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On the Average Errors of Multivariate Lagrange Interpolation*

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ABSTRACT

In this paper, we discuss the average errors of multivariate Lagrange interpolation based on the Chebyshev nodes of the first kind. The average errors of the interpolation sequence are determined on the multivariate Wiener space.

Keywords: Multivariate Lagrange Interpolation; Average Error; Chebyshev Polynomial; Wiener Sheet Measure

1. Introduction

Let F be a real separable Banach space equipped with a probability measure μ on the Borel sets of F. Let X be another normed space such that F is continuously embedded in X. By $\|\cdot\|_X$ we denote the norm in X. Any $T: F \to X$ such that $f \mapsto \|f - T(f)\|_X$ is a measurable mapping is called an approximation operator. The average error of T is defined as

$$e(T, F, \|\cdot\|_X) := \left(\int_F \|f - T(f)\|_X^2 \, \mu(df) \right)^{1/2}.$$

For $d \geq 1$, let

$$C_{0,d} = \{ f \in C[0,1]^d \mid f(x_1, \cdots, x_d) = 0, \text{ whenever } x_i = 0 \text{ for some } 1 \leq i \leq d \}.$$

The space $C_{0,d}$ equipped with the sup norm

$$\|f\| = \sup_{t \in [0,1]^d} |f(t)|.$$

The classical Wiener sheet measure w_d on $\mathfrak{B}(C_{0,d})$ is Gaussian with mean zero and covariance kernel

$$R_{w_d}(s,t) = \int_{C_{0,d}} f(s) f(t) w_d(df) = \prod_{i=1}^{d} \min\{s_i, t_i\}, \qquad s, t \in [0,1]^d. \tag{1}$$

For more detailed discussion and properties of w_d, we refer to [1].

In this paper, we let

$$F_d = \{ f \in C[-1,1]^d \mid g(x_1, \cdots, x_d) = f(2x_1 - 1, \cdots, 2x_d - 1) \in C_{0,d} \}.$$

For every measurable subset $A \in \mathfrak{B}(F_d)$, we define the measure of A by

$$\mu_d(A) := w_d\{ g(x_1, \cdots, x_d) = f(2x_1 - 1, \cdots, 2x_d - 1) \in A \}. \tag{2}$$

Let $\rho(x_1, \cdots, x_d) = \left(\prod_{i=1}^{d} \sqrt{1 - x_i^2} \right)^{-1}$, the weighted L_2-norm for $f \in F_d$ is defined as

$$\|f\| := \|f\|_{2,\rho} := \left(\int_{[-1,1]^d} |f(t)|^2 \rho(t) dt \right)^{1/2}.$$

Let

$$\xi_k := \xi_{k,n} := \cos \frac{2k-1}{2n} \pi, \quad k = 1, \cdots, n$$

is the zeros of $T_n(x) = \cos n\theta$ $(x = \cos\theta)$, the nth degree Chebyshev polynomial of the first kind. For $f \in F_d$, the well-known Lagrange interpolation polynomial of f based on $\left\{ \xi_{i_1}, \cdots, \xi_{i_d} \right\}_{i_1, \cdots, i_d = 1}^{n}$ is given by

$$L_{n,d}(f, x) = \sum_{i_1, \cdots, i_d = 1}^{n} f(\xi_{i_1}, \cdots \xi_{i_d}) l_{1,i_1}(x_1) \cdots l_{d,i_d}(x_d), \tag{3}$$

where

$$l_{j,i_j}(x_j) = \prod_{k=1 (k \neq i_j)}^{n} \frac{x_j - \xi_k}{\xi_{i_j} - \xi_k}, \quad j = 1, \cdots, d.$$

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2. Main Result

Since the polynomial interpolation operators are important approximation tool in the continuous functions space, there are a number of papers studying the convergence for interpolation polynomial, especially the interpolation polynomial based on roots of orthogonal polynomials. Xu Guiqiao [2] studied the average errors of univariate Lagrange interpolation based on the Chebyshev nodes on the Wiener space. Motivated by [2], we consider the average errors of multivariate Lagrange interpolation. We first study the bivariate Lagrange interpolation, then the general multivariate Lagrange interpolation. Our main results are the following:

