Journal of Applied Mathematics and Physics, 2013, 1, 73-77
http://dx.doi.org/10.4236/jamp.2013.14014 Published Online October 2013 (http://www.scirp.org/journal/jamp)

Positive Solutions for a Boundary Value Problem with a Derivative Argument*

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Received 2013

ABSTRACT

In this paper, several existence results of multiple positive solutions are obtained for a boundary value problem with p-Laplacian, by applying a fixed point theorem in cones. The interesting point is that the nonlinear term f is involved with the first-order derivative explicitly.

Keywords: Positive Solutions; Fixed Point Theorem; Cone; Boundary Value Problem; P-Laplacian

1. Introduction

In this paper, we will consider the existence of multiple positive solutions for the following one-dimensional p-Laplacian, multi-point boundary value problem

$$(\phi_p(u'(t)))' + f(t,u(t),u'(t)) = 0,\ t\in(0,1), \quad (1.1)$$

$$u'(0)=\sum_{i=1}^{n}\alpha_i u'(\xi_i),\quad u(1)=\sum_{i=1}^{n}\beta_i u(\xi_i), \quad (1.2)$$

where $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\xi_i\in(0,1)$ with $0<\xi_1<\xi_2<\cdots<\xi_n<1$ and α_i,β_i,f satisfy,

$(H1)$: $0\le\alpha_i,\beta_i<1\,(i=1,2,\ldots,n-2)$ satisfy

$$0\le\sum_{i=1}^{n}\alpha_i<1,\ 0\le\sum_{i=1}^{n}\beta_i<1;$$

$(H2)$: $f(t,u,v)\in C([0,1]\times[0,+\infty)\times R\to[0,+\infty))$, and $f(t,0,0)\not\equiv0$ on any subinterval of $(0,1)$.

$(H3)$: $f(t,u_1,v_1)\le f(t,u_2,v_2)$ for any $0\le t\le1, 0\le u_1\le u_2$, $v_1\le v_2$;

$(H4)$: $f(t,u,v)\le g(t)|u|^a+h(t)|v|^b+j(t)$, for $0\le t\le1$, $u\ge0$, where $g(t),h(t),j(t)\in L^1[0,1]$, $0\le a,b<p-1$.

There is much current interest in questions of multiple positive solutions of boundary value problems, and the existence and multiplicity of positive solutions for linear and nonlinear multipoint boundary value problems have been widely studied by many authors, one may see [1-9] and relevant literatures.

Among the substantial number of works dealing with nonlinear differential equations we mention the boundary value problem (1.1) and (1.2). We emphasize that the nonlinear term f is involved with the first-order derivative explicitly in this problem.

2. Definition and Theoretical Foundation

In this section, we give definition and theoretical foundation in this paper.

Definition 2.1. Let E be a real Banach space over R. A nonempty closed set $P\subset E$ is said to be a cone provided that

(*i*) $au+bv\in P$ for all $u,v\in P$ and all $a\ge0$, $b\ge0$, and

(*ii*) $u,-u\in P$ imply $u=0$.

To prove our main results, we need the following fixed point theorem in cones.

Theorem 2.1. [7] Let K be a cone in a Banach space X.

Let D be an open bounded subset of X with

$$D_k=D\cap K\neq\varnothing.$$

Assume that $A:\overline{D_k}\to K$ is completely continuous such that $x\neq Ax$ for $x\in\partial D_k$.

Then the following results hold:

$(S1)$ If $\|Ax\|\le\|x\|$, $x\in\partial D_k$, then $i_k(A,D_k)=1$.

$(S2)$ If there exists $e\in K\setminus\{0\}$ such that $x\neq Ax+\lambda e$ for all $x\in\partial D_k$ and all $\lambda>0$, then $i_k(A,D_k)=0$.

$(S3)$ Let U be open in X such that $\overline{U}\subset D_k$. If $i_k(A,D_k)=1$ and $i_k(A,U_k)=0$, then A has a fixed point in $D_k\setminus\overline{U_k}$.

*This work was supported by the National Natural Science Foundation of China (NO.11126245), Youth PhD Development Fund of Central University of Finance and Economics 121 Talent Cultivation Project (NO.QBJZH201004) and Discipline Construction Fund of Central University of Finance and Economics.

The same result holds if $i_k(A,D_k)=0$ and $i_k(A,U_k)=1$.

3. Preliminaries

Let the Banach space $E=C^1[0,1]$ be endowed with the norm $\|u\|:=\max\{\max_{0\le t\le 1}|u(t)|,\max_{0\le t\le 1}|u'(t)|\}$.

