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Uniform Exponential Attractors for Non-Autonomous Strongly Damped Wave Equations

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Abstract

In this paper, we study the existence of exponential attractors for strongly damped wave equations with a time-dependent driving force. To this end, the uniform Hölder continuity is established to the variation of the process in the phase apace. In a certain parameter region, the exponential attractor is a uniformly exponentially attracting time-dependent set in the phase apace, and is finite-dimensional no matter how complex the dependence of the external forces on time is. On this basis, we also obtain the existence of the infinite-dimensional uniform exponential attractor for the system.

Keywords

Exponential Attractor, Uniform Attractor, Strongly Damped Wave Equation

1. Introduction

In this paper, we study the following non-autonomous strongly damped wave equation on a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$:

$$\begin{cases} u_{tt} - \alpha\Delta u_t - \Delta u + \gamma u_t + f(u) = g(t), \\ u|_{\partial\Omega} = 0, \\ \big(u(\tau), u_t(\tau)\big) = (u_{0\tau}, u_{1\tau}), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a real-valued function on $\Omega \times [\tau, +\infty)$, $\alpha, \gamma > 0$, $f \in C^1(\mathbb{R}, \mathbb{R})$, $u_{0\tau}(x) \in H_0^1(\Omega)$, $u_{1\tau}(x) \in L^2(\Omega)$. Let $G(u) = \int_0^u f(s)\mathrm{d}s$, we make the following assumptions on functions $G(u)$, $f(u)$:

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$$\liminf_{|u|\to+\infty} \frac{G(u)}{u^2} \geq 0; \tag{1.2}$$

$$\liminf_{|u|\to+\infty} \frac{uf(u)-c_1G(u)}{u^2} \geq 0; \tag{1.3}$$

$$|f'(u)| \leq c_2\left(1+|u|^r\right), \quad \text{with } 0 \leq r < 2; \tag{1.4}$$

where c_1, c_2 are positive constants. And we assume that the external force g belongs to the space $L^\infty\left(\mathbb{R}, L^2(\Omega)\right)$ and satisfies

$$|g|_{L^\infty\left(\mathbb{R}, L^2(\Omega)\right)} \leq M \tag{1.5}$$

for some given (possibly large) constant M.

Wave equations, describing a great variety of wave phenomena, occur in the extensive applications of mathematical physics. Equation (1.1) can be regarded as a perturbed equation of a continuous Josephson junction where $f(u)=\sin u$, see [1]. There is a large literature on the asymptotic behavior of solutions for strongly damped wave equations (see, for instance, [1]-[9]). In [9], the author showed the uniform boundedness of the global attractor for large strongly damping and obtained an estimate of the upper bound of the Hausdorff dimension of an attractor for strongly damped wave Equation (1.1) when g is independent of t. But when the equations depend explicitly on t, the case can be complex.

Recently, motivated by [6], the authors have given a new explicit algorithm allowing to construct the exponential attractor, and this method makes it possible to consider more general processes in applications [10] [11].

An exponential attractor $\mathcal{M}$ is a compact semi-invariant set of the phase space whose fractal dimension is finite and which attracts exponentially the images of the bounded subsets of the phase space Φ. In non-autonomous dynamical systems, instead of a semigroup, we have a so-called (dynamical) process $U(t,\tau)$ depending on two parameters $t,\tau \in \mathbb{R}$ (or $t,\tau \in \mathbb{Z}$ for discrete times). The asymptotic behavior of non-autonomous dynamical systems is essentially less understood and, to the best of our knowledge, the finite-dimensionality of the limit dynamics was established only for some special (e.g. quasiperiodic) dependence of the external forces on time. Indeed, there exists, at the present time, one of the different approaches for extending the concept of a global attractor to the non-autonomous case which is based on the embedding of the non-autonomous dynamical system into a larger autonomous one by using the skew-product flow. This approach naturally leads to the so-called uniform attractor $\mathcal{A}^{un}$ which remains time-independent in spite of the fact that the dynamical system now depends explicitly on the time, see [12]. We note that however the uniform attractor reduces to an autonomous system via the skew-product flow. It seems natural to generalize the concept of an exponential attractor to the non-autonomous case, see [11] [13] [14]. But in all these articles, the uniform attractor's approach was used in order to construct an exponential attractor for the non-autonomous system considered and, consequently, an (uniform) exponential attractor remained time-independent. Since, under this approach, an exponential attractor should contain the uniform attractor, all the drawbacks of uniform attractors (artificial infinite-dimensionality and high dynamical complexity) described above are preserved for exponential attractors.

In the present article, we study exponential attractors of the system (1.1) based on the concept of a non-autonomous (pullback) attractor. Thus, in the approach, an exponential attractor is also time-dependent. To be more precise, a family $t \to \mathcal{M}(t)$ of compact semi-invariant (*i.e.*, $U(t,\tau)\mathcal{M}(t) \subset \mathcal{M}(t)$) sets of the dynamical process (1.1) is an (non-autonomous) exponential attractor if

1) The fractal dimension of all the sets $\mathcal{M}(t)$ is finite and uniformly bounded with respect to t:

$$\mathbf{dim}_F\left(\mathcal{M}(t),\Phi\right) \leq C < \infty.$$

2) There exist a positive constant β and a monotonic function Q such that, for every $t \in \mathbb{R}, s \geq 0$ and every bounded subset B of Φ,

$$\mathbf{dist}_\Phi\left(U(t+s,t)B, \mathcal{M}(t+s)\right) \leq Q\left(\|B\|_\Phi\right)\mathrm{e}^{-\beta s}. \tag{1.6}$$

We emphasize that the convergence in (1.6) is uniform with respect to $t \in \mathbb{R}$ and, consequently, under this approach, we indeed overcome the main drawback of global attractors [13].

