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Common Fixed Point Iterations of Generalized Asymptotically Quasi-Nonexpansive Mappings in Hyperbolic Spaces

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Abstract

We introduce a general iterative method for a finite family of generalized asymptotically quasi-nonexpansive mappings in a hyperbolic space and study its strong convergence. The new iterative method includes multi-step iterative method of Khan *et al.* [1] as a special case. Our results are new in hyperbolic spaces and generalize many known results in Banach spaces and *CAT*(0) spaces, simultaneously.

Keywords

Hyperbolic Space, General Iterative Method, Generalized Asymptotically Quasi-Nonexpansive Mapping, Common Fixed Point, Strong Convergence

1. Introduction

Let C be a nonempty subset of a metric space X and $T: C \to C$ be a mapping. Throughout this paper, we assume that $F(T)$, the set of fixed points of T is nonempty and $I = \{1, 2, 3, \ldots, r\}$. The mapping T is: 1) asymptotically nonexpansive if there exists a sequence of real numbers $\{u_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d\left(T^n x, T^n y\right) \le \left(1 + u_n\right) d(x, y)$ for all $x, y \in C$ and $n \ge 1$ 2) asymptotically quasi-nonexpansive if there exists a sequence of real numbers $\{u_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d\left(T^n x, p\right) \le \left(1 + u_n\right) d(x, p)$ for all $x \in C, p \in F(T)$ and $n \ge 1$ 3) generalized asymptotically quasi-nonexpansive if there exist two sequences of real numbers $\{u_n\}$ and $\{c_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0 = \lim_{n\to\infty} c_n$ such that $d\left(T^n x, p\right) \le d\left(x, p\right) + u_n d\left(x, p\right) + c_n$ for all $x \in C, p \in F(T)$ and $n \ge 1$ (iv) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d\left(T^n x, T^n x\right) \le L d(x, y)$ for all $x, y \in C$ and $n \ge 1$ (v) $(L - \gamma)-$

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uniformly Lipschitzian if there are constants $L>0,\gamma>0$ such that $d\left(T^n x,T^n x\right)\le Ld(x,y)^\gamma$ for all $x,y\in C$ and $n\ge 1$ and (vi) semi-compact if for any sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d\left(x_n,Tx_n\right)=0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\to c\in C$.

Let (X,d) be a metric space. Suppose that there exists a family F of metric segments such that any two points x,y in X are endpoints of a unique metric segment $[x,y]\in F[x,y]$ is an isometric image of the real line interval $[0,d(x,y)]$). We shall denote by $\alpha x\oplus(1-\alpha)y$ the unique point $z$of $[x,y]$ which satisfies

$$d(x,z)=(1-\alpha)d(x,y)\text{and}d(z,y)=\alpha d(x,y)\text{for}\alpha\in J=[0,1].$$

Such metric spaces are usually called convex metric spaces [2] [3]. One can easily deduce that $0x\oplus 1y=y$, $1x\oplus 0y=x$ and $\alpha x\oplus(1-\alpha)x=x$ from the definition of a convex metric space [2].

A convex metric space X is hyperbolic if

$$d(\alpha x\oplus(1-\alpha)y,\alpha z\oplus(1-\alpha)w)\le\alpha d(x,z)+(1-\alpha)d(y,w)$$

for all $x,y,z,w\in X$ and $\alpha\in J$. For $z=w$, the hyperbolic inequality reduces to convex structure [3].

$$d\left(\alpha x\oplus(1-\alpha)y,z\right)\le\alpha d(x,z)+(1-\alpha)d(y,z). \tag{1.1}$$

A nonempty subset C of a convex metric space X is convex if $\alpha x\oplus(1-\alpha)y\in C$ for all $x,y\in C$ and $\alpha\in J$.

Normed spaces and their subsets are linear hyperbolic spaces while $CAT(0)$ spaces [4]-[6] qualify for the criteria of nonlinear hyperbolic spaces [2] [7].

A convex metric space X is uniformly convex [7] if

$$\delta(r,\varepsilon)=\inf\left\{1-\frac{1}{r}d\left(a,\frac{1}{2}x\oplus\frac{1}{2}y\right):d(a,x)\le r,d(a,y)\le r,d(x,y)\ge r\varepsilon\right\}>0,$$

for any $a\in X,r>0$ and $\varepsilon>0$.

