International Journal of Modern Nonlinear Theory and Application, **2013, 2, 1-6**
http://dx.doi.org/10.4236/ijmnta.2013.21001 Published Online March 2013 (http://www.scirp.org/journal/ijmnta)

Stability Analysis of a Nonlinear Difference Equation

Fatma Bozkurt
Department of Mathematics, Faculty of Education, Erciyes University, Kayseri, Turkey
Email: fbozkurt@erciyes.edu.tr

Received January 1, 2013; revised January 31, 2013; accepted February 9, 2013

ABSTRACT

The local and global behavior of the positive solutions of the difference equation

$$y_{n+1}=\frac{\alpha\cdot e^{-y_n}+\beta\cdot e^{-y_{n-1}}}{\gamma+\alpha\cdot y_n+\beta\cdot y_{n-1}},\ n=0,1,\cdots$$

was investigated, where the parameters α, β and γ and the initial conditions are arbitrary positive numbers. Furthermore, the characterization of the stability was studied with a basin that depends on the conditions of the coefficients. The analysis about the semi-cycle of positive solutions has end the study of this work.

Keywords: Difference Equations; Stability Analysis; Semi-Cycle Solutions; Boundedness

1. Introduction

H. El-Metwally *et al*. [1] studied the global stability of the difference equation

$$x_{n+1}=\alpha+\beta x_{n-1}e^{-x_n},\ n=0,1,\cdots,\qquad(1.1)$$

where the parameters α and β are positive numbers and the initial conditions are arbitrary non-negative real numbers. This equation may be viewed as a model in Mathematical Biology, where α is the immigration rate and β the population growth rate.

In [2] was investigated the globally asymptotically stability of the difference equation

$$y_{n+1}=\frac{\alpha+\beta e^{-y_n}}{\gamma+y_{n-1}},\ n=0,1,\cdots,\qquad(1.2)$$

where the parameters α, β and γ are positive numbers and the initial conditions are arbitrary non-negative numbers. In [3] the boundedness and the global asymptotic behavior of the difference equation

$$y_{n+1}=\frac{\alpha\cdot e^{-\left(ny_n+(n-k)y_{n-k}\right)}}{\beta+ny_n+(n-k)y_{n-k}},\ n=0,1,\cdots,\qquad(1.3)$$

was shown, where α and β are positive real numbers, $k\in\{1,2,\cdots\}$ and the initial conditions $y_{-k},\cdots,y_{-1},y_0$ are arbitrary numbers. Similar studies can be shown in [4-7].

The aim in this paper is to study the local and global behavior of the positive solutions of the difference equation

$$y_{n+1}=\frac{\alpha\cdot e^{-y_n}+\beta\cdot e^{-y_{n-1}}}{\gamma+\alpha\cdot y_n+\beta\cdot y_{n-1}},\ n=0,1,\cdots,\qquad(1.4)$$

where the parameters α, β and γ and the initial conditions are arbitrary positive numbers. In Section 2, the local asymptotic stability of the equilibrium point of Equation (1.4) was investigated by using the Linearized Stability Theorem. A suitable Lyapunov function for the analysis of the global asymptotic stability behavior was used, like the idea in [8,9]. Furthermore, the characterization of the stability was examined that depends on the conditions of the coefficients (see [10]). In Section 3, the semi-cycle of positive solutions was analyzed. All this results will be shown theoretical and by simulations at the end of the paper.

2. Local and Global Asymptotic Stability Analysis

In this section, we discuss the local and global asymptotic stability of the unique positive equilibrium point of Equation (1.4) by using the theorems in [4,8-10].

The equilibrium points of Equation (1.4) are the solutions of the equation

$$\bar{y}=\frac{(\alpha+\beta)e^{-\bar{y}}}{\gamma+(\alpha+\beta)\bar{y}}.\qquad(2.1)$$

Set

$$f(y)=\frac{(\alpha+\beta)e^{-y}}{\gamma+(\alpha+\beta)y}-y\qquad(2.2)$$

for $y=0$ and $y\to\infty$ we obtain respectively,

$$f(0)=\frac{\alpha+\beta}{\gamma}>0,\ \lim_{y\to\infty} f(y)=-\infty \tag{2.3}$$

and

$$f'(y)=\frac{-(\alpha+\beta)\mathrm{e}^{-y}\left[\gamma+(\alpha+\beta)(y+1)\right]}{\left[\gamma+(\alpha+\beta)y\right]^2}-1. \tag{2.4}$$

It follows that Equation (2.1) has exactly one solution $\overline{y}$. From this result Equation (1.4) has a unique equilibrium $\overline{y}$.

