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The Global and Pullback Attractors for a Strongly Damped Wave Equation with Delays*

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ABSTRACT

In this paper, we study the global and pullback attractors for a strongly damped wave equation with delays when the force term belongs to different space. The results following from the solution generate a compact set.

Keywords: Strongly Damped; Pullback Attractor; Global Attractor; Delays

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$, we study the following initial boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \alpha \dfrac{\partial u}{\partial t} - \beta\Delta \dfrac{\partial u}{\partial t} - \Delta u + g(u) = f(x) + h(t, u_t), \\ \qquad t > \tau, \\ u|_{\partial\Omega} = 0, \\ \qquad t \ge \tau - r, \\ u(x,t) = \phi(x, t-\tau), \quad \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial \phi}{\partial t}(x, t-\tau), \\ \qquad x \in \Omega, t \in [\tau - r, \tau]. \end{cases} \tag{1.1}$$

where $f + h(t, u_t)$ is the source intensity which may depend on the history of the solution, α, β are the positive constants, ϕ is the initial value on the interval $[\tau - r, \tau]$ where $r > 0$, and u_t is defined for $\theta \in [-r, 0]$ as $u_t(\theta) = u(t + \theta)$. The assumption on $g(u)$ and $f(x)$ will be specified later.

It is well known that the long time behavior of many dynamical system generated by evolution equations can be described naturally in term of attractors of corresponding semigroups. Attractor is a basic concept in the study of the asymptotic behavior of solutions for the nonlinear evolution equations with various dissipation. There have been many researches on the long-time behavior of solutions to the nonlinear damped wave equations with delays. The existence of random attractors has been investigated by many authors, see, e.g., [1-4]. A new type of attractor, called a pullback attractor, was proposed and investigated for non-autonomous or these random dynamical systems. The pullback attractor describing this attractors to a component subset for a fixed parameter value is achieved by starting progressively earlier in time, that is, at parameter values that are carried forward to the fixed value. see [5-20]. However, to our knowledge, in the case of functional differential equations of second order in time, there is only partial results.

Recently, In [5], some results on pullback and forward attractor for the following strongly damped wave equation with delays

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \alpha \dfrac{\partial u}{\partial t} - \Delta u = f(x) + h(t, u_t), \\ \qquad t > \tau, \\ u|_{\partial\Omega} = 0, \\ \qquad t \ge \tau - r, \\ u(x,t) = \phi(x, t-\tau), \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial \phi}{\partial t}(x, t-\tau), \\ \qquad x \in \Omega, t \in [\tau - r, \tau]. \end{cases}$$

have been analyzed.

In this work, first, we apply the means in [3] to provide the existence of global attractor, for the dynamical system generated by the initial value problem

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(1.1). The key is to deal with the nonlinear terms and the delay term is difficult to be handled, so we aimed at showing that it is dissipative and the solution is bounded and continuous with respect to initial value. Hence we can discover the global attractor. Then, we aim to obtain the pullback attractor. The technology we use is introduced in [1], that is, we divide the semigroup into two: the one is asymptotically close to 0, while the other is uniformly compact, so we can get the pullback attractor.

Now, we state the general assumptions for problem (1.1) on $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\times C_H\to H$.

Let $G(u)=\int_0^u g(s)\mathrm{d}s$, then there exist positive constants $C_i\left(i=1,2,\cdots,5\right)$ such that the followings hold true

(G$_1$). $g(0)\in H$;

(G$_2$). $\left|g'(0)\right|\le C_1$;

(G$_3$). $\left|g''(z)\right|\le C_2\left(1+|z|^{\alpha}\right),\forall z\in\mathbb{R}$;

(G$_4$). $\liminf\limits_{|z|\to\infty} g(z)/z\ge 0$;

(G$_5$). $\left(g(z)-g(0)\right)z\ge C_4z^2,\forall z\in\mathbb{R}$;

(G$_6$). $\liminf\limits_{|z|\to\infty}\dfrac{g(z)z-C_3G(z)}{z^2}\ge 0$;

(G$_7$). $-C_5\le g'(z)\le 0,\forall z\in\mathbb{R}$.

For any $u\in V$, set $J(u)=\int_{\mathbb{R}^3}G(u)\mathrm{d}x$, by $\mathrm{G}_1-\mathrm{G}_7$, there are $C_6\ge 0, C_\gamma\ge 0$ and $\rho_\gamma\ge 0$, for any $\gamma>0$, we have

$$\left(g(u)u,u\right)-C_6J(u)\ge-\gamma\|u\|^2-C_\gamma;$$

$$J(u)\ge-\gamma\|u\|^2-C_\gamma;$$

$$\left(g(u)u,u\right)-\rho_\gamma\|u\|^2\ge-\gamma\|\nabla u\|^2-C_\gamma.$$

H$_1$. $\forall\xi\in C_H, t\in\mathbb{R}\to h(t,\xi)\in H$ is continuous;

H$_2$. $\forall t\in\mathbb{R}, h(t,0)=0$;

H$_3$. $\exists L_h>0$ such that $\forall t\in\mathbb{R},\forall\xi,\eta\in C_H$

$$\left|h(t,\xi)-h(t,\eta)\right|\le L_h\|\xi-\eta\|_{C_H};$$

H$_4$. $\exists m_0\ge 0, C_h>0$ such that

$$\forall m\in[0,m_0],\tau\le t,u,v\in C^0\left([\tau-r,t];H\right)$$

$$\int_\tau^t e_{ms}\left|h(s,u_s)-h(s,v_s)\right|^2\mathrm{d}s\le C_h^2\int_{\tau-r}^t e_{ms}\left|u(s)-v(s)\right|^2\mathrm{d}s;$$

H$_5$. $h\in C^1\left(\mathbb{R}\times C_H;H\right)$, and there exists $C>0$ such that, for any $(t,\xi)\in\mathbb{R}\times C_H$, the Frechet derivative $\delta h(t,\xi)\in\mathcal{L}\left(\mathbb{R}\times C_H;H\right)$ satisfies

$$\left\|\delta h(t,\xi)\right\|_{\mathcal{L}(\mathbb{R}\times C_H;H)}\le C\left(1+\|\xi\|_{C_H}\right).$$

The rest of this paper is organized as follows. In Section 2, we introduce basic concepts concerning global and pullback attractor. In Section 3, we obtain the existence of the global attractor. In Section 4, we obtain the existence of the pullback attractor.

2. Preliminaries

In this section,firstly, we recall some basic concepts about the global attractor.

Definition 2.1 ([3]) *Let* X *be a Banach space and* $\{S(t)\}_{t\ge0}$ *be a family of operators on* X. *We say that* $\{S(t)\}_{t\ge0}$ *is norm-to-weak continuous semigroup on* X, *if* $\{S(t)\}_{t\ge0}$ *satisfies*:

[1)] $S(0)=Id$ (identify);

[2)] $S(t)S(s)=S(t+s)$;

[3)] $S(t_n)x_n\rightharpoonup S(t)x$ *if* $t_n\to t$ *and* $x_n\to x$ *in* X.

***Remark*:** The strong continuous semigroup and the weak semigroup are both the norm-to-weak continuous

Definition 2.2 ([3]) *The semigroup* $S(t)_{t\ge0}$ *is called satisfying Condition (C) in* X *if and only if for any bounded set* B *of* X *and for any* $\epsilon>0$, *there exist a positive constant* t_B *and a finite dimensional subspace* X_1 *of X, such that* $\left\{PS(t)x\,\middle|\,x\in B,t\ge t_B\right\}$ *is bounded and*

$$\left\|(I-P)S(t)x\right\|_X<\epsilon\quad\text{for any } t\ge t_B \text{ and } x\in B,$$

where $P:X\to X_1$ is the canonical projector.