This article is organized as follows. In Section 2, we first provide some basic settings and show the absorbing and continuous properties in proper function space about Equation (1.1). In Section 3 and Section 4, we prove the existence of the uniform attractor and exponential attractor of Equation (1.1), respectively. Finally, we prove the existence of infinite-dimensional exponential attractor, and compare it with the non-autonomous exponential attractor in Section 5.

2. Preliminaries

We will use the following notations as that in Pata and Squassina [15]. Let A be the (strictly) positive operator on $L^2(\Omega)$ defined by

$$A=-\Delta \quad \text{with domain} \quad D(A)=H^2(\Omega)\cap H_0^1(\Omega),$$

and consider the family of Hilbert spaces $D\left(A^{s/2}\right), s\in\mathbb{R}$ with the standard inner products and norms, respectively,

$$(\cdot,\cdot)_{D\left(A^{s/2}\right)}=\left(A^{s/2}\cdot,A^{s/2}\cdot\right) \quad \text{and} \quad |\cdot|_{D\left(A^{s/2}\right)}=\left|A^{s/2}\cdot\right|.$$

Then we have

$$D\left(A^0\right)=L^2(\Omega),\ D\left(A^{1/2}\right)=H_0^1(\Omega),\ D\left(A^{-1/2}\right)=H^{-1}(\Omega)$$

and the compact, dense injections

$$D\left(A^{s/2}\right)\hookrightarrow D\left(A^{r/2}\right),\quad \forall s>r.$$

In particular, naming λ_1 the first eigenvalue of A, we get the inequlities

$$\left|A^{r/2}\phi\right|\le\lambda_1^{(r-s)/2}\left|A^{s/2}\phi\right|,\quad \forall\phi\in D\left(A^{s/2}\right).$$

We recall the continuous embedding

$$D\left(A^{s/2}\right)\hookrightarrow L^{6/(3-2s)}(\Omega),\quad \forall s\in\left[0,\frac{3}{2}\right),$$

and the interpolation results: given $s>r>q,$ for any $\varepsilon>0$, there exists $C_\varepsilon=C_\varepsilon(s,r,q)$ such that

$$\left|A^{r/2}u\right|\le\varepsilon\left|A^{s/2}u\right|+C_\varepsilon\left|A^{q/2}u\right|,\quad \forall u\in D\left(A^{s/2}\right),$$

and let

$$E=H_0^1(\Omega)\times L^2(\Omega),\quad \mathcal{V}=H^2(\Omega)\cap H_0^1(\Omega)\times H_0^1(\Omega).$$

Equation (1.1) is equivalent to the following initial value problem in the Hilbert space E

$$\begin{cases}\dot{Y}=PY+F(Y,t),\ t\ge\tau,\\ Y(\tau)=Y_\tau=\left(u_{0\tau},u_{1\tau}\right)^{\mathrm{T}}\in E,\end{cases} \tag{2.1}$$

where $Y=\left(u,u_t\right)^{\mathrm{T}}$, $F(Y,t)=\left(0,-\gamma u_t-f(u)+g(t)\right)^{\mathrm{T}}$ and

$$P=\begin{pmatrix}0 & I\\ -A & -\alpha A\end{pmatrix}.$$

It is well known (see, e.g., [3], [9]) that, under the above assumptions, Equation (2.1) possesses, for every $\tau\in\mathbb{R}$ and $Y_\tau\in E,$ a unique (mild) solution $Y\left(t,Y_\tau\right)\in C\left(\left[\tau,+\infty\right),E\right), t\ge\tau.$ Thus, Equation (1.1) defines a dynamical process $\left\{U_g(t,\tau),t\ge\tau,\tau\in\mathbb{R}\right\}$ in the phase space E by

$$U_g(t,\tau)Y_\tau:=Y(t),\ \text{where}\ Y(t)=\left(u,u_t\right)\ \text{solves}\ (2.1)\ \text{with}\ Y(\tau)=\left(u_{0\tau},u_{1\tau}\right). \tag{2.2}$$

Define a new weighted inner product and norm in E as

$$(\varphi,\phi)_E=\mu\left(A^{1/2}u_1,A^{1/2}u_2\right)+\left(v_1,v_2\right),\ \|\varphi\|_E=(\varphi,\varphi)_E^{1/2},$$

for any

$$\varphi=(u_1,v_1)^{\mathrm{T}},\phi=(u_2,v_2)^{\mathrm{T}}\in E,$$

where μ is chosen as

$$\mu=\frac{4+(\alpha\lambda_1+\gamma)\alpha+\gamma^2/\lambda_1}{4+2(\alpha\lambda_1+\gamma)\alpha+\gamma^2/\lambda_1}\in\left(\frac{1}{2},1\right). \tag{2.3}$$

Obviously, the norm $\|\cdot\|_E$ in (2.3) is equivalent to the usual norm $\|\cdot\|_{H_0^1(\Omega)\times L^2(\Omega)}$ in E.
Let

$$\varphi=(u,v)^{\mathrm{T}},v=u_t+ku,$$

where k is chosen as

$$k=\frac{\alpha\lambda_1+\gamma}{4+2(\alpha\lambda_1+\gamma)\alpha+\gamma^2/\lambda_1}, \tag{2.4}$$

and then the system (1.1) can be written as

$$\varphi_t+H(\varphi)=F(\varphi,t),\ \ \varphi(\tau)=(u_{0\tau},u_{1\tau}+ku_{0\tau})^{\mathrm{T}},\ t\ge\tau,\ \ \tau\in\mathbb{R}, \tag{2.5}$$

where

$$F(\varphi,t)=\begin{pmatrix}0\\ -f(u)+g(t)\end{pmatrix},$$

$$H(\varphi)=\begin{pmatrix}ku-v\\ Au-k(\alpha A-k)u-\gamma ku+(\gamma-k)v+\alpha Av\end{pmatrix}.$$

