phi PY) - A_{\phi QY} X + \nabla_x^\perp \phi QY - \phi[V] &= \alpha[g(X, Y)\xi - \eta(Y)X] \\ &+ \beta g(\phi X, Y)\xi - (\beta + 1)\eta(Y)\phi X. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \nabla_y \phi PX + h(Y, \phi PX) - A_{\phi QX} Y + \nabla_y^\perp \phi QX - \phi[\nabla_y X + h(X, Y)] &= \alpha[g(X, Y)\xi - \eta(X)Y] \\ &+ \beta g(\phi Y, X)\xi - (\beta + 1)\eta(X)\phi Y. \end{aligned}$$

On adding above equations, we have

$$\begin{aligned} \nabla_x \phi PY + \nabla_y \phi PX + h(X, \phi PY) + h(Y, \phi PX) - A_{\phi QY} X - A_{\phi QX} Y + \nabla_x^\perp \phi QY + \nabla_y^\perp \phi QX - \phi\nabla_x Y - \phi\nabla_y X \\ - 2\phi h(X, Y) = \alpha[2g(X, Y)\xi - \eta(Y)Y - \eta(Y)X] - (\beta + 1)[\eta(X)\phi Y + \eta(Y)\phi X]. \end{aligned}$$

Now using (6), (7) and equating horizontal, vertical and normal components in above equation, the lemma follows.

Lemma 4.2 *Let M be a CR-submanifold of a nearly trans-Sasakian manifold \bar{M} with a semi symmetric non-metric connection. Then*

$$\begin{aligned} 2(\bar{\nabla}_x \phi)Y = \nabla_x \phi Y - \nabla_y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y] + \alpha[2g(X, Y)\xi - \eta(Y)X - \eta(X)Y] \\ - (\beta + 1)[\eta(Y)\phi X + \eta(X)\phi Y] \end{aligned} \tag{23}$$

for any $X, Y \in D$.

Proof. By the use of (17), we have

$$\bar{\nabla}_x \phi Y - \bar{\nabla}_y \phi X = \nabla_x \phi Y - \nabla_y \phi X + h(X, \phi Y) - h(Y, \phi X). \tag{24}$$

Also, we have

$$\bar{\nabla}_x \phi Y - \bar{\nabla}_y \phi X = (\bar{\nabla}_x \phi)Y - (\bar{\nabla}_y \phi)X + \phi[X, Y]. \tag{25}$$

From above equations, we get

$$(\bar{\nabla}_x \phi)Y - (\bar{\nabla}_y \phi)X = \nabla_x \phi Y - \nabla_y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y]. \tag{26}$$

For a nearly trans-Sasakian manifold with a semi symmetric non-metric connection, we have

$$(\bar{\nabla}_x \phi)Y + (\bar{\nabla}_y \phi)X = \alpha[2g(X, Y)\xi - \eta(X)Y - \eta(Y)X] - (\beta + 1)[\eta(Y)\phi X + \eta(X)\phi Y]. \tag{27}$$

Combining (26) and (27), the lemma follows.

In particular, we have the following corollary.

Corollary 4.3 *Let M be a ξ -vertical CR-submanifold of a nearly trans-Sasakian manifold \bar{M} with a semi symmetric non-metric connection. Then*

$$2(\bar{\nabla}_x \phi)Y = \nabla_x \phi Y - \nabla_y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y] + 2\alpha g(X, Y)\xi \tag{28}$$

for any $X, Y \in D$.

Similarly, by Weingarten formula, we can easily get the following lemma.

Lemma 4.4 Let M be a CR-submanifold of a nearly trans-Sasakian manifold \bar{M} with a semi symmetric non-metric connection. Then

$$2(\bar{\nabla}_Y\phi)Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^\perp\phi Z - \nabla_Z^\perp\phi Y - \phi[Y, Z] + \alpha[2g(X, Y)\xi - \eta(Y)Z - \eta(Z)Y] - (\beta + 1)[\eta(Z)\phi Y + \eta(Y)\phi Z] \tag{29}$$

for any $Y, Z \in D$.

Corollary 4.5 Let M be a ξ -horizontal CR-submanifold of a nearly trans-Sasakian manifold \bar{M} with a semi symmetric non-metric connection. Then

$$2(\bar{\nabla}_Y\phi)Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^\perp\phi Z - \nabla_Z^\perp\phi Y - \phi[Y, Z] + 2\alpha g(X, Y)\xi \tag{30}$$

for any $Y, Z \in D^\perp$.