Lemma 2.1 ([3]) *Let* X *be a Banach space and* $\{S(t)\}_{t\ge0}$ *be a norm-to-weak continuous semigroup on* X. *Then* $\{S(t)\}_{t\ge0}$ *has a global attractor in* X *provided that the following conditions hold:*

1) $\{S(t)\}_{t\ge0}$ has a bounded absorbing set B_0 in X;

2) $\{S(t)\}_{t\ge0}$ satisfies Condition (C) in X.

Then, we state the *concepts* and some result about the process and the pullback attractor.

Instead of a family of *the* one-parameter map $S(t)$, we need to use a two-parameter semigroup or process $U(t,\tau)$ on the complete metric space X, $u(t,\tau)\psi$ denotes the value of the solution at time t which was equal to the initial value ψ at time τ.

The semigroup *property* is replaced by the process composition property

$$U(t,\tau)U(\tau,r)=U(t,r)\quad\text{for all } t\ge\tau\ge r,$$

and, obviously, the initial condition implies $U(\tau,\tau)=Id$.

Definition 2.3 *Let* U *be the two-parameter semigroup or process on the complete metric space* X. *A family of compact set* $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ *is said to be a pullback attractor for* U *if, for all* $\tau\in\mathbb{R}$, *it satisfies*

[1)] $U(t,\tau)\mathcal{A}(\tau)=\mathcal{A}(\tau)$ for all $t\ge\tau$, and

[2)] $\lim\limits_{s\to\infty}\mathrm{dist}_X\left(U(t,t-s)\mathcal{D},\mathcal{A}(t)\right)=0$, for all bounded $\mathcal{D}\subset X$, and all $t\in\mathbb{R}$.

Definition 2.4 *The family* $\{B(t)\}_{t\in\mathbb{R}}$ *is said to be*

1) pullback absorbing with respect to the process U, if for all $t\in\mathbb{R}$ *and* all bounded $D\subset X$, there exists $T_D(t)>0$ such that $U(t,t-s)D\subset B(t)$ for all $s\ge T_D(t)$;

2) pullback *attracting* with respect to the process U, if for all $t\in\mathbb{R}$, all bounded $D\subset X$, and all $\epsilon>0$, there exists $T_{\epsilon,D}(t)>0$ such that for all $s\ge T_{\epsilon,D}(t)$

$$\mathrm{dist}_X\left(U(t,t-s)D,B(t)\right)<\epsilon;$$

3) pullback uniformly absorbing (respectively uniformly attracting) if $T_D(t)$ in pact (a) (respectively $T_{\epsilon,D}(t)$ in part (b)) does not depend on the time t.

***Theorem* 2.1** *Let* $U(t,\tau)$ *be a two-parameter process, and suppose* $U(t,\tau):X\to X$ *is continuous for all* $t\ge\tau$. *If there exists a family of compact pullback attracting sets* $\{B(t)\}_{t\in\mathbb{R}}$, *then there exists a pullback attractor* $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, *such that* $\mathcal{A}(t)\subset\{B(t)\}$ *for all* $t\in\mathbb{R}$, *and which is given by*

$$\mathcal{A}=\overline{\bigcup_{D\subset X}\Lambda_D(t)},\ \text{where}\ \Lambda_D(t)=\bigcap_{n\in\mathbb{N}}\overline{\bigcup_{s\ge n}U(t,t-s)D}.$$

We set $E=V\times H$, where $V=H_0^1(\Omega),H=L^2(\Omega)$, which are Hilbert spaces for the usual inner product and associated norms. we denote by λ_1 the first eigenvalue of $-\Delta$ in V.

Our problem can be written as a second-order differential equation in H:

$$\begin{cases}u''+\alpha u'-\beta\Delta u'-\Delta u+g(u)=f(x)+h(t,u_t),\ t>\tau,\\ u(t)=\phi(t-\tau),\quad u'(t)=\phi'(t-\tau),\quad t\in[\tau-r,\tau].\end{cases}\tag{2.1}$$

3. Existence of the Global Attractor

In this section, our objection is to show that the well-posed of the solution and the existence of global attractor for the initial boundary value problem (1.1), we assume that $f\in L^2(\Omega)$.

Let $0<\varepsilon\le\min\left\{\frac{1}{\beta},\frac{\alpha}{4},\frac{\lambda_1}{2\alpha}\right\}$ and $\alpha>0,\beta>0$, then by the transformation $v=u'+\varepsilon u$. The initial boundary value problem (2.1) is equivalent to

$$\begin{aligned}&v'+(\alpha-\varepsilon)v+\varepsilon(\varepsilon-\alpha)u\\&-\beta\Delta v+(\beta\varepsilon-1)\Delta u+g(u)\\&=f(x)+h(t,u_t),\ t>\tau,\end{aligned}\tag{3.1}$$

with the initial value conditions

$$v(t)=\phi'(t-\tau)+\varepsilon\phi(t-\tau),\qquad t\in[\tau-r,\tau].$$

Theorem **3.1** *Assume that the hypotheses on* g *and* h *hold for all* $(u,u')^{\mathrm{T}}\in E$ *and* $f\in L^2(\Omega)$, α,β *are the positive constants. Then the initial boundary value problem* (3.1) *has the unique solution* $(u,v)^{\mathrm{T}}\in E$ *for all* $t>\tau$.

Proof. Taking the inner product of the Equation (3.1) with v in H, we find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v|^2+(\alpha-\varepsilon)(v,v)+\varepsilon(\varepsilon-\alpha)(u,v)+\beta\|v\|^2\\&+\frac{1-\beta\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\varepsilon(1-\beta\varepsilon)\|u\|^2+(g(u),v)\\&=(f(x),v)+(h(t,u_t),v).\end{aligned}\tag{3.2}$$

Since $v=u'(t)+\varepsilon u$ and $0<\varepsilon\le\min\left\{\frac{1}{2\beta},\frac{\alpha}{4},\frac{\lambda_1}{2\alpha}\right\}$, we deal with the terms in (3.2) one by one as follows

$$(\alpha-\varepsilon)(v,v)=(\alpha-\varepsilon)|v|^2\ge\frac{3\alpha}{4}|v|^2;\tag{3.3}$$

$$\begin{aligned}&\varepsilon(\varepsilon-\alpha)(u,v)\ge\frac{\varepsilon(\varepsilon-\alpha)}{\sqrt{\lambda_1}}\|u\||v|\ge-\frac{\alpha\varepsilon}{\sqrt{\lambda_1}}\|u\||v|\\&\ge-\frac{\varepsilon\alpha^2}{\lambda_1}|v|^2-\frac{\varepsilon}{4}\|u\|^2\ge-\frac{\varepsilon}{4}\|u\|^2-\frac{\alpha}{2}|v|^2;\end{aligned}\tag{3.4}$$

$$\begin{aligned}(g(u),v)&=\frac{\mathrm{d}}{\mathrm{d}t}J(u)+\varepsilon(g(u),u)\\&\ge\frac{\mathrm{d}}{\mathrm{d}t}J(u)+\varepsilon C_6J(u)-\varepsilon\gamma\|u\|^2-C_\gamma\varepsilon;\end{aligned}\tag{3.5}$$

$$(h(t,u_t),v)\le\frac{1}{\alpha}|h|^2+\frac{\alpha}{4}|v|^2;\tag{3.6}$$

$$(f,v)\le\frac{1}{\alpha}|f|^2+\frac{\alpha}{4}|v|^2\tag{3.7}$$

By (3.3)-(3.7), it follows from that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)+\left(\frac{\alpha}{2}-\frac{\varepsilon\alpha^2}{\lambda_1}+\beta\lambda_1\right)|v|^2\\&+2\left(\varepsilon(1-\beta\varepsilon)-\frac{\varepsilon}{4}-\varepsilon\gamma\right)\|u\|^2+2\varepsilon C_6J(u)\\&\le\frac{2}{\alpha}|h|^2+\frac{2}{\alpha}|f|^2+2\varepsilon C_\gamma.\end{aligned}$$

