Journal of Applied Mathematics and Physics, **2013, 1, 45-48**
Published Online November 2013 (http://www.scirp.org/journal/jamp)
http://dx.doi.org/10.4236/jamp.2013.15006

Scientific Research

An Optimal Inequality for One-Parameter Mean

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Received September 16, 2013; revised October 15, 2013; accepted October 21, 2013

ABSTRACT

In the present paper, we answer the question: for $0 < \alpha < 1$ fixed, what are the greatest value $p(\alpha)$ and the least value $q(\alpha)$ such that the inequality $J_p(a,b) < A^{\alpha}(a,b)G^{1-\alpha}(a,b) < J_q(a,b)$ holds for all $a,b>0$ with $a \neq b$? where for $p \in R$, the one-parameter mean $J_p(a,b)$, arithmetic mean $A(a,b)$ and geometric mean $G(a,b)$ of two positive real numbers a and b are defined by $J_p(a,b)=\begin{cases} a, & a \neq b, \\ \dfrac{p(a^{p+1}-b^{p+1})}{(p+1)(a^p-b^p)}, & a\neq b, p\neq -1,0, \\ \dfrac{ab(\log a-\log b)}{a-b}, & a\neq b, p=-1, \\ \dfrac{a-b}{\log a-\log b}, & a \neq b, p=0, \end{cases}$ $A(a,b)=\dfrac{a+b}{2}$ and $G(a,b)=\sqrt{ab}$, respectively.

Keywords: Optimal Inequality; One-Parameter Mean; Arithmetic Mean; Geometric Mean

1. Introduction

For $p \in R$, the one-parameter mean $J_p(a,b)$, arithmetic mean $A(a,b)$ and geometric mean $G(a,b)$ of two positive real numbers a and b are defined by

$$J_p(a,b)=\begin{cases} a, & a \neq b, \\ \dfrac{p(a^{p+1}-b^{p+1})}{(p+1)(a^p-b^p)}, & a\neq b, p\neq -1,0, \\ \dfrac{ab(\log a-\log b)}{a-b}, & a\neq b, p=-1, \\ \dfrac{a-b}{\log a-\log b}, & a \neq b, p=0, \end{cases} \quad (1)$$

$A(a,b)=\dfrac{a+b}{2}$ and $G(a,b)=\sqrt{ab}$, respectively.

There has been some literature on the one-parameter mean values $J_p(a,b)$, see [1-6]. It is well-known that the one-parameter mean $J_p(a,b)$ is continuous and strictly increases with respect to $p \in R$ for fixed $a,b>0$ with $a \neq b$. Many means are special cases of the one-parameter mean, for example:

$J_1(a,b)=\dfrac{a+b}{2}=A(a,b)$, the arithmetic mean,

$J_{1/2}(a,b)=\dfrac{a+\sqrt{ab}+b}{3}=He(a,b)$, the Heronian mean,

$J_{-1/2}(a,b)=\sqrt{ab}=G(a,b)$, the geometric mean, and

$J_{-2}(a,b)=\dfrac{2ab}{a+b}=H(a,b)$, the harmonic mean.

In [1], Gao and Niu found the greatest values p,s_1 and the least values q,s_2 such that the inequalities

$$J_p(a,b)\le A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)\le J_q(a,b)$$

and

$$G_{s_1,1}(a,b)\le A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)\le G_{s_2,1}(a,b)$$

hold for all $a,b>0$ with $a\neq b$, where $\alpha+\beta\in(0,1)$, and $G_{s,1}(a,b)=\left[\frac{(a^s+b^s)}{(a+b)}\right]^{1/(s-1)}$, as the Gini mean.

In [2], Cheune and Qi proved the logarithmic convexiity of the one-parameter mean values $J_p(a,b)$ and presented the monotonicity of $J(-r)J(r)$ for $r\in R$.