We denote $E_+=C_+^1[0,1]=\{u\in E\,|\,u(t)\ge 0,t\in[0,1]\}$, and define the cone $P\subset E$ by

$$P=\{u\in E\,|\,u(t)\ge 0, u \text{ is concave on } [0,1] \text{ and satisfy } (1.2)\}.$$

Throughout, it is assumed that $(H1)-(H4)$ hold.

Lemma 3.1. Suppose $y\in C^1[0,1]$ with $(\phi_p(y'(t)))'\in C^1[0,1]$ satisfies,

$$-(\phi_p(y'(t)))'\ge 0,\qquad t\in(0,1),$$

$$y'(0)=\sum_{i=1}^{n}\alpha_i y'(\xi_i),\quad y(1)=\sum_{i=1}^{n}\beta_i y(\xi_i).$$

Then, $y(t)$ is concave and $y(t)\ge 0$, i.e., $y\in P$, and $y'(t)\le 0$ on [0,1].

Proof. The proof is very easy since

$$0\le\sum_{i=1}^{n-2}\alpha_i,\sum_{i=1}^{n-2}\beta_i<1\cdot$$

so we omit it here. □

For any $x\in C_+[0,1]$, suppose u is a solution of the problem

$$\begin{cases}(\phi_p(u'))'+f(t,x,x')=0,\ 0\le t\le 1,\\ u'(0)=\sum_{i=1}^{n-2}\alpha_i u'(\xi_i),\ u(1)=\sum_{i=1}^{n-2}\beta_i u(\xi_i),\end{cases}$$

then we have

$$u'(t)=\phi_p^{-1}\Big(W_x-\int_0^t f(s,x(s),x'(s))ds\Big),$$

$$u(t)=-\frac{\sum_{i=1}^{n-2}\beta_i\int_{\xi_i}^{1}\phi_p^{-1}\Big(W_x-\int_0^s f(r,x(r),x'(r))dr\Big)ds}{1-\sum_{i=1}^{n-2}\beta_i}-\int_t^1\phi_p^{-1}\Big(W_x-\int_0^s f(r,x(r),x'(r))dr\Big)ds,$$

where W_x satisfy

$$\phi_p^{-1}(W_x)=\sum_{i=1}^{n-2}\alpha_i\phi_p^{-1}\Big(W_x-\int_0^{\xi_i} f(r,x(r),x'(r))dr\Big).\quad (3.1)$$

Lemma 3.2. $x\in C_+^1[0,1]$, there exists a unique W_x satisfies (3.1) and

$$W_x\in\Big[-\frac{\phi_p(\sum_{i=1}^{n}\alpha_i)}{1-\phi_p(\sum_{i=1}^{n}\alpha_i)}\int_0^1 f(r,x(r),x'(r))dr,0\Big].$$

Proof. The proof is very similar to the proof of Lemma 2.2 in [8], so we omit it here.

Lemma 3.3. For $u\in P$, there exists a constant γ_1, such that $\min_{0\le t\le 1}u(t)\ge\gamma_1\|u\|$, where

$$\gamma_1=\min\left\{\frac{\sum_{i=1}^{n}\beta_i(1-\xi_i)}{1-\sum_{i=1}^{n}\beta_i}\Big(\frac{1}{\lambda}\Big)^{\frac{1}{p-1}},\frac{\sum_{i=1}^{n}\beta_i(1-\xi_i)}{1-\sum_{i=1}^{n}\beta_i\xi_i}\right\},\quad (3.2)$$

and $\lambda>1$ is a constant such that

$$0<\xi_1<2\xi_1<3\xi_1<\cdots<(\lambda-1)\xi_1<1<\lambda\xi_1.$$

Proof. By lemma 3.1, we have

$$\min_{0\le t\le 1}u(t)=u(1),\quad \|u\|=\max\{u(0),-u'(1)\}.$$

For $u\in P$, by the concavity of u, we have

$$\frac{u(\xi_i)-u(1)}{1-\xi_i}\ge\frac{u(0)-u(1)}{1},$$

that is $u(\xi_i)-\xi_i u(1)\ge(1-\xi_i)u(0),$

to combine with the boundary condition (1.2) we have

$$\sum_{i=1}^{n-2}\beta_i u(\xi_i)-\sum_{i=1}^{n-2}\beta_i\xi_i u(1)\ge\sum_{i=1}^{n-2}\beta_i(1-\xi_i)u(0),$$

so

$$\min_{0\le t\le 1}u(t)=u(1)\ge\frac{\sum_{i=1}^{n-2}\beta_i(1-\xi_i)}{1-\sum_{i=1}^{n-2}\beta_i\xi_i}u(0).$$