Since $\varepsilon\le\min\left\{\frac{1}{\beta},\frac{\alpha}{4},\frac{\lambda_1}{2\alpha}\right\}$ and $0<\gamma<\frac{1}{4}-\frac{\beta\varepsilon}{2}$, this will imply $2\left(\varepsilon(1-\beta\varepsilon)-\frac{\varepsilon}{4}-\varepsilon\gamma\right)>\varepsilon(1-\beta\varepsilon)$, then we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)\\&+2\beta\lambda_1|v|^2+\varepsilon(1-\beta\varepsilon)\|u\|^2+2\varepsilon C_6J(u)\\&\le\frac{2}{\alpha}|h|^2+\frac{2}{\alpha}|f|^2+2\varepsilon C_\gamma.\end{aligned}\tag{3.8}$$

Set $C_0=\min\{2\beta\lambda_1,\varepsilon,2\varepsilon C_6\}$, then (3.8) can be written as following

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)$$
$$+C_0\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)$$
$$\le \frac{2}{\alpha}|h|^2+\frac{2}{\alpha}|f|^2+2\varepsilon C_\gamma.$$

As our assumptions ensure that $-C_0+\frac{2C_h^2\lambda_1^{-1}}{\alpha(1-\beta\varepsilon-2\gamma)}<0$, then we can choose $m\in(0,m_0)$ small enough such that $m-C_0+\frac{2C_h^2\lambda_1^{-1}}{\alpha(1-\beta\varepsilon-2\gamma)}<0$. For this choice, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{mt}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)$$
$$=m\mathrm{e}^{mt}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)$$
$$+\mathrm{e}^{mt}\frac{\mathrm{d}}{\mathrm{d}t}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right).$$

Hence, we can get the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{mt}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)$$
$$\le(m-C_0)\mathrm{e}^{mt}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)$$
$$+\frac{2}{\alpha}\mathrm{e}^{mt}|h|^2+\frac{2}{\alpha}\mathrm{e}^{mt}|f|^2+2\varepsilon C_\gamma\mathrm{e}^{mt}.$$

By integrating over the interval $[\tau,t]$, we deduce

$$\begin{aligned}
&\mathrm{e}^{mt}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)\\
&\le \mathrm{e}^{m\tau}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)+(m-C_0)\int_\tau^t\mathrm{e}^{ms}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)\mathrm{d}s\\
&\quad+\frac{2}{\alpha}\int_\tau^t\mathrm{e}^{ms}|h|^2\,\mathrm{d}s+\frac{2}{\alpha}\int_\tau^t\mathrm{e}^{ms}|f|^2\,\mathrm{d}s+2\varepsilon C_\gamma\int_\tau^t\mathrm{e}^{ms}\mathrm{d}s\\
&\le \mathrm{e}^{m\tau}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)+(m-C_0)\int_\tau^t\mathrm{e}^{ms}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)\mathrm{d}s\\
&\quad+\frac{2C_h^2\lambda_1^{-1}}{\alpha}\int_{\tau-r}^t|u|^2\,\mathrm{d}s+\frac{2}{m\alpha}|f|^2\left(\mathrm{e}^{mt}-\mathrm{e}^{m\tau}\right)+\frac{2\varepsilon C_\gamma}{m}\left(\mathrm{e}^{mt}-\mathrm{e}^{m\tau}\right)\\
&= \mathrm{e}^{m\tau}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)+(m-C_0)\int_\tau^t\mathrm{e}^{ms}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)\mathrm{d}s\\
&\quad+\frac{2C_h^2\lambda_1^{-1}}{\alpha}\left(\int_\tau^t\mathrm{e}^{ms}|u|^2\,\mathrm{d}s+\int_{\tau-r}^\tau\mathrm{e}^{ms}|u|^2\,\mathrm{d}s\right)+\frac{2}{m\alpha}|f|^2\left(\mathrm{e}^{mt}-\mathrm{e}^{m\tau}\right)+\frac{2\varepsilon C_\gamma}{m}\left(\mathrm{e}^{mt}-\mathrm{e}^{m\tau}\right).
\end{aligned}\tag{3.9}$$

Since

$$J(u)\ge-\gamma\|u\|^2-C_\gamma,$$

So we can have

$$\begin{aligned}
&|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\\
&=|v|^2+(1-\beta\varepsilon-2\gamma)\|u\|^2+2J(u)+2\gamma\|u\|^2\\
&\ge(1-\beta\varepsilon-2\gamma)\|u\|^2+|v|^2-2C_\gamma.
\end{aligned}\tag{3.10}$$

Noticing $0<\gamma<\frac{1}{4}-\frac{\beta\varepsilon}{2}$, we obtain

$$\|u\|^2\le\frac{|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)}{1-\beta\varepsilon-2\gamma}+\frac{2C_\gamma}{1-\beta\varepsilon-2\gamma}.\tag{3.11}$$

In the Bounded set $D\subset C_{V,H}$, for any $u\in D$, there exists a constant d such that

$$\|u\|^2+|v|^2\le d^2;\tag{3.12}$$

$$|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\le d^2.\tag{3.13}$$

(3.10)-(3.13) means that

$$\begin{aligned}
&\frac{2C_h^2\lambda_1^{-1}}{\alpha}\int_\tau^t\mathrm{e}^{ms}|u|^2\,\mathrm{d}s\\
&\le\frac{2C_h^2\lambda_1^{-1}}{\alpha(1-\beta\varepsilon-2\gamma)}\int_\tau^t\mathrm{e}^{ms}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)\mathrm{d}s\\
&\quad+\frac{4C_h^2\lambda_1^{-1}C_\gamma}{m\alpha(1-\beta\varepsilon-2\gamma)}\left(\mathrm{e}^{mt}-\mathrm{e}^{m\tau}\right);
\end{aligned}\tag{3.14}$$

$$\begin{aligned}
&\frac{2C_h^2\lambda_1^{-1}}{\alpha}\int_{\tau-r}^\tau\mathrm{e}^{ms}|u|^2\,\mathrm{d}s\\
&\le\frac{2C_h^2\lambda_1^{-1}}{\alpha(1-\beta\varepsilon-2\gamma)}\int_{\tau-r}^\tau\mathrm{e}^{ms}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)\mathrm{d}s\\
&\quad+\frac{4C_h^2\lambda_1^{-1}C_\gamma}{m\alpha(1-\beta\varepsilon-2\gamma)}\left(\mathrm{e}^{m\tau}-\mathrm{e}^{m(\tau-r)}\right)\\
&\le\frac{2C_h^2\lambda_1^{-1}rd^2}{\alpha(1-\beta\varepsilon-2\gamma)}\mathrm{e}^{m\tau}+\frac{4C_h^2\lambda_1^{-1}C_\gamma}{m\alpha(1-\beta\varepsilon-2\gamma)}\left(\mathrm{e}^{m\tau}-\mathrm{e}^{m(\tau-r)}\right).
\end{aligned}\tag{3.15}$$