In [3], Wang, Qiu and Chu obtained the greatest value r_1 and the least value r_2 such that the double inequality

$$J_{r_1}(a,b)\le \alpha A(a,b)+(1-\alpha)H(a,b)\le J_{r_2}(a,b)$$

holds for all $a,b>0$ with $a\ne b$.

In [4], Hu, Tu and Chu presented the greatest value r_1 and the least value r_2 such that the double inequality $J_{r_1}(a,b)\le T(a,b)\le J_{r_2}(a,b)$ holds for all $a,b>0$ with $a\ne b$, where

$$T(a,b)=\frac{2ab}{2\arctan\left(\frac{(a-b)}{(a+b)}\right)}$$

denotes the first Seiffert mean.

In [5], Long and Chu found the greatest value p and the least value q such that the inequality

$$J_p(a,b)\le \alpha A(a,b)+(1-\alpha)H(a,b)\le J_q(a,b)$$

holds for all $a,b>0$ with $a\ne b$.

In [6], the authors established Schur-convexities of two types of one-parameter mean values in n variables, and obtained Schur-convexities of some well-known functions.

The purpose of this paper is to answer the question: for $0<\alpha<1$ fixed, what are the greatest value $p(\alpha)$ and the least value $q(\alpha)$ such that the inequality

$$J_p(a,b)<A^{\alpha}(a,b)G^{1-\alpha}(a,b)<J_q(a,b)$$

holds for all $a,b>0$ with $a\ne b$?

2. A Preliminary Lemma

In order to prove the main theorem of this paper, we need the following lemma.

Lemma 2.1. For all $t > 1$, one has

$$m(t)=\frac{t(t+1)\log^3 t}{2(t-1)^3}<1. \qquad (2)$$

Proof. The logarithmic derivative of $m(t)$ is

$$\frac{m'(t)}{m(t)}=\left[\log m(t)\right]'=\frac{n(t)}{t(t^2-1)\log t}, \qquad (3)$$

where

$$n(t)=-(t^2+4t+1)\log t+3(t^2-1),\ \lim_{t\to1^+}n(t)=0. \qquad (4)$$

Simple calculations lead to

$$n'(t)=5t-4-\frac{1}{t}-2(t+2)\log t,\ \lim_{t\to1^+}n'(t)=0 \qquad (5)$$

$$n''(t)=3-\frac{4}{t}+\frac{1}{t^2}-2\log t,\ \lim_{t\to1^+}n(t)=0, \qquad (6)$$

$$n'''(t)=\frac{-2(t-1)^2}{t^3}<0. \qquad (7)$$

(2) follows from (3)-(7) and the fact

$$\lim_{t\to1^+}m(t)=1.$$

3. Main Result

The main result of this paper is the following theorem.

Theorem 3.1. Let $0<\alpha<1$. Then for any $a,b>0$ with $a\ne b$, we have

$$J_{\frac{\alpha-1}{2}}(a,b)<A^{\alpha}(a,b)G^{1-\alpha}(a,b)<J_{\frac{3\alpha-1}{2}}(a,b). \qquad (8)$$

Moreover, the bounds $J_{\frac{\alpha-1}{2}}(a,b)$ and $J_{\frac{3\alpha-1}{2}}(a,b)$ are optimal.

Proof. It is no loss of generality to assume that $a>b$. Let $t^2=\frac{a}{b}>1$, $p\in\left\{\frac{\alpha-1}{2},\frac{3\alpha-1}{2}\right\}$ and

$$f_1(t)=\frac{J_p(t^2,1)}{A^{\alpha}(t^2,1)G^{1-\alpha}(t^2,1)},$$

then

$$\frac{f_1'(t)}{f_1(t)}=\left[\log f_1(t)\right]'=\frac{g_1(t)}{t(t^2+1)(t^{2p}-1)(t^{2p+2}-1)}, \qquad (9)$$