The linearized equation and the characteristic equation associated with Equation (1.4) about the equilibrium $\overline{y}$ is

$$x_{n+1}+\frac{\alpha\left(\mathrm{e}^{-\overline{y}}+\overline{y}\right)}{\gamma+(\alpha+\beta)\overline{y}}x_n+\frac{\beta\left(\mathrm{e}^{-\overline{y}}+\overline{y}\right)}{\gamma+(\alpha+\beta)\overline{y}}x_{n-1}=0,\quad n=0,1,\cdots \tag{2.5}$$

and

$$\lambda^2+\frac{\alpha\left(\mathrm{e}^{-\overline{y}}+\overline{y}\right)}{\gamma+(\alpha+\beta)\overline{y}}\lambda+\frac{\beta\left(\mathrm{e}^{-\overline{y}}+\overline{y}\right)}{\gamma+(\alpha+\beta)\overline{y}}=0, \tag{2.6}$$

respectively.

Theorem 2.1. The following statements are true.

1) Every solution of Equation (1.4) is bounded if $0<y_n$.

2) The equilibrium point of Equation (1.4) is bounded if $0<\overline{y}$.

Proof.

1) Suppose that $0<y_n$. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of Equation (1.4). We have

$$0<y_{n+1}=\frac{\alpha\mathrm{e}^{-y_n}+\beta\mathrm{e}^{-y_{n-1}}}{\gamma+\alpha y_n+\beta y_{n-1}}<\frac{\alpha\mathrm{e}^{-y_n}+\beta\mathrm{e}^{-y_{n-1}}}{\gamma}<\frac{(\alpha+\beta)\mathrm{e}^0}{\gamma}=\frac{(\alpha+\beta)}{\gamma},$$

which gives that every positive solution of the Equation (1.4) is bounded. Thus 1) is true.

2) Assume that $0<\overline{y}$. Then

$$0<\overline{y}=\frac{\alpha\mathrm{e}^{-\overline{y}}+\beta\mathrm{e}^{-\overline{y}}}{\gamma+\alpha\overline{y}+\beta\overline{y}}<\frac{\alpha\mathrm{e}^{-\overline{y}}+\beta\mathrm{e}^{-\overline{y}}}{\gamma}<\frac{(\alpha+\beta)\mathrm{e}^0}{\gamma}=\frac{(\alpha+\beta)}{\gamma},$$

which implies that 2) is also true.

Theorem 2.2 Let $\alpha>\beta$. If

$$(\alpha+\beta)<\gamma\mathrm{e}^{\frac{-\gamma+\sqrt{\gamma^2+4(\alpha+\beta)\gamma}}{2(\alpha+\beta)}}, \tag{2.7}$$

then the positive equilibrium point of Equation (1.4) is locally asymptotically stable.

Proof. From the Linearized Stability Theorem, we can write

$$\left|-\frac{\alpha\left(\mathrm{e}^{-\overline{y}}+\overline{y}\right)}{\gamma+(\alpha+\beta)\overline{y}}\right|<1+\frac{\beta\left(\mathrm{e}^{-\overline{y}}+\overline{y}\right)}{\gamma+(\alpha+\beta)\overline{y}}<2. \tag{2.8}$$

The inequality (2.8) can be shown under two cases;

1) $\left|-\frac{\alpha\left(\mathrm{e}^{-\overline{y}}+\overline{y}\right)}{\gamma+(\alpha+\beta)\overline{y}}\right|<1+\frac{\beta\left(\mathrm{e}^{-\overline{y}}+\overline{y}\right)}{\gamma+(\alpha+\beta)\overline{y}}$

2) $1+\frac{\beta\left(\mathrm{e}^{-\overline{y}}+\overline{y}\right)}{\gamma+(\alpha+\beta)\overline{y}}<2$.