Theorem 1. Let

$$x=(x_1,x_2)\in[-1,1]^2,$$

$$\rho(x_1,x_2)=\frac{1}{\sqrt{1-x_1^2}\sqrt{1-x_2^2}},$$

and

$$L_{mn}(f,x)=\sum_{i=1}^{m}\sum_{j=1}^{n} f(\xi_{i,m},\xi_{j,n})l_{1,i}(x_1)l_{2,j}(x_2),$$

where

$$l_{1,i}(x_1)=\prod_{k=1(k\neq i)}^{m}\frac{x_1-\xi_{k,m}}{\xi_{i,m}-\xi_{k,m}},$$

$$l_{2,j}(x_2)=\prod_{s=1(s\neq j)}^{n}\frac{x_2-\xi_{s,n}}{\xi_{j,n}-\xi_{s,n}}.$$

Then we have

$$e^2(L_{mn},F_2,\|\cdot\|_{2,\rho})=\int_{F_2}\|f(x)-L_{mn}(f,x)\|_{2,\rho}^2\ \mu_2(df)$$

$$=\frac{\pi}{2}\left(\frac{\sin\frac{\pi}{m}\cos\frac{\pi}{2m}+\sin\frac{\pi}{2m}(1-\cos\frac{\pi}{m})}{m^2(1-\cos\frac{\pi}{m})}+\frac{\sin\frac{\pi}{n}\cos\frac{\pi}{2n}+\sin\frac{\pi}{2n}(1-\cos\frac{\pi}{n})}{n^2(1-\cos\frac{\pi}{n})}\right)$$

$$-\frac{1}{2}\frac{\sin\frac{\pi}{m}\cos\frac{\pi}{2m}+\sin\frac{\pi}{2m}(1-\cos\frac{\pi}{m})}{m^2(1-\cos\frac{\pi}{m})}\cdot\frac{\sin\frac{\pi}{n}\cos\frac{\pi}{2n}+\sin\frac{\pi}{2n}(1-\cos\frac{\pi}{n})}{n^2(1-\cos\frac{\pi}{n})}.$$

Theorem 2. Let

$$x=(x_1,\cdots,x_d)\in[-1,1]^d,$$

$$\rho(x)=\left(\prod_{i=1}^{d}\sqrt{1-x_i^2}\right)^{-1},$$

and $L_{n,d}(f,x)$ be defined by (3). Then we have

$$e^2(L_{n,d},F_d,\|\cdot\|_{2,\rho})$$

$$=\int_{F_d}\|f(x)-L_{n,d}(f,x)\|_{2,\rho}^2\ \mu_d(df)$$

$$=\frac{1}{2^{d-1}}\sum_{k=1}^{d}\binom{d}{k}(-1)^{k-1}$$

$$\cdot\left(\frac{\sin\frac{\pi}{n}\cos\frac{\pi}{2n}+\sin\frac{\pi}{2n}(1-\cos\frac{\pi}{n})}{n^2(1-\cos\frac{\pi}{n})}\right)^k\pi^{d-k}.$$

Remark 1. Let us recall some fundamental notions about the information-based complexity in the average case setting. Let F be a set with a probability measure ν, and let G be a normed linear space with norm $\|\cdot\|$. Let S be a measurable mapping from F to G which is called a solution operator. Let N be a measurable mapping from F into R^d, and let ϕ be a measurable mapping from R^d into G which are called an information operator and an algorithm, respectively. The average error of the approximation $\phi\circ N$ with respect to the measure ν is defined by

$$e(S,N,\phi):=\left(\int_F\|S(f)-\phi(N(f))\|^2\ \nu(df)\right)^{\frac{1}{2}},$$