Lemma 2.1 *For any* $\varphi=(u,v)^{\mathrm{T}}\in E,$ *we have*

$$(H(\varphi),\varphi)_E\ge\sigma_0\|\varphi\|_E^2+\frac{\alpha\lambda_1+\gamma}{2}|v|^2,$$

where

$$\sigma_0=\frac{\lambda_1\alpha+\gamma}{\gamma_1+\sqrt{\gamma_1\gamma_2}},\ \ \gamma_1=4+(\alpha\lambda_1+\gamma)\alpha+\frac{\gamma^2}{\lambda_1},\ \ \gamma_2=(\alpha\lambda_1+\gamma)\alpha+\frac{\gamma^2}{\lambda_1}. \tag{2.6}$$

Proof. Since $D(A)\times D(A)$ is dense in E, and $u=1-k\alpha$; we only need to prove lemma 2.1 for any $\varphi=(u,v)^{\mathrm{T}}\in D(A)\times D(A),$

$$(H(\varphi),\varphi)_E-\sigma_0\|\varphi\|_E^2-\frac{\alpha\lambda_1+\gamma}{2}|v|^2\ge(k-\sigma_0)\mu\|u\|^2+\left(\frac{\alpha\lambda_1+\gamma}{2}-k-\sigma_0\right)|v|^2-\frac{\gamma k}{\sqrt{\lambda_1\mu}}\sqrt{\mu}\|u\|\cdot|v|.$$

By (2.4) and (2.6), elementary computation shows

$$4(k-\sigma_0)\left(\frac{\alpha\lambda_1+\gamma}{2}-k-\sigma_0\right)\ge\frac{\gamma^2k^2}{\lambda_1\mu}.$$

The proof is completed. □

Lemma 2.2 *Let assumptions* (1.2)-(1.5) *be satisfied. For any initial data* $\varphi_\tau\in E$, *there exists a positive constant* ρ *depending only on the coefficients of* (1.3) *and* (2.4) *and* Ω *such that the following dissipative estimate holds*:

$$\|\varphi(t)\|_E\le Q\left(\|\varphi(\tau)\|_E\right)\mathrm{e}^{-\rho(t-\tau)}+M_1,\ \ \ t\ge\tau,$$

where Q is a monotonic function and where the positive number M_1 depends also on M (but is independent of the concrete choice of g).

Proof. Write $\bar{G}(u)=\int_\Omega G(u)\mathrm{d}x$. Let $\varphi=(u,v)^{\mathrm{T}}\in E$ be the solution of the system (2.5) with the initial value $\varphi(\tau)=(u_{0\tau},u_{1\tau}+ku_{0\tau})^{\mathrm{T}}\in E$. Taking the inner product $(\cdot,\cdot)_E$ of (2.5) with φ, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\varphi\|_E^2+2\bar{G}(u)\right]+\left(H(\varphi),\varphi\right)_E+k\left(f(u),u\right)=\left(g(t),v\right). \tag{2.7}$$

By (1.2), (1.3) and Poincaré inequality, there exist two positive constants $k_1,k_2\geq 0$ such that

$$\left(u,f(u)\right)-c_1\bar{G}(u)+\frac{1}{8}\|u\|^2+k_2\geq 0,\ \forall u\in H_0^1(\Omega), \tag{2.8}$$

$$\bar{G}(u)+\frac{1}{8}\|u\|^2+k_1\geq 0,\ \ \bar{G}(u)+\frac{1}{32c_1}\|u\|^2+k_1\geq 0,\ \forall u\in H_0^1(\Omega). \tag{2.9}$$

By (2.4) and (2.6), $\frac{1}{2}k<\sigma_0<k$. Let $y(t)=\|\varphi\|_E^2+2\bar{G}(u)+2k_1\geq\frac{1}{4}\|\varphi\|_E^2\geq 0$. By (2.8)-(2.9), we have

$$\left(H(\varphi),\varphi\right)_E+k\left(f(u),u\right)\geq\frac{1}{2}\rho y-k\left(k_2+c_1k_1\right)+\frac{\alpha\lambda_1+\gamma}{2}|v|^2, \tag{2.10}$$

where $\rho=\frac{1}{32}k\vartheta,\ \vartheta=\min(1,16c_1)$. By (2.7) and (2.10),

$$\frac{\mathrm{d}}{\mathrm{d}t}y+\rho y\leq\frac{1}{\alpha\lambda_1+\gamma}|g|^2+2k\left(c_1k_1+k_2\right).$$

By Gronwall's inequality, we have an absorbing property:

$$\|\varphi(t)\|_E^2\leq 4y(\tau)\mathrm{e}^{-\rho(t-\tau)}+4\left(\frac{M^2}{(\alpha\lambda_1+\gamma)\rho}+\frac{2k\left(c_1k_1+k_2\right)}{\rho}\right)\left(1-\mathrm{e}^{-\rho(t-\tau)}\right),\quad t\geq\tau.$$

This completes the proof. □

Theorem 2.1 *Given any* $b>0$ *and for the solutions of* (2.5) *with any two initial data* $\varphi_{1\tau},\varphi_{2\tau}\in E$ *such that* $\|\varphi_{1\tau}\|\leq b,\|\varphi_{2\tau}\|\leq b$, *we have the following Lipschitz continuity in* E

$$\left\|U_g(t,\tau)\varphi_{1\tau}-U_g(t,\tau)\varphi_{2\tau}\right\|_E\leq\mathrm{e}^{K_1(t-\tau)}\left\|\varphi_{1\tau}-\varphi_{2\tau}\right\|_E,\quad\forall t\geq\tau,$$

for some $K_1=K_1(b)$.