From 2), we get

$$\overline{y}>\frac{\beta\mathrm{e}^{-\overline{y}}-\gamma}{\alpha}. \tag{2.9}$$

By 1), we will have

$$0<(\alpha+\beta)\mathrm{e}^{-\overline{y}}+2(\alpha+\beta)\overline{y}+\gamma, \tag{2.10}$$

which always holds and since $\alpha>\beta$, we can also obtain

$$\overline{y}>\frac{(\alpha-\beta)\mathrm{e}^{-\overline{y}}-\gamma}{2\beta}. \tag{2.11}$$

Considering both (2.9) and (2.11), if

$$\frac{(\alpha-\beta)\mathrm{e}^{-\overline{y}}-\gamma}{2\beta}>\frac{\beta\mathrm{e}^{-\overline{y}}-\gamma}{\alpha},$$

then we have

$$(\alpha+\beta)\mathrm{e}^{-\overline{y}}<\gamma. \tag{2.12}$$

Rewriting (2.12), we get

$$(\alpha+\beta)\overline{y}^2+\gamma\overline{y}-\gamma<0. \tag{2.13}$$

In view of (2.12) and (2.13), we obtain

$$(\alpha+\beta)<\gamma\mathrm{e}^{\frac{-\gamma+\sqrt{\gamma^2+4(\alpha+\beta)\gamma}}{2(\alpha+\beta)}}.$$

Theorem 2.3. Let the conditions in Theorem 2.2 hold and assume that $\overline{y}_1$ and $\overline{y}_2$ are the equilibrium points of Equation (1.4), which parameters have the conditions $\gamma_2<\gamma_1<\frac{(\alpha+\beta)\beta}{\alpha}$. If the parameter γ decreases, then the local stability of the positive equilibrium point

$$\overline{y}=\frac{-(\alpha\gamma-(\alpha+\beta)\beta)+\sqrt{(\alpha\gamma-(\alpha+\beta)\beta)^2+4\alpha(\alpha+\beta)^2}}{2\alpha(\alpha+\beta)} \tag{2.14}$$

decreases also.

Proof. By the Linearized Stability Theorem, we have

$$\left|-\frac{\alpha\left(\mathrm{e}^{-\bar{y}}+\bar{y}\right)}{\gamma+(\alpha+\beta)\bar{y}}\right|<1+\frac{\beta\left(\mathrm{e}^{-\bar{y}}+\bar{y}\right)}{\gamma+(\alpha+\beta)\bar{y}}<2. \quad (2.15)$$

Let

$$\left|-\frac{\alpha\left(\mathrm{e}^{-\bar{y}}+\bar{y}\right)}{\gamma+(\alpha+\beta)\bar{y}}\right|=1. \quad (2.16)$$

From (2.16), computations will give us

$$\alpha(\alpha+\beta)\bar{y}^2-\left((\alpha+\beta)\beta-\alpha\gamma\right)\bar{y}-(\alpha+\beta)\gamma=0, \quad (2.17)$$

where we get the positive equilibrium point

$$\bar{y}=\frac{(\alpha+\beta)\beta-\alpha\gamma+\sqrt{\left((\alpha+\beta)\beta-\alpha\gamma\right)^2+4\alpha(\alpha+\beta)^2}}{2\alpha(\alpha+\beta)}.$$

Let us write

$$\bar{y}_1=\frac{(\alpha+\beta)\beta-\alpha\gamma_1+\sqrt{\left((\alpha+\beta)\beta-\alpha\gamma_1\right)^2+4\alpha(\alpha+\beta)^2}}{2\alpha(\alpha+\beta)} \quad (2.18)$$

and

$$\bar{y}_2=\frac{(\alpha+\beta)\beta-\alpha\gamma_2+\sqrt{\left((\alpha+\beta)\beta-\alpha\gamma_2\right)^2+4\alpha(\alpha+\beta)^2}}{2\alpha(\alpha+\beta)}, \quad (2.19)$$

where $\gamma_2<\gamma_1<\dfrac{(\alpha+\beta)\beta}{\alpha}$.

Considering both (2.18) and (2.19), we get

$$\bar{y}_1<\bar{y}_2. \quad (2.20)$$

On the other side, from (2.15), we will investigate

$$\frac{\beta\left(\mathrm{e}^{-\bar{y}}+\bar{y}\right)}{\gamma+(\alpha+\beta)\bar{y}}<1. \quad (2.21)$$

From (2.21) assume that

$$\frac{\beta\left(\mathrm{e}^{-\bar{y}_1}+\bar{y}_1\right)}{\gamma+(\alpha+\beta)\bar{y}_1}<\frac{\beta\left(\mathrm{e}^{-\bar{y}_2}+\bar{y}_2\right)}{\gamma+(\alpha+\beta)\bar{y}_2}<1. \quad (2.22)$$