and the average radius of information N with respect to ν is defined by

$$r(S,N):=\inf_{\phi}e(S,N,\phi),$$

where ϕ ranges over the set of all possible algorithms that use information N. Furthermore, let Λ_m be the class of all deterministic information operators N with cardinality m. Then, the m th minimal average radius is defined by

$$r(S,\Lambda_m):=\inf_{N\in\Lambda_m}r(S,N).$$

For $F=C_{0,d},\nu=w_d,S=I$ (the identity mapping), and Λ_m consisting of function values taken at grid points, *i e*.,

$$N(f)=[f(h_1,\cdots,h_d),\cdots,f(i_1h_1,\cdots,i_dh_d),$$
$$\cdots,f(1-h_1,\cdots,1-h_d)]$$

for some $h_1,\cdots,h_d$, by [3,p.16] we know

$$r(S,\Lambda_m)\asymp\frac{1}{m^{1/2d}}.$$

From Theorem 2 we have

$$e(L_{n,d},F_d,\|\cdot\|_{2,\rho})\asymp\frac{1}{n^{1/2}}.$$

Note that $m=n^d$, we can say that the average error of $L_{n,d}$ is weakly equivalent to the corresponding n^d th minimal average radius.

3. Proof of Theorem 1

Proof of Theorem 1. By a simple computation, we have

$$\begin{aligned}
&e^2(L_{mn},F_2,\|\cdot\|_{2,\rho})=\int_{F_2}\int_{[-1,1]^2}|f(x)-L_{mn}(f,x)|^2\rho(x)dx\mu_2(df)\\
&=\int_{[-1,1]^2}\{\int_{F_2}\left(f^2(x)-2f(x)L_{mn}(f,x)+L_{mn}^2(f,x)\right)\mu_2(df)\}\rho(x)dx\\
&=\int_{[-1,1]^2}\{\int_{F_2}f^2(x)\mu_2(df)\}\rho(x)dx-2\int_{[-1,1]^2}\{\int_{F_2}f(x)L_{mn}(f,x)\mu_2(df)\}\rho(x)dx+\int_{[-1,1]^2}\{\int_{F_2}L_{mn}^2(f,x)\mu_2(df)\}\rho(x)dx\\
&=:I_1-2I_2+I_3.
\end{aligned}\tag{4}$$

On using (1) and (2), we obtain

$$\begin{aligned}
I_1&=\int_{[-1,1]^2}\{\int_{F_2}f^2(x)\mu_2(df)\}\rho(x)dx=\int_{[-1,1]^2}\{\int_{C_{0,2}}g^2(\frac{x_1+1}{2},\frac{x_2+1}{2})w_2(dg)\}\rho(x)dx\\
&=\int_{-1}^1\frac{x_1+1}{2}\frac{1}{\sqrt{1-x_1^2}}dx_1\int_{-1}^1\frac{x_2+1}{2}\frac{1}{\sqrt{1-x_2^2}}dx_2=\frac{\pi^2}{4}.
\end{aligned}\tag{5}$$