By (H_3) and lemma 3.2, we have

$$\begin{aligned}&\lambda\Big(-W_x+\int_0^{\xi_1}f(r,x(r),x'(r))dr\Big)\\ &\ge\Big(-W_x+\int_0^{\xi_1}f(r,x(r),x'(r))dr\Big)\\ &+\Big(-W_x+\int_{\xi_1}^{2\xi_1}f(r,x(r),x'(r))dr\Big)\\ &+\cdots+\Big(-W_x+\int_{(\lambda-1)\xi_1}^{\lambda\xi_1}f(r,x(r),x'(r))dr\Big)\\ &=-\lambda W_x+\int_0^{\lambda\xi_1}f(r,x(r),x'(r))dr\\ &\ge-W_x+\int_0^1 f(r,x(r),x'(r))dr,\end{aligned}$$

therefore,

$$-W_x+\int_0^{\xi_1}f(r,x(r),x'(r))dr\ge\frac{1}{\lambda}\Big(-W_x+\int_0^1 f(r,x(r),x'(r))dr\Big).$$

On the other hand,

$$\min_{0\le t\le 1}u(t)=u(1)=\frac{\sum_{i=1}^{n}\beta_i\int_{\xi_i}^{1}\phi_P^{-1}\Big(-W_x+\int_0^s f(r,x(r),x'(r))dr\Big)ds}{1-\sum_{i=1}^{n}\beta_i}$$

$$\ge \frac{\sum_{i=1}^{n}\beta_i(1-\xi_i)}{1-\sum_{i=1}^{n}\beta_i}\phi_P^{-1}\left(-W_x+\int_0^{\xi_1} f(r,x(r),x'(r))dr\right)$$

$$\ge \frac{\sum_{i=1}^{n}\beta_i(1-\xi_i)}{1-\sum_{i=1}^{n}\beta_i}\left(\frac{1}{\lambda}\right)^{\frac{1}{p-1}}\phi_P^{-1}\left(-W_x+\int_0^{1} f(r,x(r),x'(r))dr\right)$$

$$= \frac{\sum_{i=1}^{n}\beta_i(1-\xi_i)}{1-\sum_{i=1}^{n}\beta_i}\left(\frac{1}{\lambda}\right)^{\frac{1}{p-1}}\left(-u'(1)\right).$$

From the discussion above, choose γ_1 by (3.2), we can obtain the lemma 3.3 is proved.

Lemma 3.4. For $u\in P,$ there exists a constant $N>0,$ such that $\max_{0\le t\le 1}|u'(t)|\le N.$

Proof. For $u\in P$, by the concavity of u and lemma 3.1, there is

$$u(t)-u(1)\le -u'(1)$$
$$=\max_{0\le t\le 1}|u'(t)|,\qquad t\in[0,1].$$

Taking into account that $u(1)=\sum_{i=1}^{n}\beta_i u(\xi_i)$, we have

$$\max_{0\le t\le 1}|u(t)|\le (\sum_{i=1}^{n}\beta_i)\max_{0\le t\le 1}|u(t)|+\max_{0\le t\le 1}|u'(t)|.$$

So,

$$(1-\sum_{i=1}^{n}\beta_i)\max_{0\le t\le 1}|u(t)|\le \max_{0\le t\le 1}|u'(t)|.$$

And because

$$u'(t)=\phi_p^{-1}\left(W_x-\int_0^t f(s,x(s),x'(s))ds\right),$$

in view of $(H3)$, we have

$$(|u'(t)|)^{p-1}$$
$$=-W_x+\int_0^t f(s,x(s),x'(s))ds$$
$$\le \frac{1}{1-\phi_p(\sum_{i=1}^{n}\alpha_i)}\int_0^1 f(r,x(r),x'(r))dr$$
$$\le \frac{1}{1-\phi_p(\sum_{i=1}^{n}\alpha_i)}\Big(\big(\max_{0\le t\le 1}|u(t)|\big)^a\int_0^1|g(t)|dt$$
$$+\big(\max_{0\le t\le 1}|u'(t)|\big)^b\int_0^1|h(t)|dt+\int_0^1|j(t)|dt\Big)$$

$$\le \frac{1}{1-\phi_p(\sum_{i=1}^{n}\alpha_i)}\Big(\frac{1}{(1-\sum_{i=1}^{n}\beta_i)^a}\big(\max_{0\le t\le 1}|u'(t)|\big)^a\int_0^1|g(t)|dt$$
$$+\big(\max_{0\le t\le 1}|u'(t)|\big)^b\int_0^1|h(t)|dt+\int_0^1|j(t)|dt\Big).$$

Since $0\le a,b<p-1,$ we have $\max_{0\le t\le 1}|u(t)|$ is bounded. Therefore, there exists a constant $N>0$, such that lemma 3.4 is proved.