Considering the conditions in Theorem 2.2, if furthermore (2.22) holds, than the stability of $\bar{y}_2$ is weaker than $\bar{y}_1$. By computing (2.22), we obtain

$$\begin{aligned}&\gamma\left(\mathrm{e}^{-\bar{y}_2}-\mathrm{e}^{-\bar{y}_1}\right)+(\alpha+\beta)\left(\bar{y}_1\mathrm{e}^{-\bar{y}_2}-\bar{y}_2\mathrm{e}^{-\bar{y}_1}\right)\\&+\gamma(\bar{y}_2-\bar{y}_1)>0.\end{aligned} \quad (2.23)$$

This inequality can be also written in the form

$$\begin{aligned}&\gamma\left(\bar{y}_2^2-\bar{y}_1^2\right)+\frac{\gamma^2}{\alpha+\beta}(\bar{y}_2-\bar{y}_1)+\gamma(\bar{y}_2-\bar{y}_1)\\&+(\alpha+\beta)\bar{y}_1\bar{y}_2(\bar{y}_2-\bar{y}_1)>0.\end{aligned} \quad (2.24)$$

From (2.20) we get

$$\gamma(\bar{y}_2+\bar{y}_1)+\frac{\gamma^2}{\alpha+\beta}+\gamma+(\alpha+\beta)\bar{y}_1\bar{y}_2>0, \quad (2.25)$$

which always holds. This completes the proof.

Theorem 2.4. Suppose that $\{y_n\}_{-1}^{\infty}$ is a monoton decreasing solution of Equation (1.4) and assume that the conditions in Theorem 2.2 hold. If

$$y_n>2\bar{y} \quad (2.26)$$

then the positive equilibrium point of Equation (1.4) is globally asymptotically stable.

Proof. We consider a Lyapunov function $V(n)$ defined by

$$V(n)=\{y_n-\bar{y}\}^2, n=0,1,2,\cdots \quad (2.27)$$

The change along the solutions of Equation (1.4) is

$$\begin{aligned}\Delta V(n)&=V(n+1)-V(n)\\&=\{y_{n+1}-y_n\}\{y_{n+1}+y_n-2\bar{y}\}.\end{aligned} \quad (2.28)$$

From (2.28) we can write

$$y_{n+1}-y_n=\frac{\alpha\cdot\mathrm{e}^{-y_n}+\beta\cdot\mathrm{e}^{-y_{n-1}}}{\gamma+\alpha\cdot y_n+\beta\cdot y_{n-1}}-y_n$$

and

$$y_{n+1}+y_n-2\bar{y}=\frac{\alpha\cdot\mathrm{e}^{-y_n}+\beta\cdot\mathrm{e}^{-y_{n-1}}}{\gamma+\alpha\cdot y_n+\beta\cdot y_{n-1}}+y_n-2\bar{y}.$$

It can be compute that by using the hypothesis we have $\Delta V(n)<0$, which is the condition for the global asymptotic stability of the positive equilibrium point of Equation (1.4).

3. The Semi-Cycle and Oscillation

In this section, we consider the semi-cycle and oscillation of the positive solutions of Equation (1.4).