Lemma 4.6 Let M be a CR-submanifold of a nearly trans-Sasakian manifold \bar{M} with a semi symmetric non-metric connection. Then

$$2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^\perp\phi Y - h(Y, \phi X) - \nabla_Y\phi X - \phi[X, Y] + \alpha[2g(X, Y)\xi - \eta(Y)X - \eta(X)Y] - (\beta + 1)[\eta(Y)\phi X + \eta(X)\phi Y] \tag{31}$$

for any $X \in D, Y \in D^\perp$.

Proof. As we have

$$(\bar{\nabla}_X\phi)Y = \nabla_X\phi Y - \phi(\bar{\nabla}_X Y).$$

Now, by using Gauss and Weingarten formulae in above equation, we have

$$(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^\perp\phi Y - \phi\nabla_X Y - \phi h(X, Y).$$

Also, we have

$$(\bar{\nabla}_Y\phi)X = \nabla_Y\phi X + h(Y, \phi X) - \phi\nabla_Y X - \phi h(X, Y).$$

From above equations, we get

$$(\bar{\nabla}_X\phi)Y - \bar{\nabla}_Y\phi X = -A_{\phi Y}X + \nabla_X^\perp\phi Y - \phi[X, Y] - \nabla_Y\phi X - h(Y, \phi X).$$

In view of (10) and above equation, the lemma follows.

5. Parallel Distributions

Definition 5.1 The horizontal (resp., vertical) distribution D (resp., D^\perp) is said to be parallel [1] with respect to the semi symmetric non-metric connection ∇ on M if $\nabla_X Y \in D$ (resp., $\nabla_Z W \in D^\perp$) for any $X, Y \in D$ (resp., $W, Z \in D^\perp$).

Now, we have the following proposition.

Proposition 5.2 Let M be a ξ -vertical CR-submanifold of a nearly trans-Sasakian manifold \bar{M} with a semi symmetric non-metric connection. Then

$$h(X, \phi Y) = h(Y, \phi X) \tag{32}$$

for all $X, Y \in D$.

Proof. By the parallelness of horizontal distribution D , we have

$$\nabla_X\phi Y \in D, \nabla_Y\phi X \in D \text{ for any } X, Y \in D. \tag{33}$$

For $Y \in D$, using the fact that $QX = QY = 0$, (21) gives

$$Bh(X, Y) = g(X, Y)Q\xi \text{ for any } X, Y \in D. \tag{34}$$

Therefore in view of (7), we have

$$\phi h(X, Y) = g(X, Y)Q\xi + Ch(X, Y) \text{ for any } X, Y \in D. \tag{35}$$

From (22), we have

$$h(X, \phi Y) + h(Y, \phi X) = 2\phi h(X, Y) - 2g(X, Y)Q\xi \tag{36}$$

for any $X, Y \in D$.

Now, putting $X = \phi X \in D$ and $Y = \phi Y \in D$ in (36), we get respectively

$$h(\phi X, \phi Y) - h(Y, X) = 2\phi h(\phi X, Y) - 2g(\phi X, Y)Q\xi \tag{37}$$

$$h(\phi X, \phi Y) - h(Y, X) = 2\phi h(X, \phi Y) - 2g(X, \phi Y)Q\xi \tag{38}$$

Hence from (37) and (38), we have

$$\phi h(X, \phi Y) - \phi h(Y, \phi X) = g(X, \phi Y)Q\xi - g(\phi X, Y)Q\xi. \tag{39}$$

Operating ϕ on both sides of (39) and using $\phi\xi = 0$, we get

$$h(X, \phi Y) = h(Y, \phi X) \tag{40}$$

for all $X, Y \in D$.

Now, for the distribution D^\perp , we have the following proposition.