The proof is similar to Theorem 2 in [15]. □

Theorem 2.2 *For the solutions of* (2.5) *with different external forces* g_1 *and* g_2 *satisfying* (1.5) *and with the initial data* $\varphi_{1\tau}$ *and* $\varphi_{2\tau}\in E$, *the following contiuity holds*:

$$\left\|U_{g_1}(t,\tau)\varphi_{1\tau}-U_{g_2}(t,\tau)\varphi_{2\tau}\right\|_E^2\leq C\mathrm{e}^{K_2(t-\tau)}\left(\left\|\varphi_{1\tau}-\varphi_{2\tau}\right\|_E^2+\int_\tau^t\left|g_1(s)-g_2(s)\right|^2\mathrm{d}s\right),\quad t\geq\tau,$$

where C and K_2 are independent of M,t and τ.

The proof is similar to Lemma 4 in [5].

3. Existence of the Uniform Attractor

The dissipativity property obtained in Lemma 2.2 yields the existence of an absorbing set for the process $U_g(t,\tau)$ on E. In the following section, we assume that $\sigma\geq c_6$ holds, where c_6 is specified in (3.11).

Theorem 3.1 *The process* $\{U_g(t,\tau)|t\geq\tau\}$ *possesses a uniform attractor* $\mathcal{A}^{un}$ *in* E.

Proof. We consider $g_\epsilon\in L^\infty\left(\mathbb{R},L^2(\Omega)\right)$ such that

$$\left|g-g_\epsilon\right|_{L^2(\Omega)}<\epsilon, \tag{3.1}$$

and we introduce the splitting $(u,v)=(p,q)+\left(w^1,\rho^1\right)+\left(w^2,\rho^2\right)$ where (p,q) satisfies

$$\begin{cases} p_t + kp - q = 0, \\ q_t + Ap - k(\alpha A - k)p + (\alpha A - k)q + \gamma(q - kp) + f(u) = g_\epsilon(t), \\ p(\tau) = 0, \quad q(\tau) = 0, \end{cases} \tag{3.2}$$

(w^1, ρ^1) satisfies

$$\begin{cases} w_t^1 + kw^1 - \rho^1 = 0, \\ \rho_t^1 + Aw^1 - k(\alpha A - k)w^1 + (\alpha A - k)\rho^1 + \gamma(\rho^1 - kw^1) = g(t) - g_\epsilon(t), \\ w^1(\tau) = 0, \quad \rho^1(\tau) = 0, \end{cases} \tag{3.3}$$

and (w^2, ρ^2) is the solution of

$$\begin{cases} w_t^2 + kw^2 - \rho^2 = 0, \\ \rho_t^2 + Aw^2 - k(\alpha A - k)w^2 + (\alpha A - k)\rho^2 + \gamma(\rho^2 - kw^2) = 0, \\ w^2(\tau) = u_{0\tau}, \quad \rho^2(\tau) = u_{1\tau}. \end{cases} \tag{3.4}$$

We now define the families of maps $\{U_k^1(t,\tau) | t \ge \tau\}$ and $\{U_k^2(t,\tau) | t \ge \tau\}$ in E, where

$$U_k^1(t,\tau)(u_{0\tau}, u_{1\tau}) = (p(t), q(t)) + (w^1(t), \rho^1(t)), \qquad U_k^2(t,\tau)(u_{0\tau}, u_{1\tau}) = (w^2(t), \rho^2(t)).$$

First step: We prove that (p,q) is bounded in $\mathcal{V}$ that the solution of (2.5) is starting in bounded sets of initial data φ_τ. The system (3.2) can be written as

$$\varphi_t^0 + H_0(\varphi^0) = F_0(\varphi^0), \ \varphi^0(\tau) = (0,0)^{\mathrm{T}}, \tag{3.5}$$

where $\varphi^0 = (p,q)^{\mathrm{T}}$,

$$F_1(\varphi^0) = \begin{pmatrix} 0 \\ -f(u) + g_\epsilon(t) \end{pmatrix},$$

$$H_1(\varphi^0) = \begin{pmatrix} kp - q \\ Ap - k(\alpha A - k)p + (\alpha A - k)q + \gamma(q - kp) \end{pmatrix}. \tag{3.6}$$

Similar to Lemma 2.1, we have

$$\left(H_1(\varphi^0), \varphi^0\right)_E \ge \sigma_0 \left\|\varphi^0\right\|_E^2 + \frac{\alpha\lambda_1 + \gamma}{2}|q|^2,$$

where σ_0 is as (2.6). Multiply (3.5) by (p,q), so we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varphi^0\right\|_E^2 + \sigma_0\left\|\varphi^0\right\|_E^2 + \frac{\alpha\lambda_1 + \gamma}{2}|q|^2 \le (-f(u), q) + (g_\epsilon(t), q). \tag{3.7}$$

Similar to Lemma 2.2, applying (3.7) and Young, Poincaré, Gronwall inequalities, we obtain

$$\left\|\varphi^0\right\|_E^2 \le \frac{1}{\sigma_0} C\left(|g_\epsilon|_\infty, \alpha, \lambda_1, k, \gamma, k_1, k_2, c_1\right).$$