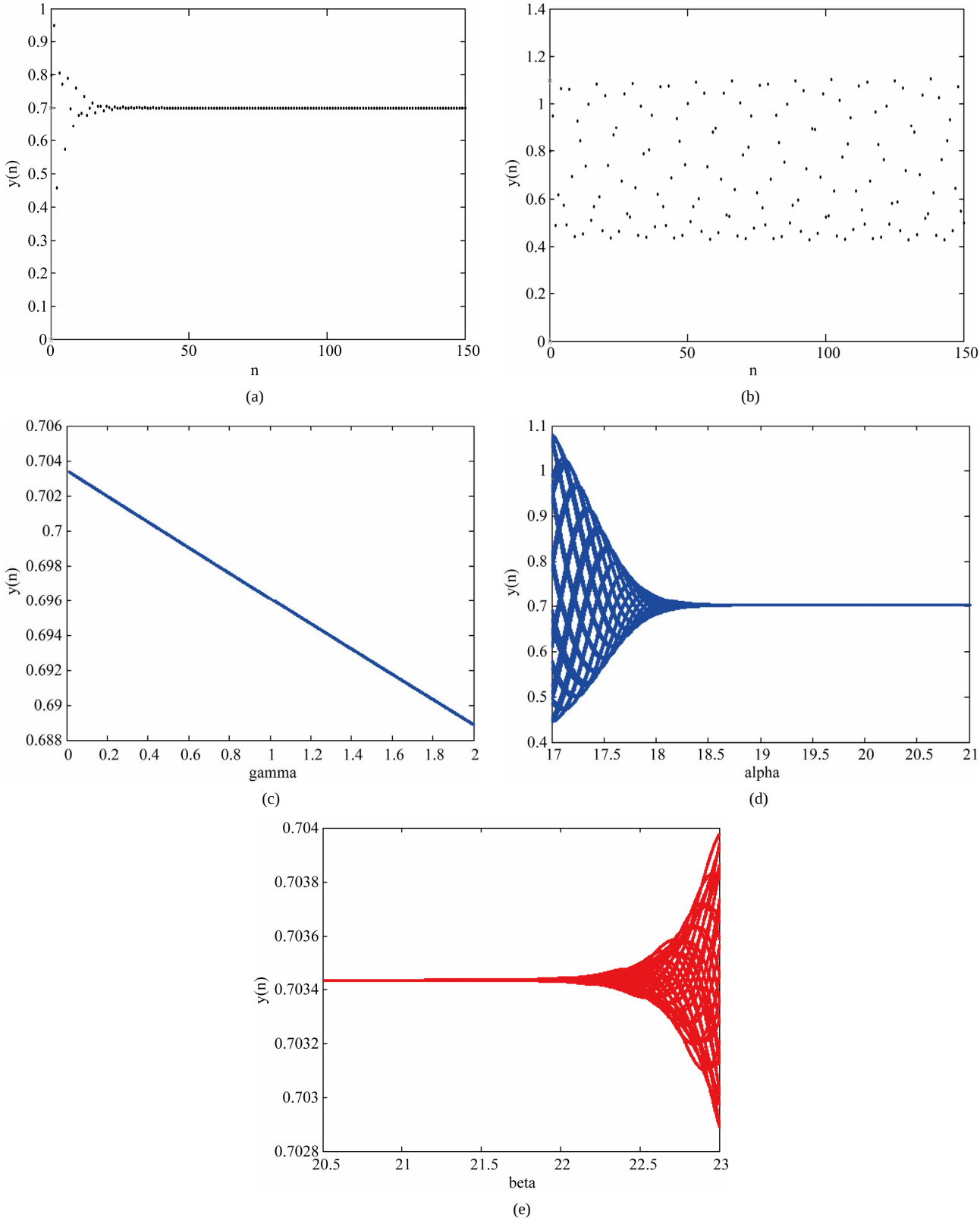

Figure 1. (a) Local stable behavior of Equation (1.4) for $\alpha = 30$, $\beta = 20$, $\gamma = 0.3$, $y(-1) = 0.8$ and $y(0) = 0.95$; (b) Unstable behavior of Equation (1.4) for $\alpha = 20$, $\beta = 30$, $\gamma = 0.3$, $y(-1) = 0.8$ and $y(0) = 0.9$; (c) Stability analysis for $\alpha = 30$, $\beta = 20$, $\gamma \in [0.01, 2]$, $y(-1) = 0.8$ and $y(0) = 0.95$; (d) Diagram of the solutions of Equation (1.4) for $\beta = 30$, $\alpha \in [17, 21]$, $\gamma \in 0.03$, $y(-1) = 0.8$ and $y(0) = 0.95$; (e) Diagram of the solution of Equation (1.4) for $\alpha = 17.2$, $\beta \in [20.5, 23]$, $\gamma \in 0.003$, $y(-1) = 0.8$ and $y(0) = 0.95$.

By [8], a positive semi-cycle of a solution $\{x(n)\}_{n=-1}^{\infty}$ of $x(n+1)=f(x(n),x(n-1))$ consists of a "string" of terms $\{x(k),x(k+1),\cdots,x(m)\}$, all greater than or equal to the equilibrium $\overline{x}^*$, with $k\geq -1$ and $m\leq\infty$ and such that either $k=-1$ or $k>-1$ and $x(k-1)<\overline{x}$ and either $m=\infty$ or $m<\infty$ and $x(m+1)<\overline{x}$.