Proposition 5.3 *Let M be a ξ -vertical CR-submanifold of a nearly trans-Sasakian manifold \bar{M} with a semi symmetric non-metric connection. If the distribution D^\perp is parallel with a semi symmetric non-metric connection on M . Then*

$$(A_{\phi Y}Z + A_{\phi Z}Y) \in D^\perp \text{ for any } Y, Z \in D^\perp. \tag{41}$$

Proof. By using Weingarten formula, we have

$$\bar{\nabla}_Y \phi Z = -A_{\phi Z}Y + \nabla_Y^\perp \phi Z$$

and

$$\bar{\nabla}_Z \phi Y = -A_{\phi Y}Z + \nabla_Z^\perp \phi Y$$

for $Y, Z \in D^\perp$. From above equations, we have

$$-A_{\phi Z}Y + \nabla_Y^\perp \phi Z - A_{\phi Y}Z + \nabla_Z^\perp \phi Y = (\bar{\nabla}_Y \phi)Z + (\bar{\nabla}_Z \phi)Y + \phi(\bar{\nabla}_Y Z + \bar{\nabla}_Z Y).$$

Using (10) and (17), we obtain

$$\begin{aligned} -A_{\phi Z}Y + \nabla_Y^\perp \phi Z - A_{\phi Y}Z + \nabla_Z^\perp \phi Y &= \alpha [2g(Y, Z)\xi - \eta(Y)Z - \eta(Z)Y] - (\beta + 1)[\eta(Y)\phi Z + \eta(Z)\phi Y] \\ &\quad + \phi \nabla_Y Z + \phi \nabla_Z Y + 2\phi h(Y, Z) \end{aligned} \tag{42}$$

for any $Y, Z \in D^\perp$. Taking inner product with $X \in D$ in (41), we get

$$g(A_{\phi Z}Y, X) + g(A_{\phi Y}Z, X) = g(\nabla_Y Z, \phi X) + g(\nabla_Z Y, \phi X). \tag{43}$$

If the distribution D^\perp is parallel, then $\nabla_Y Z \in D^\perp$ and $\nabla_Z Y \in D^\perp$ for any $Y, Z \in D^\perp$. So from above equation, we get

$$g(A_{\phi Z}Y, X) + g(A_{\phi Y}Z, X) = 0 \tag{44}$$

or

$$g(A_{\phi Z}Y + A_{\phi Y}Z, X) = 0, \tag{45}$$

which implies that $A_{\phi Z}Y + A_{\phi Y}Z \in D^\perp$ for any $Y, Z \in D^\perp$.

Definition 5.4 A CR-submanifold with a semi symmetric non-metric connection is said to be mixed totally geodesic if $h(X, Z) = 0$ for all $X \in D$ and $Z \in D^\perp$.

Definition 5.5 A normal vector field $N \neq 0$ with a semi symmetric non-metric connection is called D -parallel normal section if $\nabla_X^\perp N = 0$ for all $X \in D$.

Now, we have the following proposition.

Proposition 5.6 Let M be a mixed totally geodesic ξ -vertical CR-submanifold of a nearly trans-Sasakian manifold \bar{M} with a semi symmetric non-metric connection. Then the normal section $N \in \phi D^\perp$ is D -parallel if and only if $\nabla_X \phi N \in D$ for all $X \in D$.

6. Integrability Conditions of Distributions

In this section, we calculate the Nijenhuis tensor $N(X, Y)$ on a nearly trans-Sasakian manifold \bar{M} with a semi symmetric non-metric connection. For this, first we prove the following lemma.

Lemma 6.1 Let \bar{M} be a nearly trans-Sasakian manifold with a semi symmetric non-metric connection. Then

$$\begin{aligned}
 (\bar{\nabla}_{\phi X} \phi)Y &= 2\alpha g(\phi X, Y)\xi - \alpha\eta(Y)\phi X + (\beta + 1)\eta(Y)X - (\beta + 1)\eta(X)\eta(Y)\xi - \eta(X)\bar{\nabla}_Y \xi \\
 &\quad + \phi(\bar{\nabla}_Y \phi)X - ((\bar{\nabla}_Y \eta)X)\xi
 \end{aligned} \tag{46}$$

for any $X, Y \in T\bar{M}$.