From now onwards we assume that X is a uniformly convex hyperbolic space with the property that for every $s\ge 0,\varepsilon>0$, there exists $\eta(s,\varepsilon)>0$ depending on s and ε such that $\delta(r,\varepsilon)>\eta(s,\varepsilon)>0$ for any $r>s$.

We now translate the iterative method (1.3) [1] from normed space setting to the more general setup of hyperbolic space as follows:

$$x_1\in C, x_{n+1}=U_{rn}x_n, n\ge 1 \tag{1.2}$$

where

$$\begin{aligned}
&U_{0n}=I\left(\text{the identity mapping}\right)\\
&U_{1n}x=a_{1n}T_1^nU_{0n}x\oplus\left(1-a_{1n}\right)x\\
&U_{2n}x=a_{2n}T_2^nU_{1n}x\oplus\left(1-a_{2n}\right)x\\
&\vdots\\
&U_{rn}x=a_{rn}T_r^nU_{(r-1)n}x\oplus\left(1-a_{rn}\right)x
\end{aligned}$$

and $\{T_i:i\in I\}$ is a family of generalized asymptotically quasi-nonexpansive self-mappings of C, *i.e.*, $d\left(T_i^n x,p_i\right)\le\left(1+u_{in}\right)d\left(x,p_i\right)+c_{in}$ for all $x\in C$ and $p_i\in F\left(T_i\right), i\in I,\{u_{in}\}$ and $\{c_{in}\}$ are sequences in $[0,\infty)$ with $\sum_{n=1}^{\infty}u_{in}<\infty$ and $\sum_{n=1}^{\infty}c_{in}<\infty$ for each i.

The purpose of this paper is to:

1) establish convergence of iterative method (1.2) to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings on a hyperbolic space(uniformly convex hyperbolic space).

Our work is a significant generalization of the corresponding results in Banach spaces and $CAT(0)$ spaces.

In the sequel, we assume that $F=\bigcap_{i\in I}F\left(T_i\right)\ne\varnothing$.

2. Convergence Theorems in Hyperbolic Space

Lemma 2.1. *Let* C *be a nonempty, closed and convex subset of a hyperbolic space* X. *Then, for the sequence*

$\{x_n\}$ *in* (1.2), *there are sequences* $\{v_n\}$ *and* $\{\xi_n\}$ *in* $[0,\infty)$ *satisfying* $\sum_{n=1}^{\infty} v_n < \infty, \sum_{n=1}^{\infty} \xi_n < \infty$ *such that*

1) $d(x_{n+1}, p) \le (1+v_n) d(x_n, p) + \xi_n$, *for all* $p \in F$ *and all* $n \ge 1$

2) $d(x_{n+m}, p) \le M_1 (d(x_n, p) + \sum_{n=1}^{\infty} \xi_n)$, *for all* $p \in F$ *and* $n \ge 1, m \ge 1, M_1 > 0$.

Proof. (a) Let $p \in F$ and $v_n = \max_{i\in I} u_{in}$ for all $n \ge 1$. Since $\sum_{n=1}^{\infty} u_{in} < \infty$ for each i, therefore $\sum_{n=1}^{\infty} v_n < \infty$.

Now we have

$$d(U_{1n}x_n, p) = d\left(a_{1n}T_1^n U_{0n}x_n \oplus (1-a_{1n})x_n, p\right) \le (1-a_{1n}) d(x_n, p) + a_{1n} d\left(T_1^n x_n, p\right)$$
$$\le (1-a_{1n}) d(x_n, p) + a_{1n}\left[(1+u_{1n}) d(x_n, p) + c_{1n}\right] \le (1+u_{1n}) d(x_n, p) + c_{1n} \le (1+v_n)^1 d(x_n, p) + c_{1n}.$$