Now we multiply (3.5) by (Ap, Aq) and integrate over Ω to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|A^{\frac{1}{2}}\varphi^0\right\|_E^2 + \sigma_0\left\|A^{\frac{1}{2}}\varphi^0\right\|_E^2 + \frac{\alpha\lambda_1 + \gamma}{2}\left|A^{\frac{1}{2}}q\right|^2 \le (-f(u), Aq) + (g_\epsilon(t), Aq), \tag{3.8}$$

with

$$(-f(u), Aq) = \left(-A^{\frac{1}{2}} f(u), A^{\frac{1}{2}}(p_t + kp)\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(A^{\frac{1}{2}} f(u), A^{\frac{1}{2}} p\right) + \left(A^{\frac{1}{2}} f'(u) u_t, A^{\frac{1}{2}} p\right) - k\left(A^{\frac{1}{2}} f(u), A^{\frac{1}{2}} p\right).$$

Then from (3.8) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|A^{\frac{1}{2}}\varphi^{0}\right\|_{E}^{2}+2\left(A^{\frac{1}{2}}f(u),A^{\frac{1}{2}}p\right)\right]+\sigma\left\|A^{\frac{1}{2}}\varphi^{0}\right\|_{E}^{2}+k\left(A^{\frac{1}{2}}f(u),A^{\frac{1}{2}}p\right)$$
$$\leq-\frac{\alpha\lambda_{1}+\gamma}{2}\left|A^{\frac{1}{2}}q\right|^{2}+\left(A^{\frac{1}{2}}f'(u)u_{t},A^{\frac{1}{2}}p\right)+\left(A^{\frac{1}{2}}g_{\epsilon},A^{\frac{1}{2}}q\right),$$

i.e.,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|A^{\frac{1}{2}}\varphi^{0}\right\|_{E}^{2}+2\left(A^{\frac{1}{2}}f(u),A^{\frac{1}{2}}p\right)\right]+\sigma\left\|A^{\frac{1}{2}}\varphi^{0}\right\|_{E}^{2}+k\left(A^{\frac{1}{2}}f(u),A^{\frac{1}{2}}p\right)$$
$$\leq\left(A^{\frac{1}{2}}f'(u)u_{t},A^{\frac{1}{2}}p\right)+\frac{\left|A^{\frac{1}{2}}g_{\epsilon}\right|^{2}}{2(\alpha\lambda_{1}+\gamma)},\tag{3.9}$$

for the first term on the right-hand side of (3.9), we have

$$\left|A^{\frac{1}{2}}f'(u)u_{t}\right|\leq c_{3}\left|f'(u)u_{t}\right|_{L^{6}(\Omega)}\leq c_{4}\left(\int_{\Omega}\left(1+|u|^{6r}\right)dx\right)^{\frac{1}{6}}\cdot\left(\int_{\Omega}|u_{t}|^{6}\,dx\right)^{\frac{1}{6}}$$
$$\leq c_{5}\left(\int_{\Omega}\left(1+|u|^{2r}\right)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\Omega}|u_{t}|^{2}\,\mathrm{d}x\right)^{\frac{1}{2}}.\tag{3.10}$$

By (3.9), (3.10), and Lemma 2.2, there is $T_{1}(Q)\geq\tau$, such that for all $t\geq T_{1}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|A^{\frac{1}{2}}\varphi^{0}\right\|_{E}^{2}+2\left(A^{\frac{1}{2}}f(u),A^{\frac{1}{2}}p\right)\right]+2(\sigma-c_{6})\left\|A^{\frac{1}{2}}\varphi^{0}\right\|_{E}^{2}+2k\left(A^{\frac{1}{2}}f(u),A^{\frac{1}{2}}p\right)\leq\frac{\left|A^{\frac{1}{2}}g_{\epsilon}\right|^{2}}{\alpha\lambda_{1}+\gamma}+2M_{1},\tag{3.11}$$

let $\rho'=\min\{\sigma-c_{6},k\}\geq0$, using the Gronwall's lemma, we have

$$\left\|A^{\frac{1}{2}}\varphi^{0}\right\|_{E}^{2}+2\left(A^{\frac{1}{2}}f(u),A^{\frac{1}{2}}p\right)$$
$$\leq\left(\left\|A^{\frac{1}{2}}\varphi^{0}(0)\right\|_{E}^{2}+2\left(A^{\frac{1}{2}}f(u(0)),A^{\frac{1}{2}}p(0)\right)\right)\mathrm{e}^{-2\rho't}+\frac{\left|A^{\frac{1}{2}}g_{\epsilon}\right|^{2}}{(\alpha\lambda_{1}+\gamma)\rho'}+\frac{2M_{1}}{\rho'}=\frac{\left|A^{\frac{1}{2}}g_{\epsilon}\right|^{2}}{(\alpha\lambda_{1}+\gamma)\rho'}+\frac{2M_{1}}{\rho'}.\tag{3.12}$$

By (1.2), (1.3) and (2.8), (2.9), from (3.12), we obtain

$$\left\|A^{\frac{1}{2}}\varphi^{0}\right\|_{E}^{2}\leq\frac{1}{\rho}C\left(\left|A^{\frac{1}{2}}g_{\epsilon}\right|_{\infty},\alpha,\lambda_{1},k,\gamma,k_{1},k_{2},c_{1},c_{2}\right).\tag{3.13}$$

Lemma 2.2 and (3.13) imply that (p,q) is bounded in $\mathcal{V}$.