Proof. From the definition of nearly trans-Sasakian manifold with a semi symmetric non-metric connection \bar{M} , we have

$$(\bar{\nabla}_{\phi X} \phi)Y = 2\alpha g(\phi X, Y)\xi - \alpha\eta(Y)\phi X + (\beta + 1)\eta(Y)X - (\beta + 1)\eta(X)\eta(Y)\xi - (\bar{\nabla}_Y \phi)\phi X. \tag{47}$$

Also, we have

$$(\bar{\nabla}_Y \phi)\phi X = \eta(X)\bar{\nabla}_Y \xi - \phi(\bar{\nabla}_Y \phi)X + ((\bar{\nabla}_Y \eta)X)\xi. \tag{48}$$

Now, using (48) in (47), we get

$$\begin{aligned}
 (\bar{\nabla}_{\phi X} \phi)Y &= 2\alpha g(\phi X, Y)\xi - \alpha\eta(Y)\phi X + (\beta + 1)\eta(Y)X - (\beta + 1)\eta(X)\eta(Y)\xi - \eta(X)\bar{\nabla}_Y \xi \\
 &\quad + \phi(\bar{\nabla}_Y \phi)X - ((\bar{\nabla}_Y \eta)X)\xi
 \end{aligned} \tag{49}$$

for any $X, Y \in T\bar{M}$, which completes the proof of the lemma.

On a nearly trans-Sasakian manifold with a semi symmetric non-metric connection \bar{M} , Nijenhuis tensor is given by

$$N(X, Y) = (\bar{\nabla}_{\phi X} \phi)Y - (\bar{\nabla}_{\phi Y} \phi)X - \phi(\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_Y \phi)X \tag{50}$$

for any $X, Y \in T\bar{M}$.

From (46) and (50), we get

$$\begin{aligned}
 N(X, Y) &= 4\alpha g(\phi X, Y)\xi - \alpha[\eta(Y)\phi X - \eta(X)\phi Y] + (\beta + 1)[\eta(Y)X - \eta(X)Y] \\
 &\quad - \eta(X)\bar{\nabla}_Y \xi + \eta(Y)\bar{\nabla}_X \xi + 2g(X, \phi Y)\xi + 2\phi(\bar{\nabla}_Y \phi)X - 2\phi(\bar{\nabla}_X \phi)Y.
 \end{aligned} \tag{51}$$

In view of (10), we have

$$\phi(\bar{\nabla}_X \phi)Y = -\alpha\eta(Y)\phi X - \alpha\eta(X)\phi Y + (\beta + 1)[\eta(Y)X + \eta(X)Y] - 2(\beta + 1)\eta(X)\eta(Y)\xi - \phi(\bar{\nabla}_Y \phi)X.$$

Using above equation in (51), we obtain

$$\begin{aligned}
 N(X, Y) &= 4\alpha g(\phi X, Y)\xi + \alpha\eta(Y)\phi X + 3\alpha\eta(X)\phi Y - (\beta + 1)\eta(Y)X - 3(\beta + 1)\eta(X)Y \\
 &\quad + 2g(X, \phi Y)\xi + 4\phi(\bar{\nabla}_Y \phi)X + 4(\beta + 1)\eta(X)\eta(Y)\xi - \eta(X)\bar{\nabla}_Y \xi + \eta(Y)\bar{\nabla}_X \xi
 \end{aligned} \tag{52}$$

for any $X, Y \in T\bar{M}$.

Proposition 6.2 Let M be a ξ -vertical CR-submanifold of a nearly trans-Sasakian manifold \bar{M} with a semi symmetric non-metric connection. Then the distribution D is integrable if the following conditions are satisfied:

$$S(X, Z) \in D, \quad h(X, Z) = h(\phi X, Z) \tag{53}$$

for any $X, Z \in D$.

Proof. The torsion tensor $S(X, Y)$ of the almost contact metric structure (ϕ, ξ, η, g) is given by

$$S(X, Y) = N(X, Y) + 2d\eta(X, Y)\xi = N(X, Y) + 2g(\phi X, Y)\xi. \quad (54)$$

Thus, we have

$$S(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2g(\phi X, Y)\xi \quad (55)$$

for any $X, Y \in TM$.

Suppose that the distribution D is integrable. So for $X, Y \in D$, $Q[X, Y] = 0$. If $S(X, Y) \in D$, then from (52) and (54), we have

$$4\alpha g(\phi X, Y)Q\xi + 4(\phi Q\nabla_Y \phi X + \phi h(Y, \phi X) + Q\nabla_Y X + h(X, Y)) = 0 \quad (56)$$

for any $X, Y \in D$ and $\xi \in D^\perp$.