Assume that $d(U_{kn}x_n, p) \le (1+v_n)^k d(x_n, p) + (1+v_n)^{k-1} \sum_{i=1}^{k} c_{in}$ holds for some $k > 1$.
Consider

$$d\left(U_{(k+1)n}x_n, p\right) = d\left(a_{(k+1)n}T_{k+1}^n U_{kn}x_n \oplus \left(1-a_{(k+1)n}\right)x_n, p\right) \le \left(1-a_{(k+1)n}\right) d(x_n, p) + a_{(k+1)n} d\left(T_{k+1}^n U_{kn}x_n, p\right)$$
$$\le \left(1-a_{(k+1)n}\right) d(x_n, p) + a_{(k+1)n}\left(1+u_{(k+1)n}\right) d(U_{kn}x_n, p) + a_{(k+1)n}c_{(k+1)n}$$
$$\le \left(1-a_{(k+1)n}\right) d(x_n, p) + a_{(k+1)n}c_{(k+1)n} + a_{(k+1)n}\left(1+u_{(k+1)n}\right) d(U_{kn}x_n, p)$$
$$\le \left(1-a_{(k+1)n}\right) d(x_n, p) + a_{(k+1)n}c_{(k+1)n} + a_{(k+1)n}(1+v_n)\left[(1+v_n)^k d(x_n, p) + (1+v_n)^{k-1}\sum_{i=1}^{k} c_{in}\right]$$
$$\le \left(1-a_{(k+1)n}\right)(1+v_n)^{k+1} d(x_n, p) + a_{(k+1)n}(1+v_n)c_{(k+1)n} + a_{(k+1)n}(1+v_n)^{k+1} d(x_n, p) + a_{(k+1)n}(1+v_n)^{k-1}\sum_{i=1}^{k+1} c_{in}$$
$$\le (1+v_n)^{k+1} d(x_n, p) + (1+v_n)^k \sum_{i=1}^{k+1} c_{in}$$

By mathematical induction, we have

$$d\left(U_{jn}x_n, p\right) \le (1+v_n)^j d(x_n, p) + (1+v_n)^{j-1}\sum_{i=1}^{j} c_{in}, 1 \le j \le r. \tag{2.1}$$

Now, by (1.2) and (2.1), we obtain

$$d(x_{n+1}, p) = d\left(a_{rn}T_r^n U_{(r-1)n}x \oplus (1-a_{rn})x, p\right)$$
$$\le a_{rn} d\left(T_r^n U_{(r-1)n}x_n, p\right) + (1-a_{rn}) d(x_n, p)$$
$$\le a_{rn}\left[(1+u_{rn}) d\left(U_{(r-1)n}x_n, p\right) + c_{rn}\right] + (1-a_{rn}) d(x_n, p)$$
$$\le a_{rn}(1+u_{rn})\left[(1+v_n)^{r-1} d(x_n, p) + (1+v_n)^{r-2}\sum_{i=1}^{r-1} c_{in}\right] + a_{rn}c_{rn} + (1-a_{rn}) d(x_n, p)$$
$$\le a_{rn}(1+v_n)^r d(x_n, p) + a_{rn}(1+v_n)^{r-1}\sum_{i=1}^{r} c_{in}c_{in} + (1-a_{rn}) d(x_n, p)$$
$$\le \left[1-a_{rn} + a_{rn}(1+v_n)^r\right] d(x_n, p) + a_{rn}(1+v_n)^{r-1}\sum_{i=1}^{r} c_{in}$$
$$= \left[1-a_{rn} + a_{rn}\sum_{k=1}^{r}\left(1 + \frac{(r(r-1)\ldots(r-k+1))}{k!}v_n^k\right)\right] d(x_n, p) + a_{rn}(1+v_n)^{r-1}\sum_{i=1}^{r-1} c_{in}$$
$$\le (1+v_n)^r d(x_n, p) + (1+v_n)^{r-1}\sum_{i=1}^{r} c_{in} \le (1+v_n)^r d(x_n, p) + \xi_n,$$

Where $M = \sup M = \sup(1+v_n)^{r-1}, \xi_n = M\sum_{i=1}^{r} c_{in}$ and $\sum_{n=1}^{\infty} \xi_n < \infty$.