For notational convenience, we denote

$$\theta=\frac{\phi_p(\sum_{i=1}^{n}\alpha_i)}{1-\phi_p(\sum_{i=1}^{n}\alpha_i)},\ \eta=\frac{1}{1-\phi_p(\sum_{i=1}^{n}\alpha_i)},$$

$$\frac{1}{l}=\max\left\{\frac{p-1}{p}\left(\frac{\sum_{i=1}^{n}\beta_i}{1-\sum_{i=1}^{n}\beta_i}\left(\eta^{\frac{p}{p-1}}-(\xi_i+\theta)^{\frac{p}{p-1}}\right)+\left(\eta^{\frac{p}{p-1}}-\theta^{\frac{p}{p-1}}\right)\right),\eta^{\frac{1}{p-1}}\right\},$$

$$\frac{1}{L}=\frac{p-1}{p}\left(1-\sum_{i=1}^{n}\beta_i\right)^{-1}\sum_{i=1}^{n}\beta_i(1-\xi_i^{\frac{p}{p-1}}),$$

then define

$$K=\{u\mid u\in E,\min_{0\le t\le 1}u(t)\ge\gamma\|u\|,-N\le u'\le 0,\}$$

where $\gamma=\gamma_1\gamma_2$, γ_1 is defined by (3.2), $\gamma_2=\frac{l}{L}$, N is defined in lemma 3.4.

It is easy to obtain that, $0<l,L<\infty$ and $L\gamma=L\gamma_1\gamma_2=\gamma_1 l<l.$

We define an operator $T:K\to E$ by

$$(Tu)(t)=\frac{-\sum_{i=1}^{n}\beta_i\int_{\xi_i}^{1}\phi_p^{-1}\left(W_u-\int_0^s f(r,u(r),u'(r))dr\right)ds}{1-\sum_{i=1}^{n}\beta_i}$$
$$-\int_t^1\phi_p^{-1}\left(W_u-\int_0^s f(r,u(r),u'(r))dr\right)ds,\quad (3.3)$$

where W_u is defined in (3.1). Then a standard argument shows that $T:K\to E$ is completely continuous.

Define

$$K_N=\{x\in K\mid \|x\|<N\},$$
$$\Omega_N=\{x\in K\mid \min_{0\le t\le 1}x(t)<\gamma N,-N<x'(t)<0\}$$
$$=\{x\mid x\in C^1[0,1],x\ge 0,\gamma\|x\|$$
$$\le\min_{0\le t\le 1}x(t)<\gamma N,-N<x'(t)<0\}.$$

Lemma 3.5. [9] Ω_N has the following properties:

Ω_N is open relative to K;

$$K_{\gamma N}\subset\Omega_N\subset K_N;$$

$x\in\partial\Omega_N$ if and only if $\min_{0\le t\le 1}x(t)=\gamma N$, $\min_{0\le t\le 1}x'(t)=-N$;

If $x\in\partial\Omega_N$, then $\gamma N\le x(t)\le N$, $-N\le x'(t)\le 0$ for $t\in[0,1]$. □

For notational convenience, we introduce the following notations

$$f_{-N,0}^{\gamma N,N}=\min\left\{\frac{f(t,u,v)}{\phi_p(N)}:t\in[0,1],u\in[\gamma N,N],v\in[-N,0]\right\},$$

$$f_{-N,0}^{0,N}=\max\left\{\frac{f(t,u,v)}{\phi_p(N)}:\ t\in[0,1],u\in[0,N],v\in[-N,0]\right\},$$

$$f_{-N,0}^{\delta}=\lim_{u\to\alpha}\max\left\{\frac{f(t,u,v)}{\phi_p(u)}:\ t\in[0,1],v\in[-N,0]\right\},$$

$$f_{\delta}^{-N,0}=\lim_{u\to\alpha}\min\left\{\frac{f(t,u,v)}{\phi_p(u)}:\ t\in[0,1],v\in[-N,0]\right\},$$

$$(\delta=\infty\ or\ 0^+).$$

Lemma 3.6. If f satisfies

$$f_{-N,0}^{0,N} < \phi_p(l), \qquad (3.4)$$

then $i_k(T, K_N) = 1$.

