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On Harmonic Index and Diameter of Graphs*

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ABSTRACT

The harmonic index of a graph G is defined as $H(G)=\sum_{uv\in E(G)}\frac{2}{d(u)+d(v)}$, where $d(u)$ denotes the degree of a vertex u in G. It has been found that the harmonic index correlates well with the Randi c′ index and with the π-electronic energy of benzenoid hydrocarbons. In this work, we give several relations between the harmonic index and diameter of graphs.

Keywords: Harmonic Index; Diameter

1. Introduction

All graphs considered in the following will be simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of graph G is the number of its vertices. The leaf of a graph is a vertex with degree one. For undefined terminology and notations we refer the reader to [1]. For a graph G, the harmonic index $H(G)$ is defined as

$$H(G)=\sum_{uv\in E(G)}\frac{2}{d(u)+d(v)}.$$

It has been found that the harmonic index correlates well with the Randić index [2,3] and the π-electronic energy of benzenoid hydrocarbons [4,5]. Favaron *et al.* [6] considered the relation between harmonic index and the eigenvalues of graphs. Zhong [7] found the minimum and maximum values of the harmonic index for connected graphs and trees, and characterized the corresponding extremal graphs. Recently, Wu *et al.* [8] gave the minimum value of the Harmonic index among the graphs with the minimum degree at least two. In this work, we shall give some relations between the harmonic index and diameter of graphs.

2. Main Results

First, we shall give two sharp upper bounds of the relationship involving the harmonic index and diameter of connected graphs. In [7], Zhong gave the following result:

Lemma 2.1 ([7]) Let G be a graph with order n, then $H(G)\le\frac{n}{2}$, where the equality holds if and only if G is a regular graph.

Since the complete graph K_n is a regular graph with diameter one, we have:

Theorem 2.2 Let G be a connected graph with order $n(\ge 4)$ diameter $D(T)$, then

$$H(T)-D(T)\le\frac{n}{2}-1,\ \frac{H(T)}{D(T)}\le\frac{n}{2}$$

where equalities hold if and only if G is the complete graph K_n.

An edge x_1x_2 is called local maximum if its weight $\frac{2}{d(x_1)+d(x_2)}$ is maximum in its neighborhood, *i.e.*,

$$\frac{2}{d(x_1)+d(x_2)}\ge\frac{2}{d(x_i)+d(u)}$$

for any edge x_iu for $i=1,2$.

Lemma 2.3 Let x_1x_2 be a local maximum edge in graph G, then

$$H(G)-H(G-x_1x_2)>0.$$

Proof. We have

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$$H(G)-H(G-x_1x_2)=\frac{2}{d(x_1)+d(x_2)}+\sum_{u\in N(x_1)\setminus\{x_2\}}\left(\frac{2}{d(x_1)+d(u)}-\frac{2}{d(x_1)+d(u)-1}\right)$$
$$+\sum_{v\in N(x_2)\setminus\{x_1\}}\left(\frac{2}{d(x_2)+d(v)}-\frac{2}{d(x_2)+d(v)-1}\right)\ge\frac{2}{d(x_1)+d(x_2)}+(d(x_1)-1)\left(\frac{2}{d(x_1)+d(x_2)}-\frac{2}{d(x_1)+d(x_2)-1}\right)$$
$$+(d(x_2)-1)\left(\frac{2}{d(x_1)+d(x_2)}-\frac{2}{d(x_1)+d(x_2)-1}\right)=\frac{2}{(d(x_1)+d(x_2)-1)(d(x_1)+d(x_2))}>0,$$

where $N(x_i)$ denotes the vertex set adjoining to x_i for $i=1,2$.

If x_1x_2 is a leaf of G, *i.e.*, at least one of $\{x_1,x_2\}$ has degree one, we can see that it is a local maximum edge. Thus, by Lemma 2.3,

Corollary 2.4 If x_1x_2 is a leaf in graph G, then

$$H(G)-H(G-x_1x_2)>0.$$

Theorem 2.5 Let T be a tree with order $n(\ge 4)$ diameter $D(T)$, then

$$H(T)-D(T)\ge\frac{5}{6}-\frac{n}{2},\ \frac{H(T)}{D(T)}\ge\frac{1}{2}+\frac{1}{3(n-1)}$$

where equalities hold if and only if T is a path P_n.

Proof. If T is a path, we have $H(T)=\frac{n}{2}-\frac{1}{6}$ and $D(T)=n-1$. It is obvious that both equalities hold. Now we assume that T is not a path, then $D(T)\le n-2$ and there are at least three pendent vertices in T. Assume $P=u_0u_1\cdots u_D$ be the longest path in T. Then at least one pendent vertex, say v_1, is not contained in P. Now we start an operation on T, *i.e.*, we continually delete pendent vertices which are not contained in P until the resulting tree is P. Assume $v_1,\cdots,v_k$ are the vertices in the order they were deleted, we have

$$H(T)>H(T-v_1)>\cdots>H\left(T-\bigcup_{i=1}^{k}v_i\right)=H(P)=\frac{D}{2}+\frac{1}{3}$$

by Corollary 2.4 and

$D(T)=D(T-v_1)=\cdots=D\left(T-\bigcup_{i=1}^{k}v_i\right)=D$. Thus, we have

$$H(T)-D(T)>H(P)-D(P)$$
$$\ge\frac{5}{6}-\frac{D+1}{2}\ge\frac{5}{6}-\frac{n-1}{2}>\frac{5}{6}-\frac{n}{2}$$

and

$$\frac{H(T)}{D(T)}>\frac{H(P)}{D(P)}=\frac{\frac{D+1}{2}-\frac{5}{6}}{D}>\frac{\frac{n}{2}-\frac{1}{6}}{n-1}.$$

This result seems true for any connected graph with order n and we propose as a conjecture as follows:

Conjecture 2.6 Let G be a connected graph with order $n(\ge 4)$ diameter $D(G)$, then

$$H(G)-D(G)\ge\frac{5}{6}-\frac{n}{2},\ \frac{H(G)}{D(G)}\ge\frac{1}{2}+\frac{1}{3(n-1)}.$$

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