From [2], we have

$$\begin{aligned}
I_2&=\int_{[-1,1]^2}\{\int_{F_2}f(x)L_{mn}(f,x)\mu_2(df)\}\rho(x)dx=\sum_{i=1}^m\sum_{j=1}^n\int_{-1}^1\int_{-1}^1\int_{F_2}f(x_1,x_2)f(\xi_{i,m},\xi_{j,n})\mu_2(df)\cdot l_{1,i}(x_1)l_{2,j}(x_2)\rho(x_1,x_2)dx_1dx_2\\
&=\sum_{i=1}^m\sum_{j=1}^n\int_{-1}^1\int_{-1}^1\int_{C_{0,2}}g(\frac{x_1+1}{2},\frac{x_2+1}{2})\cdot g(\frac{\xi_{i,m}+1}{2},\frac{\xi_{j,n}+1}{2})w_2(dg)\cdot l_{1,i}(x_1)l_{2,j}(x_2)\rho(x_1,x_2)dx_1dx_2\\
&=\sum_{i=1}^m\sum_{j=1}^n\int_{-1}^1\int_{-1}^1\frac{1+\min\{x_1,\xi_{i,m}\}}{2}\frac{1+\min\{x_2,\xi_{j,n}\}}{2}\cdot l_{1,i}(x_1)l_{2,j}(x_2)\rho(x_1,x_2)dx_1dx_2\\
&=\sum_{i=1}^m\int_{-1}^1\frac{1+\min\{x_1,\xi_{i,m}\}}{2}\frac{l_{1,i}(x_1)}{\sqrt{1-x_1^2}}dx_1\cdot\sum_{j=1}^n\frac{1+\min\{x_2,\xi_{j,n}\}}{2}\frac{l_{2,j}(x_2)}{\sqrt{1-x_2^2}}dx_2\\
&=\frac{1}{4}\left(\pi-\frac{\sin\frac{\pi}{m}\cos\frac{\pi}{2m}+\sin\frac{\pi}{2m}(1-\cos\frac{\pi}{m})}{m^2(1-\cos\frac{\pi}{m})}\right)\cdot\left(\pi-\frac{\sin\frac{\pi}{n}\cos\frac{\pi}{2n}+\sin\frac{\pi}{2n}(1-\cos\frac{\pi}{n})}{n^2(1-\cos\frac{\pi}{n})}\right),
\end{aligned}\tag{6}$$

and

$$\begin{aligned}
I_3&=\int_{[-1,1]^2}\{\int_{F_2}L_{mn}^2(f,x)\mu_2(df)\}\rho(x)dx\\
&=\sum_{i=1}^m\sum_{j=1}^n\sum_{k=1}^m\sum_{s=1}^n\int_{-1}^1\int_{-1}^1\int_{F_2}f(\xi_{i,m},\xi_{j,n})\cdot f(\xi_{k,m},\xi_{s,n})\mu_2(df)l_{1,i}(x_1)l_{1,k}(x_1)\cdot l_{2,j}(x_2)l_{2,s}(x_2)\rho(x_1,x_2)dx_1dx_2\\
&=\sum_{i=1}^m\sum_{j=1}^n\sum_{k=1}^m\sum_{s=1}^n\int_{-1}^1\int_{-1}^1\int_{C_{0,2}}g(\frac{\xi_{i,m}+1}{2},\frac{\xi_{j,m}+1}{2})\cdot g(\frac{\xi_{k,m}+1}{2},\frac{\xi_{s,n}+1}{2})w_2(dg)\cdot l_{1,i}(x)l_{1,k}(x)l_{2,j}(y)l_{2,s}(y)\rho(x_1,x_2)dx_1dx_2\\
&=\sum_{i=1}^m\sum_{k=1}^m\int_{-1}^1\frac{1+\min\{\xi_{i,m},\xi_{k,m}\}}{2}\frac{l_{1,i}(x_1)l_{1,k}(x_1)}{\sqrt{1-x_1^2}}dx_1\cdot\sum_{j=1}^n\sum_{s=1}^n\int_{-1}^1\frac{1+\min\{\xi_{j,n},\xi_{s,n}\}}{2}\frac{l_{2,j}(x_2)l_{2,s}(x_2)}{\sqrt{1-x_2^2}}dx_2\\
&=\frac{\pi^2}{4}.
\end{aligned}\tag{7}$$