(b) We know that $1+t \le \exp t$ for $t \ge 0$. Thus, by part (a), we have

$$d(x_{n+m},p)\le(1+v_{n+m-1})^r d(x_{n+m-1},p)+\xi_{n+m-1}$$
$$\le \exp(rv_{n+m-1})d(x_{n+m-1},p)+\xi_{n+m-1}$$
$$\le \exp(rv_{n+m-1}+rv_{n+m-2})d(x_{n+m-2},p)+\xi_{n+m-1}+\xi_{n+m-2}$$
$$\vdots$$
$$\le \exp\left(r\sum_{i=n}^{n+m-1}v_i\right)d(x_n,p)+\sum_{i=n+1}^{n+m-1}v_i\sum_{i=n}^{n+m-1}\xi_i$$
$$\le \exp\left(r\sum_{i=1}^{\infty}v_i\right)\left(d(x_n,p)+\sum_{i=1}^{\infty}\xi_i\right)=M_1\left(d(x_n,p)+\sum_{i=1}^{\infty}\xi_i\right),$$
where $M_1=\exp\left(r\sum_{i=1}^{\infty}v_i\right)$.

Theorem 2.2. Let C be a nonempty, closed and convex subset of a complete hyperbolic space X. Then the sequence $\{x_n\}$ in (1.2) converges strongly to a point in F if and only if $\liminf_{n\to\infty} d(x_n,F)=0$, where $d(x,F)=\inf_{p\in F} d(x,p)$.

Proof. We only prove the sufficiency. By Lemma 2.1 (a), we have
$d(x_{n+1},p)\le(1+v_n)^r d(x_n,p)+\xi_n$ for all $p\in F$ and $n\ge1$. Therefore,

$$d(x_{n+1},F)\le\left(1+\sum_{k=1}^{r}\left(\left(r(r-1)\dots(r-k+1)\right)/k!\right)v_n^k\right)d(x_n,F)+\xi_n$$

As $\sum_{n=1}^{\infty}v_n<\infty$, so $\sum_{n=1}^{\infty}\sum_{k=1}^{r}\left(\left(r(r-1)\dots(r-k+1)\right)/k!\right)v_n^k<\infty$. Now $\sum_{n=1}^{\infty}\xi_n<\infty$ in Lemma 2.1 (a), so by Lemma 1.1 [1] and $\liminf_{n\to\infty} d(x_n,F)=0$, we get that $\lim_{n\to\infty} d(x_n,F)=0$. Let $\varepsilon>0$. From the proof of Lemma 2.1 (b), we have

$$d(x_{n+m},x_n)\le d(x_{n+m},F)+d(x_n,F)\le(1+M_1)d(x_n,F)+M_1\sum_{i=n}^{\infty}\xi_i \tag{2.2}$$

Since $\lim_{n\to\infty} d(x_n,F)=0$ and $\sum_{i=n}^{\infty}\xi_i<\infty$, therefore there exists a natural number n_0 such that
$d(x_n,F)\le\varepsilon/2(1+M_1)$ and $\sum_{i=n}^{\infty}\xi_i<\varepsilon/2M_1$ for all $n\ge n_0$.
So for all integers $n\ge n_0, m\ge1$, we obtain from (2.2) that

$$d(x_{n+m},x_n)\le(1+M_1)\left(\frac{\varepsilon}{2(1+M_1)}\right)+M_1\left(\frac{\varepsilon}{2M_1}\right)=\varepsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X and so converges to $q\in X$. Finally, we show that $q\in F$. For any $\bar{\varepsilon}>0$, there exists natural number n_1 such that $d(x_n,F)=\inf_{p\in F} d(x_n,p)<\bar{\varepsilon}/3$ and $d(x_n,q)<\bar{\varepsilon}/2$ for all $n\ge n_1$.

There must exist $p^*\in F$ such that $d(x_n,p^*)<\bar{\varepsilon}/2$ for all $n\ge n_1$, in particular, $d(x_{n1},p^*)<\bar{\varepsilon}/2$ and $d(x_{n1},q)<\bar{\varepsilon}/2$.

Hence $d(p^*,q)\le d(x_{n1},p^*)+d(x_{n1},q)<\bar{\varepsilon}$. Since $\bar{\varepsilon}$ is arbitrary, therefore $d(p^*,q)=0$. That is, $q=p^*\in F$.

Theorem 2.3. Let C be a nonempty, closed and convex subset of a complete convex metric space X, If $\lim_{n\to\infty} d(x_n,T_i x_n)=0$ for the sequence $\{x_n\}$ in (1.2), $i\in I$ and one of the mappings is semi-compact, then $\{x_n\}$ converges strongly to $p\in F$.