Proof. From the definition of T and by (3.4), for $u(t) \in \partial K_N$,

$$(Tu)(t) \le \frac{\sum_{i=1}^{n} \beta_i \int_{\xi_i}^{1} \phi_p^{-1}\left(-W_u + \int_0^s f(r,u(r),u'(r))dr\right)ds}{1-\sum_{i=1}^{n}\beta_i} + \int_0^1 \phi_q\left(-W_u + \int_0^s f(r,u(r),u'(r))dr\right)ds$$

$$< lN\frac{p-1}{p}\left(\frac{\sum_{i=1}^{n}\beta_i}{1-\sum_{i=1}^{n}\beta_i}\left(\eta^{\frac{p}{p-1}} - (\xi_i+\theta)^{\frac{p}{p-1}}\right) + \left(\eta^{\frac{p}{p-1}} - \theta^{\frac{p}{p-1}}\right)\right) \le N,$$

and

$$|(Tu)'(t)| = \phi_p^{-1}\left(-w_u + \int_0^t f(r,u(r),u'(r))dr\right) < lN\eta^{\frac{1}{p-1}} \le N.$$

So, we have $\|Tu\| < \|u\|$ for $u(t) \in \partial K_N$. By Theorem 2.1, we have $i_k(T, K_N) = 1$. □

Lemma 3.7. If f satisfies

$$f_{-N,0}^{\gamma N,N} > \phi_p(L\gamma), \qquad (3.5)$$

then $i_k(T, \Omega_N) = 0$.

Proof. Let $e(t) \equiv 1$ for $t \in [0,1]$. Then $e \in \partial K_1$. we claim that

$$u \ne Tu + \rho e, \quad u \in \partial\Omega_N, \quad \rho \ge 0.$$

In fact, if the assumption fails, there exist $u_0 \in \partial\Omega_N$ and $\rho_0 \ge 0$ such that $u_0 = Tu_0 + \rho_0 e$.

From the definition of T and by lemma 3.5, for $t \in [0,1]$, we have

$$u_0(t) = Tu_0(t) + \rho_0 e \; u_0(t) = Tu_0(t) + \rho_0 e$$

$$\ge \frac{\sum_{i=1}^{n}\beta_i \int_{\xi_i}^{1} \phi_q(-W_{u_0} + \int_0^s f(r,u_0(r),u_0'(r))dr)ds}{1-\sum_{i=1}^{n}\beta_i} + \rho_0$$

$$> \frac{\sum_{i=1}^{n}\beta_i}{1-\sum_{i=1}^{n}\beta_i}\frac{p-1}{p}L\gamma N(1-\xi_i^{\frac{p}{p-1}}) + \rho_0$$

$$> \gamma N + \rho_0.$$

This implies $\gamma N > \gamma N + \lambda_0$, which is a contradiction.

So, by Theorem 2.1, we obtain $i_k(T, \Omega_N) = 0$. □

4. Multiple Positive Solutions to (1.1), (1.2)

Our main results are as follows.

Theorem 4.1. Suppose $(H1)-(H4)$ hold, and

$(H5)$: there exist $\rho_1, \rho_2, \rho_3 \in (0, N)$, with $\rho_1 < \gamma\rho_2 < \rho_2 < \rho_3$ such that

$$f_{-\rho_1,0}^{0,\rho_1} < \phi_p(l), \quad f_{-\rho_2,0}^{\gamma\rho_2,\rho_2} > \phi_p(L\gamma), \quad f_{-\rho_3,0}^{0,\rho_3} \le \phi_p(l).$$

Then boundary value problem (1.1) (1.2) has three positive solutions in K.

Suppose $(H1)-(H4)$ hold, and

$(H6)$: there exist $\rho_1, \rho_2, \rho_3 \in (0, N)$, with $\rho_1 < \rho_2 < \gamma\rho_3$ such that

$$f_{-\rho_1,0}^{\gamma\rho_1,\rho_1} > \phi_p(L\gamma), \quad f_{-\rho_2,0}^{0,\rho_2} < \phi_p(l), \quad f_{-\rho_3,0}^{\gamma\rho_3,\rho_3} \ge \phi_p(L\gamma).$$

Then boundary value problem (1.1) (1.2) has two positive solutions in K.