On combining (4)-(7), we obtain

$$e^2(L_{mn},F_2,\|\cdot\|_{2,\rho})=\frac{\pi^2}{4}-\frac{1}{2}(\pi-\frac{\sin\frac{\pi}{m}\cos\frac{\pi}{2m}+\sin\frac{\pi}{2m}(1-\cos\frac{\pi}{m})}{m^2(1-\cos\frac{\pi}{m})})\cdot(\pi-\frac{\sin\frac{\pi}{n}\cos\frac{\pi}{2n}+\sin\frac{\pi}{2n}(1-\cos\frac{\pi}{n})}{n^2(1-\cos\frac{\pi}{n})})+\frac{\pi^2}{4}$$

$$=\frac{\pi}{2}(\frac{\sin\frac{\pi}{m}\cos\frac{\pi}{2m}+\sin\frac{\pi}{2m}(1-\cos\frac{\pi}{m})}{m^2(1-\cos\frac{\pi}{m})}+\frac{\sin\frac{\pi}{n}\cos\frac{\pi}{2n}+\sin\frac{\pi}{2n}(1-\cos\frac{\pi}{n})}{n^2(1-\cos\frac{\pi}{n})})$$

$$-\frac{1}{2}\frac{\sin\frac{\pi}{m}\cos\frac{\pi}{2m}+\sin\frac{\pi}{2m}(1-\cos\frac{\pi}{m})}{m^2(1-\cos\frac{\pi}{m})}\cdot\frac{\sin\frac{\pi}{n}\cos\frac{\pi}{2n}+\sin\frac{\pi}{2n}(1-\cos\frac{\pi}{n})}{n^2(1-\cos\frac{\pi}{n})},$$

we complete the proof of Theorem 1.

4. Proof of Theorem 2

Proof of Theorem 2. Similar to the proof of Theorem 1,

$$\begin{aligned}e^2(L_{n,d},F_d,\|\cdot\|_{2,\rho})&=\int_{F_d}\|f(x)-L_{n,d}(f,x)\|_{2,\rho}^2\;\mu_d(df)\\&=\int_{[-1,1]^d}\{\int_{F_d}(f^2(x)-2f(x)L_{n,d}(f,x)+L_{n,d}^2(f,x))\mu_d(df)\}\rho(x)dx\\&=\int_{[-1,1]^d}\{\int_{F_d}f^2(x)\mu_d(df)\}\rho(x)dx-2\int_{[-1,1]^d}\{\int_{F_d}f(x)L_{n,d}(f,x)\mu_d(df)\}\rho(x)dx+\int_{[-1,1]^d}\{\int_{F_d}L_{n,d}^2(f,x)\mu_d(df)\}\rho(x)dx\\&=:J_1-2J_2+J_3.\end{aligned}\tag{8}$$

Form (1) and (2),

$$\begin{aligned}J_1&=\int_{[-1,1]^d}\{\int_{F_d}f^2(x)\mu_d(df)\}\rho(x)dx=\int_{[-1,1]^d}\{\int_{C_{0,d}}g^2(\frac{x_1+1}{2},\cdots,\frac{x_d+1}{2})w_d(df)\}\rho(x)dx\\&=\prod_{k=1}^d\int_{-1}^1\frac{x_k+1}{2}\frac{1}{\sqrt{1-x_k^2}}dx_k=\left(\frac{\pi}{2}\right)^d.\end{aligned}\tag{9}$$

By a simple computation similar to (6)-(7) we obtain

$$\begin{aligned}J_2&=\int_{[-1,1]^d}\{\int_{F_d}f(x)L_{n,d}(f,x)\mu_d(df)\}\rho(x)dx\\&=\sum_{i_1,\cdots,i_d=1}^n\int_{[-1,1]^d}\{\int_{F_d}f(x_1,\cdots,x_d)\cdot f(\xi_{i_1},\cdots,\xi_{i_d})\mu_d(df)\}l_{1,i_1}(x_1)\cdots l_{d,i_d}(x_d)\cdot\rho(x_1,\cdots,x_d)dx_1\cdots dx_d\\&=\sum_{i_1,\cdots,i_d=1}^n\int_{[-1,1]^d}\{\int_{C_{0,d}}g(\frac{x_1+1}{2},\cdots,\frac{x_d+1}{2})\cdot g(\frac{\xi_{i_1}+1}{2},\cdots,\frac{\xi_{i_d}+1}{2})w_d(dg)\cdot l_{1,i_1}(x_1)\cdots l_{d,i_d}(x_d)\rho(x_1,\cdots,x_d)dx_1\cdots dx_d\\&=\sum_{i_1,\cdots,i_d=1}^n\int_{[-1,1]^d}\prod_{k=1}^d\frac{1+\min\left\{x_k,\xi_{i_k}\right\}}{2}\cdot l_{1,i_1}(x_1)\cdots l_{d,i_d}(x_d)\rho(x_1,\cdots,x_d)dx_1\cdots dx_d\\&=\prod_{k=1}^d\sum_{i_k=1}^n\int_{-1}^1\frac{1+\min\left\{x_k,\xi_{i_k}\right\}}{2}\frac{l_{k,i_k}(x_k)}{\sqrt{1-x_k^2}}dx_k=\frac{1}{2^d}(\pi-\frac{\sin\frac{\pi}{n}\cos\frac{\pi}{2n}+\sin\frac{\pi}{2n}(1-\cos\frac{\pi}{n})}{n^2(1-\cos\frac{\pi}{n})})^d,\end{aligned}\tag{10}$$