where

$$\begin{aligned}g_1(t)&=(1-\alpha)t^{4p+4}+(\alpha+1)t^{4p+2}+(\alpha-2p-1)t^{2p+4}\\&\quad+(2p+1-\alpha)t^{2p}-(\alpha+1)t^2+\alpha-1\\&=(1-\alpha)x^{2p+2}+(\alpha+1)x^{2p+1}+(\alpha-2p-1)x^{p+2}\\&\quad+(2p+1-\alpha)x^{p}-(\alpha+1)x+\alpha-1\\&=h_1(x),\end{aligned} \qquad (10)$$

where $x=t^2>1$. Simple calculations lead to

$$\lim_{x\to1^+}h_1(x)=0, \qquad (11)$$

$$\begin{aligned}h_1'(x)&=2(p+1)(1-\alpha)x^{2p+1}+(2p+1)(\alpha+1)x^{2p}\\&\quad+(p+2)(\alpha-2p-1)x^{p+1}\\&\quad+p(2p+1-\alpha)x^{p-1}-\alpha-1,\end{aligned} \qquad (12)$$

$$\lim_{t\to 1^+} h_1'(x)=0, \tag{13}$$

$h_1''(x)=x^{p-2}h_2(x),$
where

$$\begin{aligned} h_2(x)&=2(2p+1)(p+1)(1-\alpha)x^{p+2}\\ &+2p(2p+1)(\alpha+1)x^{p+1}\\ &+(p+1)(p+2)(\alpha-2p-1)x^2\\ &+(p-1)p(2p+1-\alpha), \end{aligned} \tag{14}$$

$$\lim_{x\to 1^+} h_2(x)=0, \tag{15}$$

$h_2'(x)=2(p+1)xh_3(x),$
where

$$\begin{aligned} h_3(x)&=(p+2)(2p+1)(1-\alpha)x^p\\ &+p(2p+1)(\alpha+1)x^{p-1}\\ &+(p+2)(\alpha-2p-1), \end{aligned} \tag{16}$$

$$\lim_{x\to 1^+} h_3(x)=p(2p-3\alpha+1) \tag{17}$$

$$h_3''(x)=p(2p+1)x^{p-2}h_4(x) \tag{18}$$

where

$$h_4(x)=(p+2)(1-\alpha)x+(p-1)(\alpha+1) \tag{19}$$

$$\lim_{x\to 1^+} h_4(x)=2p-3\alpha+1, \tag{20}$$

$$h_4'(x)=(p+2)(1-\alpha). \tag{21}$$

We now distinguish between two cases.

Case 1. $p=\frac{3\alpha-1}{2}$. We first consider the case $\alpha=\frac{1}{3}$ since in this case the one-parameter mean $J_p(a,b)$ has different expression from others. The result

$$A^{\frac{1}{3}}(t,1)G^{\frac{2}{3}}(t,1)<J_0(t,1)$$

follows from Lemma 2.1 since

$$A^{\frac{1}{3}}(t,1)G^{\frac{2}{3}}(t,1)/J_0(t,1)=m^3(t,1)<1,$$

In the following we assume $\alpha\neq\frac{1}{3}$.

From (21) we see that $h_4'(x)>0$ for $x>1$, which implies $h_4(x)$ is strictly increasing for $x>1$. From (20) we know that $h_4(x)>0$ for all $x>1$. (18) implies

$$h_3'(x)\begin{cases}<0, & \text{for } 0<\alpha<\frac{1}{3},\\ >0, & \text{for } \frac{1}{3}<\alpha<1,\end{cases}$$