Second step: Let $\varphi^{1}=(w^{1},\rho^{1})$, we will prove that there exists $K>0$ independence of ϵ such that

$$\left\|\varphi^{1}\right\|_{E}^{2}\leq K\epsilon.$$

Multiply (3.3) by (w^{1},ρ^{1}), we thus obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varphi^{1}\right\|_{E}^{2}+\sigma_{0}\left\|\varphi^{1}\right\|_{E}^{2}+\frac{\alpha\lambda_{1}+\gamma}{2}\left|\rho^{1}\right|^{2}\leq\left|g(t)-g_{\epsilon}(t)\right|_{\infty}\left|\rho^{1}\right|,$$

due to Gronwall and Poincaré inequalities, then

$$\left\|\varphi^1\right\|_E^2 \le \frac{\left|g(t)-g_\epsilon(t)\right|_\infty^2}{\sigma_0(\alpha\lambda_1+\gamma)} \le \frac{4\epsilon^2}{\sigma_0(\alpha\lambda_1+\gamma)}. \tag{3.14}$$

Since the embedding $\mathcal{V}\hookrightarrow E$ is compact, (3.13), (3.14) and the following lemma imply that $\left\{U_k^1(t,\tau)\middle| t\ge\tau\right\}$ is compact in E.

Lemma 3.1 (see [16]) *Let* X *be a complete metric space and* Λ *be a subset in* X, *such that*

$$\Lambda \subset K_\epsilon + B\left(0, C(\epsilon)\right), \quad \forall\epsilon,$$

with $\lim_{\epsilon\to 0} C(\epsilon)=0$ and K_ϵ is compact in X, then Λ is compact in X.

Third step: Let $\varphi^2=\left(w^2,\rho^2\right)$, the same arguments in the Equation (3.4) lead to

$$\left\|\varphi^2\right\|_E^2 \le \left(\mu\left\|u_{0\tau}\right\|^2+\left|u_{1\tau}\right|^2\right)\exp\left(-2\sigma_0(t-\tau)\right). \tag{3.15}$$

Then from (3.15), Lemma 2.2, and the compactness of $U_k^1(t,\tau)$, the system (2.5) exists a uniform attractor $\mathcal{A}^{un}$ in E.

It is easy to see that the process

$$U_k(t,\tau)=U_k^1(t,\tau)+U_k^2(t,\tau):\left(u_{0\tau},v_\tau=u_{1\tau}+ku_{0\tau}\right)^{\mathrm{T}} \to \left(u(t),u_t(t)+ku(t)\right)^{\mathrm{T}}, \quad E\to E,$$

defined by (2.5) has the following relation with $U_g(t,\tau)$:

$$U_k(t,\tau)=R_kU_g(t,\tau)R_{-k}, \tag{3.16}$$

where R_k is an isomorphism of E:

$$R_k:\{u,u_t\}\to\{u,u_t+ku\}.$$

Since the process $\left\{U_k(t,\tau)\middle| t\ge\tau\right\}$ possesses a uniform attractor $\mathcal{A}^{un}\subset E$, by (3.16), $\left\{U_g(t,\tau)\middle| t\ge\tau\right\}$ also possesses a uniform attractor $\mathcal{A}=R_k\mathcal{A}^{un}$. □

4. Existence of Exponential Attractors

The main result of this section is the following theorem.

Theorem 4.1 *Let the function f and the external force g satisfy the above assumptions. Then, for every external force g enjoying* (1.5), *there exists an exponential attractor* $t\to\mathcal{M}_g(t)$ *of the dynamical process* (1.1) *which satisfies the following properties*:

1) The sets $\mathcal{M}_g(t)$ are semi-invariant with respect to $U_g(t,\tau)$ and translation-invariant with respect to time-shifts:

$$U_g(t,\tau)\mathcal{M}_g(\tau)\subset\mathcal{M}_g(t), \qquad \mathcal{M}_g(t+s)=\mathcal{M}_{T_{sg}}(t), \tag{4.1}$$

where $t,s,\tau\in\mathbb{R}, t\ge\tau$ and $\{T_h,h\in\mathbb{R}\}$ is the group of temporal shifts, $\left(T_{hg}\right)(t)=g(t+h)$.

2) They satisfy a uniform exponential attraction property as follows: there exist a positive constant β_2 and a monotonic function Q (both depending only on M) such that, for every bounded subset B of E, we have

$$dist_E\left(U_g(t,\tau)B,\mathcal{M}_g(t)\right)\le Q\left(\|B\|_E\right)\mathrm{e}^{-\beta_2(t-\tau)}, \quad \forall t\ge\tau\in\mathbb{R}. \tag{4.2}$$

3) The sets $\mathcal{M}_g(t)$ are compact finite-dimensional subsets of E:

$$dim_F\left(\mathcal{M}_g(t),E\right)\le C_1, \quad t\in\mathbb{R}, \tag{4.3}$$

where the constant C_1 is independent of t and g.

4) The map $g\to\mathcal{M}_g(t)$ is Hölder continuous in the following sense:

$$dist_E^{symm}\left(\mathcal{M}_{g_1}(t),\mathcal{M}_{g_2}(t)\right)\le C_2\left(\int_\tau^t \mathrm{e}^{-\beta_3(t-s)}\left|g_1(s)-g_2(s)\right|^2\mathrm{d}s\right)^\eta, \tag{4.4}$$

where the positive constants C_2,β_3 and η are independent of g_1,g_2 and t, $dist_E^{symm}$ denotes the symmetric Hausdorff distance. In particular, the function $t \to \mathcal{M}_g(t)$ is uniformly Hölder continuous in the Hausdorff metric:

$$dist_E^{symm}\left(\mathcal{M}_g(t+s), \mathcal{M}_g(t)\right) \le C_3 |s|^{\eta_1}, \tag{4.5}$$

where C_3 and η_1 are also independent of g, t and s.