Hence, by (3.12)-(3.14) and the choice of

$m-C_0+\frac{2C_h^2\lambda_1^{-1}}{\alpha(1-\beta\varepsilon-2\gamma)}<0$, (3.9) can be rewritten

$$\begin{aligned}
&\mathrm{e}^{mt}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)\\
&\le \mathrm{e}^{m\tau}d^2+\left(m-C_0+\frac{2C_h^2\lambda_1^{-1}}{\alpha(1-\beta\varepsilon-2\gamma)}\right)\\
&\quad\times\int_\tau^t \mathrm{e}^{ms}\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)\mathrm{d}s\\
&\quad+\frac{2}{m\alpha}|f|^2\left(\mathrm{e}^{mt}-\mathrm{e}^{m\tau}\right)+\frac{4C_h^2\lambda_1^{-1}C_\gamma}{m\alpha(1-\beta\varepsilon-2\gamma)}\left(\mathrm{e}^{m\tau}-\mathrm{e}^{m(\tau-r)}\right)\\
&\quad+\frac{4C_h^2\lambda_1^{-1}C_\gamma}{m\alpha(1-\beta\varepsilon-2\gamma)}\left(\mathrm{e}^{mt}-\mathrm{e}^{m\tau}\right)+\frac{2C_h^2\lambda_1^{-1}rd^2}{\alpha(1-\beta\varepsilon-2\gamma)}\mathrm{e}^{m\tau}\\
&\le \mathrm{e}^{m\tau}d^2\left(1+\frac{2C_h^2\lambda_1^{-1}r}{\alpha(1-\beta\varepsilon-2\gamma)}\right)+\frac{2}{m\alpha}|f|^2\left(\mathrm{e}^{mt}-\mathrm{e}^{m\tau}\right)\\
&\quad+\frac{4C_h^2\lambda_1^{-1}C_\gamma}{m\alpha(1-\beta\varepsilon-2\gamma)}\left(\mathrm{e}^{mt}-\mathrm{e}^{m\tau}\right)\\
&\quad+\frac{4C_h^2\lambda_1^{-1}C_\gamma}{m\alpha(1-\beta\varepsilon-2\gamma)}\left(\mathrm{e}^{m\tau}-\mathrm{e}^{m(\tau-r)}\right).
\end{aligned}\tag{3.16}$$

So we can get by (3.16)

$$\begin{aligned}
&\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)\\
&\le \mathrm{e}^{m\tau}\mathrm{e}^{-mt}d^2\left(1+\frac{2C_h^2\lambda_1^{-1}r}{\alpha(1-\beta\varepsilon-2\gamma)}\right)\\
&\quad+\frac{2}{m\alpha}|f|^2\left(1-\mathrm{e}^{m\tau}\mathrm{e}^{-mt}\right)\\
&\quad+\frac{4C_h^2\lambda_1^{-1}C_\gamma}{m\alpha(1-\beta\varepsilon-2\gamma)}\left(1-\mathrm{e}^{m\tau}\mathrm{e}^{-mt}\right)\\
&\quad+\frac{4C_h^2\lambda_1^{-1}C_\gamma}{m\alpha(1-\beta\varepsilon-2\gamma)}\left(\mathrm{e}^{m\tau}\mathrm{e}^{-mt}-\mathrm{e}^{m(\tau-r)}\mathrm{e}^{-mt}\right).
\end{aligned}$$

which implies,for $t>\tau$

$$\begin{aligned}
&\left(|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\right)\\
&\le \mathrm{e}^{m\tau}\mathrm{e}^{-mt}d^2\left(1+\frac{2C_h^2\lambda_1^{-1}r}{\alpha(1-\beta\varepsilon-2\gamma)}\right)\\
&\quad+\frac{2}{m\alpha}|f|^2+\frac{4C_h^2\lambda_1^{-1}C_\gamma}{m\alpha(1-\beta\varepsilon-2\gamma)}.
\end{aligned}\tag{3.17}$$

If we denote

$$\rho_0^2=\frac{2}{m\alpha}|f|^2+\frac{4C_h^2\lambda_1^{-1}C_\gamma}{m\alpha(1-\beta\varepsilon-2\gamma)},\quad \widehat{\rho_0}^{\,2}=1+\frac{2C_h^2\lambda_1^{-1}r}{\alpha(1-\beta\varepsilon-2\gamma)},$$

then (3.17) yields that

$$\begin{aligned}
&|v|^2+(1-\beta\varepsilon)\|u\|^2+2J(u)\\
&\le \rho_0^2+\widehat{\rho_0}^{\,2}d^2\mathrm{e}^{m(\tau-t)},\quad \forall t>\tau.
\end{aligned}\tag{3.18}$$

which means that the initial boundary value problem (3.1) has the solution $(u,u')^{\mathrm{T}}\in E$.

Now, we prove the uniqueness of the solution. Assume that $u(\cdot)=u(\cdot;\tau,\phi)$ and $v(\cdot)=v(\cdot;\tau,\psi)$ are the two solutions of the initial boundary value problem (3.1), ϕ,ψ are the corresponding initial value,we denote $w(\cdot)=u(\cdot)-v(\cdot)$. Therefore we have

$$w''+\alpha w'-\beta\Delta w-\Delta w+g(u)-g(v)=h(t,u_t)-h(t,v_t).$$

we take the inner product of the above equation with w' and we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|w'|^2+\|w\|^2\right)+\alpha|w'|^2+\beta\|w\|^2+\left(g(u)-g(v),w'\right)\\
&=\left(h(t,u_t)-h(t,v_t),w'\right).
\end{aligned}\tag{3.19}$$

Since

$$\left|\left(g(u)-g(v),w'\right)\right|\le C_5|w||w'|;$$

$$2\left(h(t,u_t)-h(t,v_t),w'\right)\le\left|h(t,u_t)-h(t,v_t)\right|^2+|w'|^2.$$

So (3.20) can yields that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(|w'|^2+\|w\|^2\right)\\
&\le 2C_5|w||w'|+\left|h(t,u_t)-h(t,v_t)\right|^2+|w|^2
\end{aligned}\tag{3.20}$$

$$\le\left|h(t,u_t)-h(t,v_t)\right|^2+C_7\left(|w'|^2+\|w'\|^2\right).\tag{3.21}$$

$$\begin{aligned}
&\int_\tau^t\left|h(t,u_t)-h(t,v_t)\right|^2\mathrm{d}s\le C_h^2\int_{\tau-r}^\tau|u-v|^2\mathrm{d}s\\
&\le\lambda_1^{-1}C_h^2r\|\phi-\psi\|_{C_{V,H}}^2+\lambda_1^{-1}C_h^2\int_\tau^t\|w\|^2\mathrm{d}s.
\end{aligned}$$

Integrating (3.21) over the interval $[\tau,t]$, we can get

$$\begin{aligned}
&|w'(t)|^2+\|w(t)\|^2\\
&\le|w'(\tau)|^2+\|w(\tau)\|^2+C_7\int_\tau^t\left(|w'|^2+\|w\|^2\right)\\
&\quad+\lambda_1^{-1}C_h^2r\|\phi-\psi\|_{C_{V,H}}^2+\lambda_1^{-1}C_h^2\int_\tau^t\|w\|^2\mathrm{d}s\\
&\le\left(1+\lambda_1^{-1}C_h^2r\right)\|\phi-\psi\|_{C_{V,H}}^2\\
&\quad+\int_\tau^t\left(\left(\lambda_1^{-1}C_h^2+C_7\right)\|w\|^2+C_7\right)|w'|^2\mathrm{d}s.
\end{aligned}$$