Replacing Y by ϕZ for $Z \in D$, we get

$$4\alpha g(\phi X, \phi Z)Q\xi + 4(\phi Q\nabla_{\phi Z} \phi X + \phi h(\phi Z, \phi X) + Q\nabla_{\phi Z} X + h(X, \phi Z)) = 0. \quad (57)$$

Interchanging X and Z for $X, Z \in D$ in (57), we have

$$4\alpha g(\phi Z, \phi X)Q\xi + 4(\phi Q\nabla_{\phi X} \phi Z + \phi h(\phi X, \phi Z) + Q\nabla_{\phi X} Z + h(Z, \phi X)) = 0. \quad (58)$$

Subtracting above equations, we get

$$\phi Q[\phi X, \phi Z] + Q[X, \phi Z] + h(X, \phi Z) - h(Z, \phi X) = 0 \quad (59)$$

for any $X, Z \in D$ and the assertion follows.

Now, we prove the following proposition.

Proposition 6.3 *Let M be a CR-submanifold of a nearly trans-Sasakian manifold \bar{M} with a semi symmetric non-metric connection. Then*

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{1}{3}\phi P[Y, Z] + \alpha(\eta(Y)Z - \eta(Z)Y) \quad (60)$$

for any $Y, Z \in D^\perp$.

Proof. For $Y, Z \in D^\perp$ and $X \in TM$, we have

$$\begin{aligned} 2g(A_{\phi Z}Y, X) &= 2g(h(X, Y), \phi Z) \\ 2g(A_{\phi Z}Y, X) &= g(h(X, Y), \phi Z) + g(h(X, Y), \phi Z) \\ 2g(A_{\phi Z}Y, X) &= g(\bar{\nabla}_X Y + \bar{\nabla}_Y X, \phi Z) \\ 2g(A_{\phi Z}Y, X) &= -g(\phi(\bar{\nabla}_X Y + \bar{\nabla}_Y X), Z) \\ 2g(A_{\phi Z}Y, X) &= -g\left[(\bar{\nabla}_Y \phi X + \bar{\nabla}_X \phi Y) - \alpha(2g(X, Y)\xi - \eta(X)Y - \eta(Y)X) \right. \\ &\quad \left. + (\beta + 1)(\eta(Y)\phi X + \eta(X)\phi Y), Z\right] \\ 2g(A_{\phi Z}Y, X) &= -g(\bar{\nabla}_Y \phi X, Z) - g(\bar{\nabla}_X \phi Y, Z) + 2\alpha g(X, Y)\eta(Z) - \alpha g(X, Z)\eta(Y) - \alpha g(Y, Z)\eta(X) \\ 2g(A_{\phi Z}Y, X) &= g(\bar{\nabla}_Y Z, \phi X) + g(A_{\phi Y}Z, X) + 2\alpha g(X, Y)\eta(Z) - \alpha g(X, Z)\eta(Y) - \alpha g(Y, Z)\eta(X). \end{aligned} \quad (61)$$

The above equation is true for all $X \in TM$, therefore transvecting the vector field X both sides, we obtain

$$2A_{\phi Z}Y = A_{\phi Y}Z - \phi\bar{\nabla}_Y Z - \alpha g(Y, Z)\xi - \alpha\eta(Y)Z + 2\alpha\eta(Z)Y. \quad (62)$$

Interchanging the vector fields Y and Z , we get

$$2A_{\phi Y}Z = A_{\phi Z}Y - \phi\bar{\nabla}_Z Y - \alpha g(Y, Z)\xi - \alpha\eta(Z)Y + 2\alpha\eta(Y)Z. \quad (63)$$

From (62) and (63), we get

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{1}{3}\phi P[Y, Z] + \alpha(\eta(Y)Z - \eta(Z)Y) \quad (64)$$

for any $Y, Z \in D^\perp$, which completes the proof.

Proposition 6.4 Let M be a CR-submanifold of a nearly trans-Sasakian manifold \bar{M} with a semi symmetric non-metric connection. Then the distribution D^\perp is integrable if and only if

$$A_{\phi Y}Z - A_{\phi Z}Y = \alpha(\eta(Y)Z - \eta(Z)Y) \quad (65)$$

for $Y, Z \in D^\perp$.

Proof. Proof of the theorem is similar as proof of the theorem 5.4 of [2].

Corollary 6.5 Let M be a ξ -horizontal CR-submanifold of a nearly trans-Sasakian manifold \bar{M} with a semi symmetric non-metric connection. Then the distribution D^\perp is integrable if and only if

$$A_{\phi Y}Z - A_{\phi Z}Y = 0 \quad (66)$$

for $Y, Z \in D^\perp$.

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