Proof. Let T_l be semi-compact for some $1\le l\le r$. Then there exists a subsequence $\{x_i\}$ of $\{x_n\}$ such that $x_i\to p\in C$. Hence

$$d(p,T_l p)=\lim_{n_j\to\infty} d(x_i,T_l x_i)=0.$$

Thus $p\in F$ and so by Theorem 2.2, $\{x_n\}$ converges strongly to a common fixed point of the family of mappings.

3. Results in a Uniformly Convex Hyperbolic Space

Lemma 3.1. *Let* C *be a nonempty, closed and convex subset of a uniformly convex hyperbolic space* X. *Then, for the sequence* $\{x_n\}$ *in* (1.2) *with* $a_{in} \in [\delta, 1-\delta]$ *for some* $\delta \in \left(0, \frac{1}{2}\right)$, *we have*

(*a*) $\lim_{n\to\infty} d(x_n, p)$ *exists for all* $p \in F$

(*b*) $\lim_{n\to\infty} d(x_n, T_j x_n) = 0$, *for each* $j \in I$.

Proof. (a) Let $p \in F$ and $v_n = \max_{i\in I} u_{in}$, for all $n \geq 1$. By Lemma 1.1 [1] and Lemma 2.1 (a), it follows that $\lim_{n\to\infty} d(x_n, p)$ exists. Assume that

$$\lim_{n\to\infty} d(x_n, p) = c. \tag{3.1}$$

(b) The inequality (2.1) together with (3.1) gives that

$$\limsup_{n\to\infty} d(U_{jn}x_n, p) \leq c, 1 \leq j \leq r. \tag{3.2}$$

Note that

$$\begin{aligned}
d(x_{n+1}, p) &= d(U_{rn}x_n, p) = d\left(a_{rn}T_r^n U_{(r-1)n}x_n \oplus (1-a_{rn})x_n, p\right) \\
&\leq a_{rn}\left[(1+v_n)d\left(U_{(r-1)n}x_n, p\right) + c_{rn}\right] + (1-a_{rn})d(x_n, p) \\
&= a_{rn}(1+v_n)d\left(a_{(r-1)n}T_{r-1}^n U_{(r-2)n}x_n \oplus 1-a_{(r-1)n}x_n, p\right) + a_{rn}c_{rn} + (1-a_{rn})d(x_n, p) \\
&\leq a_{rn}(1+v_n)\left[a_{(r-1)n}d\left(T_{r-1}^n U_{(r-2)n}x_n, p\right) + \left(1-a_{(r-1)n}\right)d(x_n, p)\right] + a_{rn}c_{rn} + (1-a_{rn})d(x_n, p) \\
&\leq a_{rn}a_{(r-1)n}(1+v_n)^2 d\left(U_{(r-2)n}x_n, p\right) + \left(1-a_{rn}a_{(r-1)n}\right)(1+v_n)^2 d(x_n, p) \\
&\quad + a_{rn}a_{(r-1)n}(1+v_n)^2 c_{(r-1)n} + a_{rn}(1+v_n)^2 c_{rn} \\
&\leq \prod_{i=j+1}^{r} a_{in}(1+v_n)^{r-j} d(U_{jn}x_n, p) + \left(1-\prod_{i=j+1}^{r} a_{in}\right)(1+v_n)^{r-j} d(x_n, p) \\
&\quad + \prod_{i=j+1}^{r} a_{in}(1+v_n)^{r-j} c_{(j+1)n} + \prod_{i=j+2}^{r} a_{in}(1+v_n)^{r-j} c_{jn} + \ldots + a_{rn}(1+v_n)^{r-j} c_{rn}.
\end{aligned}$$

and therefore, we have

$$d(x_n, p) \leq \left(\frac{d(x_n, p)}{\delta^{r-j}}\right) - \left(\frac{d(x_{n+1}, p)}{\delta^{r-j}(1+v_n)^{r-j}}\right) + d(U_{jn}x_n, p) + c_{(j+1)n} + \left(\frac{c_{jn}}{\delta}\right) + \cdots + \frac{c_{rn}}{\delta^{r-j+1}}$$

Hence

$$c \leq \liminf_{n\to\infty} d(U_{jn}x_n, p), \quad 1 \leq j \leq r. \tag{3.3}$$

Using (3.2) and (3.3), we have $\lim_{n\to\infty} d(U_{jn}x_n, p) = c$.