Set $\gamma_1=\max\left\{\lambda_1^{-1}C_h^2+C_7,C_7\right\}$, then we have

$$\begin{aligned}
&|w'(t)|^2+|w(t)|^2\\
&\le\left(1+\lambda_1^{-1}C_h^2r\right)\|\phi-\psi\|_{C_{V,H}}^2+\gamma_1\int_\tau^t\left(\|w\|^2+|w'|^2\right)\mathrm{d}s.
\end{aligned}$$

Combining the Gronwall Lemma, we get

$$\begin{aligned}&\left|w'(t)\right|^2+\left\|w(t)\right\|^2\\&\le\left(1+\lambda_1^{-1}C_h^2 r\right)\left\|\phi-\psi\right\|_{C_{V,H}}^2 \mathrm{e}^{\gamma_1(t-\tau)},\quad \text{for all } t\ge\tau.\end{aligned}\tag{3.22}$$

If ϕ,ψ stand for the same initial value, there has

$$\left|w'(t)\right|^2+\left\|w(t)\right\|^2\le 0.$$

that shows that

$$\left|w'(t)\right|^2=0,\quad \left\|w(t)\right\|^2=0.$$

that is

$$w(t)=0.$$

therefore

$$u=v.$$

we get the uniqueness of the solution. So the proof of the theorem 3.1. has been completed.

By the theorem 3.1,we obtain the global smooth solution (u,u') continuously depends on the initial value (ϕ,ϕ'), the initial boundary value problem (1.1) generates a continuous semigroup

$$\left\{S(t)\right\}_{t\ge\tau}, S(t):E\to E;(u,u')=S(t)(\phi,\phi').$$

Then $B_{\rho_0}=\left\{(u,u')\middle|\left\|(u,u')\right\|_X\le\rho_0\right\}$ is a bounded absorbing set for the semigroup $\left\{S(t)\right\}_{t\ge\tau}$ generated by (1.1).

Under the assumption on g and f, we can get the nonlinear term $g(u)$ is compact and continuous, $f(x)$ is continuous. Next, our object is to show that the C^0 semigroup $\left\{S(t)\right\}_{t\ge 0}$ satisfies cindition *C*.

Theorem **3.2** *Assume that the hypotheses on* g *and* h *hold for all* $(u,u')^{\mathrm{T}}\in E$, α,β *are positive constants. Then the* C^0 *semigroup* $\left\{S(t)\right\}_{t\ge\tau}$ *associated with initial value problem* (3.1) *satisfies cinditionC, that is, there exists* $m\in\mathrm{N}$ *and* $T=T(B,R)$ *, for any* $N\ge m, t\ge T$ *such that*

$$\left|v_2\right|^2+(1-\beta\varepsilon)\left\|u_2\right\|^2\le C,\quad C \text{ is the positive constant.}$$

Proof. Let λ_j be the eigenvalues of $-\Delta u$ and w_j be the corresponding eigenvectors, $j=1,2,\cdots$, without loss of generality, we can assume that $\lambda_1<\lambda_2<\cdots$, and $\lim_{m\to\infty}\lambda_m=\infty$.

It is well known that $\left\{w_j\right\}_{j=1}^{\infty}$ form an orthogonal basis of H_0^1. We write

$$H_m=span\left\{w_1,w_2,\cdots,w_m\right\}$$

Since $f\in H_0^1$ and f is compact, for any $\varepsilon>0$, there exists some $m\in\mathbf{N}$ such that

$$\left\|(I-P_m)f\right\|\le\frac{\varepsilon}{2},\tag{3.23}$$

$$\left\|(I-P_m)g\right\|\le\frac{\varepsilon}{2}, \text{ for all } u\in B_R(0,R)\tag{3.24}$$

where $P_m:H_0^1\to H_m$ is orthogonal projection and R is the radius of the absorbing set. For any $(u,u_t)\in E$, we write

$$\begin{aligned}(u,u_t)&=(P_m u,P_m u_t)+\left((I-P_m)u,(I-P_m)u_t\right)\\&=(u_1,u_{1t})+(u_2,u_{2t}).\end{aligned}$$

We note that

$$h_2=(I-P_m)h,\ g_2=(I-P_m)g,\ f_2=(I-P_m)f,$$

Taking the inner product of the second equation of (3.1) with v_2 in $L^2(\mathcal{D})$, After a computation like in the proof of Theorem 3.1, we can yield that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|v_2\right|^2+(1-\beta\varepsilon)\left\|u_2\right\|^2\right)+(\alpha-\varepsilon)\left|v_2\right|^2\\&+\varepsilon(\varepsilon-\alpha)(u_2,v_2)+\beta\left\|v_2\right\|^2\\&+\varepsilon(1-\beta\varepsilon)\left\|u_2\right\|^2+\left(g_2(u),v_2\right)\\&=\left(f_2(x),v_2\right)+\left(h_2(t,u_t),v_2\right).\end{aligned}\tag{3.25}$$

This is the same as in the proof of the Theorem 3.1, except for a replacement of λ_1 with λ_{m+1}. Combined with (3.23) , (3.24) and (3.4), then we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|v_2\right|^2+(1-\beta\varepsilon)\left\|u_2\right\|^2\right)+\left(\frac{3\alpha}{4}+\beta\lambda_{m+1}-\frac{5\varepsilon}{2}\right)\left|v_2\right|^2\\&+\varepsilon(1-\beta\varepsilon)\left\|u_2\right\|^2\le\frac{L_h^2}{2\varepsilon}+\varepsilon^2.\end{aligned}$$

Choose $k_\varepsilon=\min\left\{\frac{3\alpha}{4}+\beta\lambda_{m+1}-\frac{5\varepsilon}{2},1\right\}$, we can get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|v_2\right|^2+(1-\beta\varepsilon)\left\|u_2\right\|^2\right)+k_\varepsilon\left(\left|v_2\right|^2+\left\|u_2\right\|^2\right)\le\frac{L_h^2}{2\varepsilon}+\varepsilon^2.$$

By Gronwall lemma, we can obtain

$$\left|v_2\right|^2+(1-\beta\varepsilon)\left\|u_2\right\|^2\le\frac{L_h^2+\varepsilon^3}{2\varepsilon k_\varepsilon}$$

for all $t>\tau, N\ge m$ and $(u,u')\in E$. This shows that Condition *C* is satisfied, and the proof is completed.

Due to Lemma 2.1, Theorem 3.1 and Theorem 3.2, we obtain the following Theorem

Theorem 3.3 *Assume that the hypotheses on* g *and* h *hold for all* $(u,u')^{\mathrm{T}}\in E$, α,β *are positive constants. Then the* C^0 *semigroup* $\left\{S(t)\right\}_{t\ge\tau}$ *associated with initial value problem (3.1) has a global attractor in E.*

4. Existence of the Pullback Attractor

In this subsection, we assume that $f\in H$, we aim to study the pullback attractor for the initial value problem

(1.1).