Proof. Firstly, we construct a family of exponential attractors for the discrete dynamical processes associated with Equation (2.5). According to Lemma 2.2, it only remains to construct the required exponential attractors for initial data belonging to the ball

$$B = B_{R_0} = \left\{\varphi \in E, \|\varphi\| \le R_0\right\},$$

where R_0 is a sufficiently large number depending only on M given in (1.5), is a uniform absorbing set for all the processes $U_g(t,\tau)$ generated by Equation (1.1). Moreover, from Theorem 2.1, Theorem 2.2 and Theorem 3.1, it follows Lipschitz continuity and smooth properties for the difference of two solutions $\varphi_1(t)$ and $\varphi_2(t)$. Thus, by Theorem 2.1 in [13], the family of discrete dynamical processes $U_g^\tau(m,l) = U_g(\tau + mT, \tau + lT), m, l \in \mathbb{Z}, m \ge l$ possess exponential attractors $l \to \mathcal{M}_g(l,\tau), l \in \mathbb{Z}$. For obtaining exponential attractors of the family of dynamical processes $U_g(t,\tau), t \ge \tau \in \mathbb{R}$, we need the Hölder continuity of the processes $U_g(t,\tau)$ with respect to the time, see the following lemma.

Lemma 4.1 *Let the above assumptions on Equation* (1.1) *hold. Then, for every* $\varphi_\tau \in E$, *we have*

$$\left\|U_g(t+s,\tau)\varphi_\tau - U_g(t,\tau)\varphi_\tau\right\|_E \le C|s|^{1/2}, \tag{4.6}$$

where the constant C depends on M, and is independent of $t \ge \tau \in \mathbb{R}, s \ge 0$. Moreover, for every $T > 0$, we also have

$$\left\|U_g(t+s,s)\varphi_\tau - U_g(t,\tau)\varphi_\tau\right\|_{\mathcal{V}} \le C_T' e^{K(t-\tau)} |s|^{q'}, \ t \ge T, \ 0 \le s \le T/2, \tag{4.7}$$

where q' is a positive number and the positive constant C_T' depends on T but is independent of t, τ and s.

Proof. Note that there is a $s_0 > 0$ such that

$$e^{-\rho s} \le s^{1/2}, \quad \forall s \ge s_0,$$

and $\varphi(t)$ is uniformly bounded in E and Lemma 2.2, which imply the Hölder continuity (4.6). In order to verify (4.7), we note that, due to (4.6) and Theorem 2.2, for every $\varphi \in E$, we have

$$\begin{aligned}
&\left\|U_g(t+s,s)\varphi_\tau - U_g(t,\tau)\varphi_\tau\right\|_E \\
&\le \left\|U_g(t+s,t)\left(U_g(t,s)\varphi_\tau\right) - U_g(t,s)\varphi_\tau\right\|_E + \left\|U_g(t,s)\varphi_\tau - U_g(t,s)\left(U_g(s,\tau)\varphi_\tau\right)\right\|_E \\
&\le C_T |s|^{1/2} + C_T e^{K(t-s)} \left\|\varphi_\tau - U_g(s,\tau)\varphi_\tau\right\|_E \le C_T' e^{K(t-\tau)} |s|^{1/2},
\end{aligned}$$

where all the constants depend on T, but are independent of M, t, s and τ. Using the previously mentioned interpolation inequality in Section 2 finishes the proofs of estimate (4.7). □

Now, we can define the exponential attractors for continuous time by the following formula

$$\mathcal{M}_g(t) = U_g(t,\tau)\mathcal{M}_g(\tau)$$

with respect to τ. The proofs of the semi-invariance with respect to $U_g(t,\tau)$ and translation-invariance with respect to time-shifts is similar to [11] [13]. Estimate (4.2) follows in a standard way from Lemma 2.2, Theorem 3.1 for the processes $U_g(t,\tau)$. Thus, it only remains to verify the finiteness of the fractal dimension of $\mathcal{M}_g(t)$. In order to prove this, we first note that, according to the Hölder continuities Theorem 2.1 in [13] and (4.7), we have

$$dist_E^{symm}\left(U_g(t,\tau-s_1)\mathcal{M}_g(0,\tau-s_1), U_g(t,\tau-s_2)\mathcal{M}_g(0,\tau-s_2)\right) \le C|s_1 - s_2|^{\eta_2},$$

for all $s_1, s_2 \in [0, t-\tau]$, $t \in \mathbb{R}$ and $C > 0, \eta_2 > 0$. Since the map $U_g(t,\tau - s)$ are uniformly Lipschitz continuous, Theorem 3.1 in [13] implies that

$$H_{\varepsilon}\left(\mathcal{M}_g(t), E\right) \le \frac{C}{\eta_2}\log_2\frac{1}{\varepsilon} + C',$$

for a given $\varepsilon > 0$, and some constant C and C' which are independent of t. The proof of Theorem 4.1 is completed. □

5. Infinite-Dimensional (Uniform) Exponential Attractor and Non-Autonomous Exponential Attractor

Finally, we compare the non-autonomous exponential attractor $t \to \mathcal{M}_g(t)$ obtained above with the so-called infinite-dimensional (uniform) exponential attractor constructed in [11] [13]. To the existence of the uniform attractor for strongly damped wave equations, we use the results in [4] and [5] as a model example.