Proof. Because the proof is similar to the proof of theorem in [9]. So we omit it here. □

Corollary 4.1. Suppose $(H1)-(H4)$ hold. In addition, if there exists $0 < \rho < N$ such that

$(H7)$:

$$0 \le f_{-\rho,0}^{0} < \phi_p(l), \quad f_{-\rho,0}^{\gamma\rho,\rho} > \phi_p(L\gamma), \quad 0 \le f_{-N,0}^{\infty} < \phi_p(l).$$

Then boundary value problem (1.1) (1.2) has three positive solutions in K.

Suppose $(H1)-(H4)$ hold, and if there exists $0 < \rho < N$ such that

$(H8)$:

$$\phi_p(L) < f_0^{-\rho,0} \le \infty, \quad f_{-\rho,0}^{0,\rho} < \phi_p(l), \quad \phi_p(L) < f_\infty^{-N,0} \le \infty.$$

Then boundary value problem (1.1) (1.2) has two positive solutions in K. □

It is similar to the conclusion of Theorem 4.1 we obtain the following theorem

Theorem 4.2. Suppose $(H1)-(H4)$ hold, and if one of the following conditions holds:

$(H9)$: There exist $\rho_1, \rho_2 \in (0, N)$ with $\rho_1 < \gamma\rho_2$ such that

$$f_{-\rho_1,0}^{0,\rho_1} \le \phi_p(l), \quad f_{-\rho_2,0}^{\gamma\rho_2,\rho_2} \ge \phi_p(L\gamma),$$

$(H10)$: There exist $\rho_1, \rho_2 \in (0, N)$ with $\rho_1 < \rho_2$ such that

$$f_{-\rho_1,0}^{\gamma\rho_1,\rho_1} \ge \phi_p(L\gamma), \quad f_{-\rho_2,0}^{0,\rho_2} \le \phi_p(l).$$

Then boundary value problem (1.1)(1.2) has a positive solution in K.

REFERENCES

[1] D. Guo, "Multiple Positive Solutions on Nonlinear Integral Equations and Applications," *Applicable Analysis*, Vol. 23, No. 1-2, 1986, pp. 77-84. http://dx.doi.org/10.1080/00036818608839632

[2] X. He, W. Ge and M. Peng, "Multiple Positive solutions

for One-Dimensional p-Laplacian Boundary Value Problems," *Applied Mathematics Letters*, Vol. 15, No. 8, 2002, pp. 937-943. http://dx.doi.org/10.1016/S0893-9659(02)00067-8

[3] R. Y. Ma and D. M. Cao, "Positive Solutions to an m-Point Boundary Value Problem," *Applied Mathematics—A Journal of Chinese Universities*, Vol. 17, No. 1, 2002, pp. 24-30. http://dx.doi.org/10.1007/s11766-002-0021-y

[4] B. Liu and J. S. Yu, "Solvability of Multi-Point Boundary Value Problems at Resonance(I)," *India J. Pure Appl. Math.*, Vol. 33, 2002, pp. 475-494.

[5] J. Henderson and H. Wang, "Positive Solutions for Nonlinear Eigenvalue Problems," *Journal of Mathematical Analysis and Applications*, Vol. 208, No. 1, 1997, pp. 252-259. http://dx.doi.org/10.1006/jmaa.1997.5334

[6] R. Ma and N. Castaneda, "Existence of Solutions of Nonlinear m-Point Boundary Value Problems," *Journal of Mathematical Analysis and Applications*, Vol. 256, No. 2, 2001, pp. 556-567. http://dx.doi.org/10.1006/jmaa.2000.7320

[7] D. Guo and V. Lakshmikantham, "Nonlinear Problems in Abstract Cones," Academic Press, New York, 1988.

[8] D. Ma, Z. Du and W. Ge, "Existence and Iteration of Monotone Positive Solutions for Multipoint Boundary Value Problems with p-Laplacian Operator," *Computers & Mathematics with Applications*, Vol. 50, No. 5-6, 2005, pp. 729-739. http://dx.doi.org/10.1016/j.camwa.2005.04.016

[9] Y. Wang and C. Hou, "Existence of Triple Positive Solutions for Multi-Point Boundary Value Problems with a One-Dimensional p-Laplacian," *Computers & Mathematics with Applications*, Vol. 54, No. 6, 2007, pp. 793-807. http://dx.doi.org/10.1016/j.camwa.2006.10.038