That is, $\lim_{n\to\infty} d\left(a_{jn}T_j^n U_{(j-1)n}x_n \oplus (1-a_{jn})x_n, p\right) = c$ for $1 \leq j \leq r$.

This together with (3.1), (3.2) and Lemma 2.5 [8] gives that

$$\lim_{n\to\infty} d\left(T_j^n U_{(j-1)n}x_n, x_n\right) = 0 \quad \text{for} \quad 1 \leq j \leq r. \tag{3.4}$$

If $j = 1$, we have by (3.4), $\lim_{n\to\infty} d(T_1^n x_n, x_n) = 0$.

In case $j \in \{2, 3, 4, \cdots, r\}$, we observe that

$$d\left(x_n, U_{(j-1)n}x_n,\right) = d\left(x_n, a_{(j-1)n}T_{j-1}^n U_{(j-2)n}x_n \oplus \left(1-a_{(j-1)n}\right)x_n\right) \leq a_{(j-1)n}d\left(T_{j-1}^n U_{(j-2)n}x_n, x_n\right) \to 0 \tag{3.5}$$

Since T_j is $(L-\gamma)-$ uniformly Lipschitzian, therefore the inequality

$$d\left(T_j^n x_n, x_n\right) \le d\left(T_j^n x_n, T_j^n U_{(j-1)n} x_n\right) + d\left(T_j^n U_{(j-1)n} x_n, x_n\right) \le Ld\left(x_n, U_{(j-1)n} x_n\right)^\gamma + d\left(T_j^n U_{(j-1)n} x_n, x_n\right),$$

together with (3.4) and (3.5) gives that $\lim_{n\to\infty} d\left(T_j^n x_n, x_n\right) = 0.$

Hence,

$$d\left(T_j^n x_n, x_n\right) \to 0 \text{as} n \to \infty \text{ for } 1 \le j \le r. \tag{3.6}$$

Note that $d\left(x_n, x_{n+1}\right) = d\left(x_n, a_{rn} T_r^n U_{(r-1)n} x_{nn} \oplus (1-a_{rn}) x_n\right) \le a_{rn} d\left(x_n, T_r^n U_{(r-1)n} x_n\right) \to 0.$

Let us observe that:

$$\begin{aligned} d\left(x_n, T_j x_n\right) &\le d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, T_j^{n+1} x_{n+1}\right) + d\left(T_j^{n+1} x_{n+1}, T_j^{n+1} x_n\right) + d\left(T_j^{n+1} x_n, T_j x_n\right) \\ &\le d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, T_j^{n+1} x_{n+1}\right) + Ld\left(x_{n+1}, x_n\right)^\gamma + Ld\left(T_j^n x_n, x_n\right)^\gamma. \end{aligned}$$

So by $(L-\gamma)-$ uniformly Lipschitzian property of T_j, (3.5) and (3.6), we get $\lim_{n\to\infty} d\left(x_n, T_j x_n\right) = 0, 1 \le j \le r$.

Theorem 3.2. *Under the hypotheses of Lemma* 3.1, *assume that, for some* $1 \le j \le r$, T_j^m *is semi-compact for some positive integer m. Then* $\{x_n\}$ *in* (1.2), *converges strongly to a point in* F.

Proof. Fix $j \in I$ and suppose T_j^m is semi-compact for some $m \ge 1$. By Lemma 3.1 (b), we obtain

$$\begin{aligned} d\left(T_j^m x_n, x_n\right) &\le d\left(T_j^m x_n, T_j^{m-1} x_n\right) + d\left(T_j^{m-1} x_n, T_j^{m-2} x_n\right) + \cdots + d\left(T_j^2 x_n, T_j x_n\right) + d\left(T_j x_n, x_n\right) \\ &\le d\left(T_j x_n, x_n\right) + (m-1) Ld(T_j x_n, x_n)^\gamma \to 0. \end{aligned}$$

Since $\{x_n\}$ is bounded and T_j^m is semi-compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to q \in C$. Hence, by Lemma 3.1 (b), we have $d(q, T_i q) = \lim_{n\to\infty} d\left(x_{n_j}, T_i x_{n_j}\right) = 0, i \in I$.

Thus $q \in F$ and so by Theorem 2.2, $\{x_n\}$ converges strongly to a common fixed point q of the family $\{T_i : i \in I\}$.

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