from which we know $h_3(x)$ is strictly decreasing for $\alpha\in\left(0,\frac{1}{3}\right)$ and strictly increasing for $\alpha\in\left(\frac{1}{3},1\right)$. This result together with (17) implies $h_3(x)<0$ for $\alpha\in\left(0,\frac{1}{3}\right)$ and $h_3(x)>0$ for $\alpha\in\left(\frac{1}{3},1\right)$. The same reasoning applies to $h_2(x),h_1''(x),h_1'(x),h_1(x)$ as well, and using (15), (14), (12), (11), (9) and (8), we know $g_1(t)<0$ for $\alpha\in\left(0,\frac{1}{3}\right)$ and $g_1(t)>0$ for $\alpha\in\left(\frac{1}{3},1\right)$. (8) implies $f_1'(t)>0$ for all $t>1$. Thus $f_1(t)$ is strictly increasing for $t>1$, which together with

$$\lim_{t\to 1^+} f_1(t)=1 \tag{22}$$

implies right-hand side inequality of (8).

Case 2. $p=\frac{\alpha-1}{2}$. From (21) we know $h_4'(x)>0$ for all $x>1$, which implies that $h_4(x)$ is strictly increasing for $x>1$. By (20) one has $h_4(1^+)=-2\alpha<0$, and by (19) one has

$$\lim_{x\to+\infty} h_4(x)=+\infty.$$

Thus there exists $\xi_1>1$ such that $h_4(x)<0$ for $x\in(1,\xi_1)$ and $h_4(x)>0$ for $x\in(\xi_1,+\infty)$. (18) implies $h_3'(x)>0$ for $x\in(1,\xi_1)$ and $h_3'(x)<0$ for $x\in(\xi_1,+\infty)$. Thus $h_3(x)$ is strictly increasing for $x\in(1,\xi_1)$ and strictly decreasing for $x\in(\xi_1,+\infty)$. By (17) $h_3(1^+)>0$ and by

$$\lim_{x\to+\infty} h_3(x)=0$$

we know $h_3(x)>0$ for all $x>1$. The same reasoning applies to $h_2'(x),h_2(x),h_1'(x),h_1(x)$ and $g_1(t)$ as well, and applying (9)-(16), we have $g_1(t)>0$ for all $t>1$. (9) implies $f_1'(t)<0$, thus $f_1(t)$ is strictly decreasing for $t>1$. The left-hand side inequality of (8) follows from (22).

Next we prove that the bounds $J_{\frac{3\alpha-1}{2}}(a,b)$ and $J_{\frac{\alpha-1}{2}}(a,b)$ are optimal.

For any $\varepsilon>0$ and $t>0$ sufficiently small,

$$\begin{aligned} &\log\frac{J_{\frac{3\alpha-1}{2}-\varepsilon}(1+t,1)}{A^{\alpha}(1+t,1)G^{1-\alpha}(1+t,1)}\\ &=\log\frac{4t+(3\alpha-2\varepsilon-1)t^2}{4t+(3\alpha-2\varepsilon-3)t^2}-\log\left(\frac{t+2}{2}\right)^{\alpha}(1+t)^{\frac{1-\alpha}{2}}\\ &=2t^2-\alpha\left(\frac{t}{2}-\frac{t^2}{8}\right)-\frac{1-\alpha}{2}\left(t-\frac{t^2}{2}\right)\\ &=\frac{(18-\alpha)t-4}{8}t+o(t)<0. \end{aligned}$$

This implies

$$J_{\frac{3\alpha-1}{2}-\varepsilon}(t,1) < A^{\alpha}(t,1)G^{1-\alpha}(t,1)$$

for t sufficiently close to 1.

For any $\varepsilon > 0$, since

$$\lim_{t\to+\infty}\frac{J_{\frac{\alpha-1}{2}+\varepsilon}(t,1)}{A^{\alpha}(t,1)G^{1-\alpha}(t,1)} = +\infty,$$

then there exists $T > 1$ such that

$$J_{\frac{\alpha-1}{2}+\varepsilon}(t,1) > A^{\alpha}(t,1)G^{1-\alpha}(t,1)$$

For $t > T$.

4. Acknowledgements

This research is supported by NSF of Hebei Province (No. A2011201011).

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