From Theorem 3.1, the initial value problem (1.1) generates a family two-parameter semigroup $U(\cdot,\cdot)$ in $C_{V,H}$, which can be defined by

$$U(t,\tau)(\phi)=u_t(\cdot;\tau,\phi),\quad t\ge\tau,\phi\in C_{V,H}$$

Lemma 4.1 *Let* ϕ,ψ *be* the *two initial values for the problem* (1.1), $\tau\in\mathbb{R}$ *is the initial time, Denote by* $u(\cdot)=u(\cdot;\tau,\phi)$ *and* $v(\cdot)=v(\cdot;\tau,\psi)$ *the corresponding solutions to* (1.1). *Then, there exists a constant* $\gamma_1>0$ *which is independent of initial value value and time, such that the following estimates hold:*

$$\begin{aligned}&\left|u'(t)-v'(t)\right|^2+\left\|u(t)-v(t)\right\|^2\\&\le\left(1+\lambda_1^{-1}C_h^2r\right)\|\phi-\psi\|_{C_{V,H}}^2\,\mathrm{e}^{\gamma_1(t-\tau)},\text{ for all } t\ge\tau;\end{aligned}\tag{4.1}$$

$$\begin{aligned}&\left\|u_t-v_t\right\|^2\\&\le\left(1+\lambda_1^{-1}C_h^2r\right)\|\phi-\psi\|_{C_{V,H}}^2\,\mathrm{e}^{\gamma_1(t-\tau)},\text{ for all } t\ge\tau+r.\end{aligned}\tag{4.2}$$

Proof. We denote $w=u-v$, by (3.22), we can get (4.1) easily.

If we consider $t\ge\tau+r$, then $t+\theta\ge\tau$ for any $\theta\in[-r,0]$, and

$$\begin{aligned}&\left|w'(t+\theta)\right|^2+\left\|w(t+\theta)\right\|^2\\&\le\left(1+\lambda_1^{-1}C_h^2r\right)\|\phi-\psi\|_{C_{V,H}}^2\,\mathrm{e}^{\gamma_1(t-\tau+\theta)}\\&\le\left(1+\lambda_1^{-1}C_h^2r\right)\|\phi-\psi\|_{C_{V,H}}^2\,\mathrm{e}^{\gamma_1(t-\tau)}.\end{aligned}$$

Thus, $\|w_t\|^2\le\left(1+\lambda_1^{-1}C_h^2r\right)\|\phi-\psi\|_{C_{V,H}}^2\,\mathrm{e}^{\gamma_1(t-\tau)},\forall t\ge\tau+r$.

Theorem **4.1** *The mapping* $U(t,\tau):C_{V,H}\to C_{V,H}$ *is continuous for any* $t\ge\tau$.

Proof. Let $\phi,\psi\in C_{V,H}$ be the initial value for the problem (1.1) and $t\ge\tau$. Denote by $u(\cdot)=u(\cdot;\tau,\phi)$ and $v(\cdot)=v(\cdot;\tau,\psi)$ the corresponding solutions to (1.1). Then, writing again $w=u-v$ we obtain the following. If $t\in[\tau-r,\tau]$, then $w(t)=\phi(t-\tau)-\psi(t-\tau)$ and

$$\begin{aligned}\left|w'(t)\right|^2+\left|w(t)\right|^2&\le\|\phi-\psi\|_{C_V}^2+\|\phi'-\psi'\|_{C_H}^2\\&\le\left(1+\lambda_1^{-1}C_h^2r\right)\|\phi-\psi\|_{C_{V,H}}^2\,\mathrm{e}^{\gamma_1(t-\tau+r)}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\left|w'(t)\right|^2+\left\|w(t)\right\|^2\\&\le\left(1+\lambda_1^{-1}C_h^2r\right)\|\phi-\psi\|_{C_{V,H}}^2\,\mathrm{e}^{\gamma_1(t-\tau+r)},\forall t\ge\tau-r,\end{aligned}$$

whence

$$\|w_t\|^2\le\left(1+\lambda_1^{-1}C_h^2r\right)\|\phi-\psi\|_{C_{V,H}}^2\,\mathrm{e}^{\gamma_1(t-\tau+r)},\ \forall t\ge\tau,$$

which implies the continuity of $U(t,\tau)$.

Theorem 4.2 *Assume that the hypotheses on* g *and* h *hold with* $m_0>0$, α,β *are the positive constants.*

Suppose *in addition that* $\sqrt{2}C_h\sqrt{\lambda_1}<\alpha_1\sqrt{1-2\gamma}$. *Then exists a family* $\{B(t)\}_{t\in\mathbb{R}}$ *of bounded sets in* $C_{V,H}$ *which is uniformly pullback absorbing fir the process* $U(\cdot,\cdot)$. *Moreover,* $B(t)=B^0$ *for all* $t\in\mathbb{R}$, *where* B^0 *is the bounded set in* $C_{V,H}$.

Proof. By (3.18), we can have

$$\begin{aligned}&\left\|u(t;\tau,\phi)\right\|^2+\left|u'(t;\tau,\phi)\right|^2\\&\le\rho_0^2+\widehat{\rho_0}^2d^2\mathrm{e}^{m(\tau-t)},\quad\forall t\ge\tau.\end{aligned}$$

and, in particular,

$$\left\|u(t;\tau,\phi)\right\|^2+\left|u'(t;\tau,\phi)\right|^2\le\rho_0^2+\widehat{\rho_0}^2d^2,\quad\forall t\ge\tau.\tag{4.3}$$

Moreover, as $u(t;\tau,\phi)=\phi(t-\tau)$ and $u'(t;\tau,\phi)=\phi'(t-\tau)$ for $t\in[\tau-r,\tau]$, then inequality (4.3) holds true for $t\ge\tau-r$.

If we take now $t\ge\tau+r$, then for all $\theta\in[-r,0]$ we have $t+\theta\ge\tau$ and so

$$\left\|u(t;\tau,\phi)\right\|^2+\left|u'(t;\tau,\phi)\right|^2\le\rho_0^2+\widehat{\rho_0}^2d^2\mathrm{e}^{m(\tau-t)},\tag{4.4}$$

or, in other words,

$$\begin{aligned}&\left\|U(t,\tau)\phi\right\|_{C_{V,H}}^2\\&\le\rho_0^2+\widehat{\rho_0}^2d^2\mathrm{e}^{m(\tau-t)},\ \forall t\ge\tau+r,\phi\in D.\end{aligned}$$

Therefore, there exists $T_D\ge r$ such that

$$\left\|U(t,t-s)\phi\right\|_{C_{V,H}}^2\le\rho_0^2,\quad\forall t\in\mathbb{R},s\ge T_D,\phi\in D.$$

which means that the ball $B_{C_{V,H}}(0,\rho_0)=B^0\subset C_{V,H}$ is uniformly pullback absorbing for the process $U(\cdot,\cdot)$.

Remark: On the one hand, observe that if $t_0\in\mathbb{R}$ and $t\ge t_0$, then $u(t+\theta;t_0-s,\phi)=u(t+\theta;t-(s+t-t_0),\phi)$ and $u'(t+\theta;t_0-s,\phi)=u'(t+\theta;t-(s+t-t_0),\phi)$ with $s+t-t_0\ge s$. As a sequence of (4.4) we have

$$\left\|U(t,t_0-s)\phi\right\|_{C_{V,H}}^2\le\rho_0^2,\quad\forall t_0\in\mathbb{R},t\ge t_0,s\ge T_D,\phi\in D.$$

or ,we have $\forall t_0\in\mathbb{R},t\ge t_0,\theta\in[-r,0],s\ge T_D,\phi\in D$

$$\left\|u(t+\theta;t_0-s,\phi)\right\|^2+\left|u'(t+\theta;t_0-s,\phi)\right|^2\le\rho_0^2.$$

On the other hand, (4.3) implies, $\forall t_0\in\mathbb{R},t\ge t_0,s\in\mathbb{R},t\ge t_0-s-r,\phi\in D$,

$$\left\|u(t;t_0-s,\phi)\right\|^2+\left|u'(t;t_0-s,\phi)\right|^2\le\rho_0^2+\widehat{\rho_0}^2d^2,\quad\forall t\ge\tau.$$

Theorem 4.3 *Under the assumption in Theorem* 4.1. *Then there* exists *a compact set* $B^2\subset C_{V,H}$ *which is uniformly pullback attracting for the process* $U(\cdot,\cdot)$, *and consequently, there exits the pullback attractor.*

$\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$. *Moreover,* $\{\mathcal{A}(t)\}_{t\in\mathbb{R}} \subset C_{D(A),V}$ *for all* $t\in\mathbb{R}$.