Let $g \in L^{\infty}\left(\mathbb{R}, L^2(\Omega)\right)$ be some external force. Let $\mathcal{H}(g)$ be the hull of g in $L^{\infty}\left(\mathbb{R}, L^2(\Omega)\right)$, *i.e.*,

$$\mathcal{H}(g) := \left[T(h)g \,\middle|\, h \in \mathbb{R}\right]_{L^{\infty}\left(\mathbb{R}, L^2(\Omega)\right)},$$

where $[\cdot]_{L^{\infty}\left(\mathbb{R}, L^2(\Omega)\right)}$ denotes the closure in $L^{\infty}\left(\mathbb{R}, L^2(\Omega)\right)$. Evidently, $T(h)\mathcal{H}(g) = \mathcal{H}(g)$ for any $h \in \mathbb{R}$. Using the standard skew product flow in [4] and [5], for every external forces g satisfying (1.5), we can embed the dynamical process $U_g(t,\tau)$ into the autonomous dynamical system $\mathbb{S}(t)$ acting on the extended phase space $\Phi := E \times \mathcal{H}(g)$ via

$$\mathbb{S}(t)(Y_0, \sigma) := \left(U_{\sigma}(t,0)Y_0, T(t)\sigma\right), \ Y_0 \in E, \quad \sigma \in \mathcal{H}(g), \quad t \ge 0.$$

It is known that $\mathbb{S}(t)$ is a semigroup. If this semigroup possesses the global attractor $\mathbb{A} = \mathbb{A}(g) \subset \Phi$, then, its projection $\mathcal{A}^{un}(g) := \prod_1 \mathbb{A}(g)$ onto the first component of the Cartesian product is called the uniform attractor associated with problem (1.1).

It is also known that the uniform attractor $\mathcal{A}^{un}(g)$ exists under the relatively weak assumption that the hull $\mathcal{H}(g)$ is compact in $L^{\infty}\left(\mathbb{R}, L^2(\Omega)\right)$, but, unfortunately, for more or less general external forces g, its Hausdorff and fractal dimensions are infinite. Instead, the following estimate for its Kolmogorov's ε-entropy holds, see [4].

Proposition 5.1 *Let the above assumptions hold and the hull* $\mathcal{H}(g)$ *of the initial external forces be compact. Then, Equation* (1.1) *possesses the uniform attractor* $\mathcal{A}^{un}(g)$ *and its* ε*-entropy can be estimated in terms of the* ε*-entropy of the hull* $\mathcal{H}(g)$ *as follows*:

$$\mathbb{H}_{\varepsilon}\left(\mathcal{A}^{un}(g)\right) \le C_2 \log_2\frac{\varepsilon_0}{\alpha\varepsilon} + \mathbb{H}_{\varepsilon_0}\left(\mathcal{A}^{un}(g)\right) + \mathbb{H}_{\varepsilon/K'}\left(\Pi_{\left[0, a\log_2\frac{\varepsilon_0}{\alpha\varepsilon}\right]}\mathcal{H}(g)\right), \quad \forall \varepsilon > \varepsilon_0, \tag{5.1}$$

for some positive constants C_2, ε_0, K' and a depending on f.

Definition 5.1 [11] [13] A set $\mathcal{M}^{un}(g)$ is an (uniform) exponential attractor of Equation (1.1) if the following properties are satisfied:

1) Entropy estimate: $\mathcal{M}^{un}(g)$ is a compact subset of the phase space E which satisfies estimate (5.1) (possibly, for larger constants C_2, K' and a).

2) Semi-invariance: for every $Y_0 \in \mathcal{M}^{un}(g)$, there exists $h \in \mathcal{H}(g)$ such that $U_h(t,0)Y_0 \subset \mathcal{M}^{un}(g)$ for all $t \ge 0$.

3) Uniform exponential attraction property: there exists a positive constant $\hat{\rho}$ and a monotonic function Q such that, for every $h \in \mathcal{H}(g)$ and every bounded subset $B \subset E$, we have

$$dist_E\left(U_h(t,\tau)B, \mathcal{M}^{un}(g)\right) \le Q\left(\|B\|_E\right)e^{-\hat{\rho}t}, \quad \forall t \ge \tau \in \mathbb{R}. \tag{5.2}$$

[13] points out that a uniform exponential attractor $\mathcal{M}^{un}(g)$ can be constructed if the (non-autonomous) exponential attractor $t \to \mathcal{M}_g(t)$ has been constructed, so we have

Theorem 5.1 *Let the assumptions of Theorem* 4.1 *hold and let, in addition, the hull* $\mathcal{H}(g)$ *of some external* *forces satisfying* (1.5) *be compact. Then, there exists a uniform exponential attractor* $\mathcal{M}^{un}(g)$ *for problem* (1.1) *which can be constructed as follows*:

$$\mathcal{M}^{un}(g) := \left[\bigcup_{t\in\mathbb{R}} \mathcal{M}_g(t)\right]_E = \bigcup_{h\in\mathcal{H}(g)} \mathcal{M}_h(0).$$

Remark 1 When $\alpha = 0$, Equation (1.1) reduces to the following damped wave equation on a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$:

$$\begin{cases} u_{tt} - \Delta u + \gamma u_t + f(u) = g(t), \\ u|_{\partial\Omega} = 0, \\ (u(\tau), u_t(\tau)) = (u_{0\tau}, u_{1\tau}), \end{cases} \tag{5.3}$$

Equation (1.1) reduces to the damped wave equation modeling the Josephson junction in superconduction which was studied by many authors (see [1] [6] [17]). We assume that the function f satisfy (1.2)-(1.4). The Equation (5.3) also possesses a finite dimensional exponential attractor.

Remark 2 When $\gamma = 0$, Theorem 4.1 remains valid for the following strongly damped wave equation was studied by many authors (cf. [7] [18]):

$$\begin{cases} u_{tt} - \Delta u_t - \Delta u + f(u) = g(t), \\ u|_{\partial\Omega} = 0, \\ (u(\tau), u_t(\tau)) = (u_{0\tau}, u_{1\tau}), \end{cases}$$

if we assume that the function f satisfy (1.2)-(1.4).

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