and

$$
\begin{aligned}
J_3 &= \int_{[-1,1]^d} \{ \int_{F_d} L_{n,d}^2 (f,x)\mu_d(df)\}\rho(x)dx \\
&= \sum_{i_1,\cdots,i_d=1}^{n} \sum_{j_1,\cdots,j_d=1}^{n} \int_{[-1,1]^d} \{ \int_{F_d} f(\xi_{i_1},\cdots,\xi_{i_d}) \cdot f(\xi_{j_1},\cdots,\xi_{j_d})\mu_d(df)\} \prod_{k=1}^{d} l_{k,i_k}(x_k) \cdot \prod_{s=1}^{d} l_{s,j_s}(x_s)\rho(x_1,\cdots,x_d)dx_1\cdots dx_d \\
&= \sum_{i_1,\cdots,i_d=1}^{n} \sum_{j_1,\cdots,j_d=1}^{n} \int_{[-1,1]^d} \{ \int_{C_{0,d}} g(\frac{\xi_{i_1}+1}{2},\cdots,\frac{\xi_{i_d}+1}{2}) \cdot g(\frac{\xi_{j_1}+1}{2},\cdots,\frac{\xi_{j_d}+1}{2}) w_d(dg)\} \prod_{k=1}^{d} l_{k,i_k}(x_k) \cdot \prod_{s=1}^{d} l_{s,j_s}(x_s)\rho(x_1,\cdots,x_d)dx_1\cdots dx_d \\
&= \prod_{k=1}^{d} \sum_{i_k=1}^{n} \sum_{j_k=1}^{n} \int_{-1}^{1} \frac{1+\min\{\xi_{i_k},\xi_{j_k}\}}{2} \frac{l_{k,i_k}(x_k) l_{k,j_k}(x_k)}{\sqrt{1-x_k^2}} dx_k = \left(\frac{\pi}{2}\right)^d .
\end{aligned} \tag{11}
$$

On combining (8)-(11), we obtain

$$
\begin{aligned}
e^2(L_{n,d},F_d,\|\cdot\|_{2,\rho}) &= \left(\frac{\pi}{2}\right)^d - \frac{1}{2^{d-1}}\left(\pi - \frac{\sin\frac{\pi}{n}\cos\frac{\pi}{2n} + \sin\frac{\pi}{2n}(1-\cos\frac{\pi}{n})}{n^2(1-\cos\frac{\pi}{n})}\right)^d + \left(\frac{\pi}{2}\right)^d \\
&= \frac{1}{2^{d-1}} \sum_{k=1}^{d} \binom{d}{k} (-1)^{k-1} \cdot \left(\frac{\sin\frac{\pi}{n}\cos\frac{\pi}{2n} + \sin\frac{\pi}{2n}(1-\cos\frac{\pi}{n})}{n^2(1-\cos\frac{\pi}{n})}\right)^k \pi^{d-k}.
\end{aligned}
$$

We complete the proof of Theorem 2.

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