Proof. For each $\varepsilon\in\mathbb{R}$, the norm $\|\phi\|_\varepsilon^2 = \|\phi\|_{C_V}^2 + \|\phi' + \varepsilon\phi\|_{C_H}^2$, $\phi \in C_{V,H}$ is equivalent to $\|\cdot\|_0 := \|\cdot\|_{C_{V,H}}$. This allows us to obtain absorbing ball for the original norm by proving the existence of absorbing balls for this new norm for some suitable value of ε. Indeed, let us denote $B_\varepsilon(0,\rho) = \{\phi\in C_{V,H} : \|\phi\|_\varepsilon < \rho\}$. Noticing that for $c_1 = \max\{2, 1+2\varepsilon^2\lambda_1^{-1}\}$ it follows that

$$\begin{aligned}\|\phi\|_{C_{V,H}}^2 &= \|\phi\|_{C_V}^2 + \|\phi' + \varepsilon\phi - \varepsilon\phi\|_{C_H}^2\\ &\le \|\phi\|_{C_V}^2 + 2\|\phi'+\varepsilon\phi\|_{C_H}^2 + 2\varepsilon^2\|\phi\|_{C_H}^2\\ &\le \left(1+2\varepsilon^2\lambda_1^{-1}\right)\|\phi\|_{C_V}^2 + 2\|\phi'+\varepsilon\phi\|_{C_H}^2\\ &\le c_1\|\phi\|_\varepsilon^2,\end{aligned}$$

we then have $B_\varepsilon(0,\rho)\subset B_0\left(0, c_1^{1/2}\rho\right)$.

Let $D\subset C_{V,H}$ be a bounded set, i.e. there exists $d>0$ such that for any $\phi\in D$ it holds

$$\|\phi\|_\varepsilon^2 = \|\phi\|_{C_V}^2 + \|\phi'+\varepsilon\phi\|_{C_H}^2 \le c_1 d^2.$$

Denote by $u(\cdot) = u(\cdot;\tau,\phi)$ the solution of the problem (2.1), and consider the problems:

$$\begin{cases} v'' + \alpha v' - \beta\Delta v' - \Delta v + g(u) = f(x) + h(t,u_t), & t>\tau,\\ v(t) = 0, \quad v'(t) = 0, & t\in[\tau-r,\tau].\end{cases} \tag{4.5}$$

$$\begin{cases} w'' + \alpha w' - \beta\Delta w' - \Delta w = 0, & t>\tau,\\ w(t) = \phi(t-\tau), \quad w'(t) = \phi'(t-\tau), & t\in[\tau-r,\tau].\end{cases} \tag{4.6}$$

From the uniqueness of the solution of problems (2.1), (4.5) and (4.6) it follows that

$$u(\cdot) = v(\cdot) + w(\cdot),\ \forall\tau\in\mathbb{R}, \text{and } \forall t\ge\tau-r.$$

Consequently, $U(t,\tau)$ can be written as

$$U(t,\tau)(\phi) = U_1(t,\tau)(\phi) + U_2(t,\tau)(\phi),\quad \forall\phi\in C_{V,H},\ t\ge\tau-r.$$

where $U_1(t,\tau)(\phi) = v_t(\cdot) = v_t(\cdot;\tau,\phi)$ and $U_2(t,\tau)(\phi) = w_t(\cdot) = w_t(\cdot;\tau,\phi)$ are the solutions of (4.5) and (4.6) respectively.

First, thanks to (4.4), but with $g=f=h-0$, it follows that

$$\begin{aligned}&\|w_t(\cdot;\tau,\phi)\|_{C_V}^2 + |w_t'(\cdot;\tau,\phi)|_{C_V}^2\\ &\le c_1 d^2 e^{m(r+\tau-t)},\quad \forall t\ge\tau+r,\ \phi\in D.\end{aligned} \tag{4.7}$$

Furthermore, for $t_0\in\mathbb{R}, t\ge t_0$ and $s\ge T_D\ge r$,

$$w(t;t_0-s,\phi) = w(t;t-(s-t_0+t),\phi).$$

with $s+t-t_0\ge s\ge T_D\ge r$. Thus, Equation (4.7) implies in particular

$$w(t;t_0-s,\phi)\le c_1 d^2 e^{m(r+t_0-s-t)} \le c_1 d^2 e^{m(r-s)},\quad \forall t_0\in\mathbb{R}, t\ge t_0, s\ge T_D, \phi\in D.$$

Then we can obtain that

$$\|U_2(t,t-s)\phi\|_{C_{V,H}}^2 \le c_1 d^2 e^{m(r-s)},\ \forall t\in\mathbb{R}, s\ge r, \phi\in D,$$

whence,

$$\lim_{s\to+\infty}\sup_{t\in\mathbb{R}}\sup_{\phi\in D}\|U_2(t,t-s)\phi\|_{C_{V,H}}^2 = 0.$$

Next, fix $t_0\in\mathbb{R}, s\ge T_D, \phi\in D$ and denote

$$\begin{aligned}&u(t) = u(t;t_0-s,\phi),\\ &v(t) = v(t;t_0-s,\phi),\ t\ge t_0-s-r,\\ &F(t) = f + h(t,u_t) - g,\quad t\ge t_0-s.\end{aligned}$$

Then, for $t\ge t_0$,

$$\begin{aligned}|F(t)| &\le |f| + |g| + L_h\|u_t\|\\ &\le |f| + |g| + L_h\lambda_1^{-\frac12}\rho_0 = K_1,\end{aligned} \tag{4.8}$$

and for $t\ge t_0-s$, we have

$$\begin{aligned}|F(t)| &\le |f| + |g| + L_h\|u_t\|\\ &\le |f| + |g| + L_h\lambda_1^{-\frac12}\left(\rho_0^2 + \hat\rho_0^2 d^2\right)^{1/2}\\ &\le K_1 + L_h\lambda_1^{-\frac12}\hat\rho_0 d.\end{aligned} \tag{4.9}$$

Then, we deduce from the assumption on h that $F'(t) = \langle\delta h(t,u_t),(1+u_t')\rangle - \langle\delta g, u'\rangle$ and $F'(t)\le K\left(1+\|u_t\|_{C_H}\right)\left(1+\|u_t'\|_{C_H}\right) + C_5|u'|$. Arguing as we did in order to obtain (4.8) and (4.9), we have

$$|F'(t)| \le K\left(1+\lambda_1^{-\frac12}\rho_0\right)(1+\rho_0) + C_5\rho_0 = K_2,\quad \forall t\ge t_0, \tag{4.10}$$

and

$$\begin{aligned}F'(t)\le{}& K\left(1+\left(\rho_0^2+\hat\rho_0^2 d^2\right)^{1/2}\right)\left(1+\lambda_1^{-\frac12}\left(\rho_0^2+\hat\rho_0^2 d^2\right)^{1/2}\right)\\ &+ C_5\left(\rho_0^2+\hat\rho_0^2 d^2\right)^{1/2} = K_3(d),\quad \forall t\ge t_0-s.\end{aligned} \tag{4.11}$$

Let us denote $y(t) = \left\|v'(t) + \frac{\alpha}{2}v(t)\right\|^2 + |Av(t) - F(t)|^2$ and make use of the estimates in Theorem 4.2. On the one hand, for all $t\ge t_0-s$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(y(t)\right)+\frac{\alpha}{2}y(t)\le\alpha\left|F(t)\right|^2+\frac{4}{\alpha}\left|F'(t)\right|^2+\frac{\alpha^3}{8}\left\|v(t)\right\|^2$$

$$\le\alpha\left(K_1+L_h\lambda_1^{-\frac{1}{2}}\hat{\rho}_0 d\right)^2+\frac{4}{\alpha}K_3(d)^2+\frac{\alpha^3}{8}\left\|v(t)\right\|^2.$$

but, as (4.4) and (4.7) ensure

$$\left\|v(t)\right\|^2\le 2\left\|u(t)\right\|^2+2\left\|w(t)\right\|^2\le 2\rho_0^2+2\hat{\rho}_0^2 d^2+2c_1 d^2.$$

if we denote by

$$K_4(d)=\left(K_1+L_h\lambda_1^{-\frac{1}{2}}\hat{\rho}_0 d\right)^2$$

$$+\frac{4}{\alpha}K_3(d)^2+\frac{\alpha^3}{4}\left(\rho_0^2+\hat{\rho}_0^2 d^2+c_1 d^2\right).$$

then, in particular,

$$y'(t)+\frac{\alpha}{2}y(t)\le K_4 d,\ \forall t\in[t_0-s,t_0].$$

Noticing that $y(t_0-s)=\left|F(t_0-s)\right|^2$, the Gronwall lemma leads us to

$$y(t_0)\le\frac{2}{\alpha}K_4(d)+\left(K_1+L_h\lambda_1^{-\frac{1}{2}}\hat{\rho}_0 d\right)^2=K_5(d).$$

On the other hand, if $t\ge t_0$, we deduce that

$$\left\|v(t)\right\|^2\le 2\left\|u(t)\right\|^2+2\left\|w(t)\right\|^2\le 2\rho_0^2+2c_1 d^2\mathrm{e}^{m(r-s)},$$

and, from (4.8) and (4.10),

$$y'(t)+\frac{\alpha}{2}y(t)\le\alpha K_1+\frac{4}{\alpha}K_2+\frac{\alpha^3}{8}\left\|v(t)\right\|^2$$

$$\le\alpha K_1+\frac{4}{\alpha}K_2+\frac{\alpha^3}{4}\rho_0^2+\frac{\alpha^3}{4}c_1 d^2\mathrm{e}^{m(r-s)}$$

$$=K_6+K_7 d^2\mathrm{e}^{-ms},\quad\forall t\ge t_0.$$

Once again, the Gronwall lemma implies that

$$y(t)\le y(t_0)\mathrm{e}^{\frac{\alpha}{2}(t_0-t)}+\frac{2}{\alpha}K_6+\frac{2}{\alpha}K_7 d^2\mathrm{e}^{-ms}$$

$$\le K_5(d)\mathrm{e}^{\frac{\alpha}{2}(t_0-t)}+\frac{2}{\alpha}K_6+\frac{2}{\alpha}K_7 d^2\mathrm{e}^{-ms},\quad\forall t\ge t_0.$$

Then, there exists $T_D'\ge T_D$ such that, if $s\ge T_D'$,

$$y(t)\le K_5(d)\mathrm{e}^{\frac{\alpha}{2}(t_0-t)}+\frac{3}{\alpha}K_6,\ \forall t_0\in\mathbb{R},t\ge t_0.$$

Recalling that $y(t)=y(t;t_0-s,\phi)$, if we fix $t\ge t_0$, take $s=T_D'$ and denote $\tilde{s}=t-t_0+T_D'$ we have, provided $t-t_0$ is large enough, that

$$y(t;t_0-T_D',\phi)=y\left(t;t-\left(t-t_0+T_D'\right),\phi\right)$$

$$=y(t;t-\tilde{s},\phi)\le\frac{4}{\alpha}K_6.$$

In conclusion, there exists $T_D''>0$ such that for all $t\in\mathbb{R}$, and all $s\ge T_D'+T_D''$,

$$y(t;t-s,\phi)\le\frac{4}{\alpha}K_6,\quad\forall\phi\in D.$$

Denoting $\hat{T}_D=T_D'+T_D''+r$, we have for all $\phi\in D,t\in\mathbb{R},s\ge\hat{T}_D$

$$\left\|v'(t;t-s,\phi)+\frac{\alpha}{2}v(t;t-s,\phi)\right\|^2$$

$$+\left|Av(t;t-s,\phi)-F(t;t-s,\phi)\right|^2\le\frac{4}{\alpha}K_6,$$

where

$$F(t;t-s,\phi)=f+h\left(t,u_t(t;t-s,\phi)\right)-g\left(u(t;t-s,\phi)\right).$$

But as for all $\phi\in D,t\in\mathbb{R}$ and $s\ge T_D$, we get $\left\|v(t;t-s,\phi)\right\|^2\le\rho_0^2$ and $\left|F(t;t-s,\phi)\right|^2\le K_7^2=2\left|f\right|^2+2C_5^2\rho_0^2+2L_h\lambda_1^{-1}\rho_0^2$, and, consequently, for all $\phi\in D,t\in\mathbb{R}$ and $s\ge T_D$,

$$\left\|v'(t;t-s,\phi)\right\|^2+\left|Av(t;t-s,\phi)\right|^2$$

$$\le\frac{4}{\alpha}K_6\le\frac{8}{\alpha}K_6+\frac{\alpha^2}{2}\rho_0^2+2K_7^2,$$

which shows that

$$\left\|v_t(\cdot;t-s,\phi)\right\|_{C_{D(A),V}}^2\le\rho_1^2=\frac{4}{\alpha}K_6\le\frac{8}{\alpha}K_6+\frac{\alpha^2}{2}\rho_0^2+2K_7^2,$$

for all $\phi\in D,t\in\mathbb{R}$ and $s\ge\hat{T}_D$. This means that the all $B^1=B_{C_{D(A),V}}(0,\rho_1)$ is the bounded set in $C_{D(A),V}$ which , in addition, is uniformly absorbing for the family of operators $U(\cdot,\cdot)$. As B^1 is the bounded set in $C_{D(A),V}$, then there exists $T_{B^1}\ge r$ such that

$$U_1(t,t-s)B^1\subset B^1,\forall t\in\mathbb{R},s\ge T_{B^1},$$

and, therefore, the bounded set $B^2\subset C_{D(A),V}$ given

$$B^2=\bigcup_{t\in\mathbb{R}}\bigcup_{s\ge T_{B^1}}U_1(t,t-s)B^1\subset B^1,$$

is uniformly pullback absorbing for $U_1(\cdot,\cdot)$ in $C_{V,\underline{H}}$.

By Ascoli-Arzelà theorem, we can prove that $\overline{B}^2$ is compact, so $\left\{B(t)\equiv\overline{B}^2\right\}_{t\in\mathbb{R}}$ is a family of compact subsets in $C_{V,H}$, which is also uniformly pullback attracting for $U(\cdot,\cdot)$, and the proof has been completed.

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