A negative semi-cycle of a solution $\{x(n)\}_{n=-1}^{\infty}$ of $x(n+1)=f(x(n),x(n-1))$ consists of a "string" of terms $\{x(k),x(k+1),\cdots,x(m)\}$, all less than the equilibrium $\overline{x}^*$, with $k\geq -1$ and $m\leq\infty$ and such that either $k=-1$ or $k>-1$ and $x(k-1)\geq\overline{x}$ and either $m=\infty$ or $m<\infty$ and $x(m+1)\geq\overline{x}$.

Theorem B ([9]) Assume that $f\in C\left[(0,\infty)x(0,\infty),(0,\infty)\right]$ and that $f(x,y)$ is decreasing in both arguments. Let $\overline{y}$ be a positive equilibrium of $y(n+1)=f(y(n),y(n-1))$. Then every oscillatory solution of the difference equation $y(n+1)=f(y(n),y(n-1))$ has semi-cycle of length at most two.

Theorem 3.2. Let $f(x,y)=\dfrac{\alpha\cdot e^{-x}+\beta\cdot e^{-y}}{\gamma+\alpha\cdot x+\beta\cdot y}$ be a function such that $f\in C\left[(0,\infty)x(0,\infty),(0,\infty)\right]$. Then every oscillatory solution of Equation (1.4) has semi-cycle of length at most two.

Proof. By the Theorem B, we can write Equation (1.4) such as

$$f(x,y)=\frac{\alpha\cdot e^{-x}+\beta\cdot e^{-y}}{\gamma+\alpha\cdot x+\beta\cdot y}. \tag{3.1}$$

The first derivative of (3.1) with respect to *x* and *y* are

$$\frac{\partial f}{\partial x}=\frac{-\alpha e^{-x}\left(\gamma+\alpha\cdot x+\beta\cdot y\right)-\alpha\left(\alpha\cdot e^{-x}+\beta\cdot e^{-y}\right)}{\left(\gamma+\alpha\cdot x+\beta\cdot y\right)^2} \tag{3.2}$$

and

$$\frac{\partial f}{\partial y}=\frac{-\beta e^{-x}\left(\gamma+\alpha\cdot x+\beta\cdot y\right)-\beta\left(\alpha\cdot e^{-x}+\beta\cdot e^{-y}\right)}{\left(\gamma+\alpha\cdot x+\beta\cdot y\right)^2}, \tag{3.3}$$

respectively. These derivatives are less than zero if

$$-\alpha e^{-x}\left(\gamma+\alpha\cdot x+\beta\cdot y\right)-\alpha\left(\alpha\cdot e^{-x}+\beta\cdot e^{-y}\right)<0 \tag{3.4}$$

and

$$-\beta e^{-x}\left(\gamma+\alpha\cdot x+\beta\cdot y\right)-\beta\left(\alpha\cdot e^{-x}+\beta\cdot e^{-y}\right)<0 \tag{3.5}$$

respectively. Since the parameters are positive and the variables x and y are in a positive interval, (3.4) and (3.5) will be always hold. This completes the proof.

Example 1 In this Example, **Figure 1(a)** show the local stability of Equation (1.4) for the parameters $\alpha=30$, $\beta=20$, $\gamma=0.3$, $y(-1)=0.8$ and $y(0)=0.95$ by using the conditions in Theorem 2.2. The parameters $\alpha=20$, $\beta=30$ $\gamma=0.3$ and the initial conditions $y(-1)=0.8$ and $y(0)=0.95$ are selected to show in **Figure 1(b)** the unstable behavior of the solutions of Equation (1.4). In **Figure 1(c)**, we can show that by decreasing of the parameter γ the local stability get be weaker. At last, **Figures 1(d)** and **(e)** show the diagram of the solutions of Equation (1.4) for $\alpha\in[17,21]$, $\beta=25$, $\gamma=0.003$, $y(-1)=0.8$ and $y(0)=0.95$ and for $\alpha=17.2$, $\gamma=0.003$, $\beta\in[20.5,23]$, $y(-1)=0.8$ and $y(0)=0.95$, respectively. This give us the relation between the parameters α and β, which have an important role by the stability analysis of Equation (1.4), as shown in Theorem 2.2 and Theorem 2.4.

REFERENCES

[1] H. El-Metwally, E. A. Grove, G. Ladas, R. Levins and M. Radin, "On the Difference Equation $x_{n+1}=\alpha+\beta x_{n-1}e^{-x_n}$," *Nonlinear Analysis*, Vol. 47, No. 7, 2001, pp. 4623-4634. doi:10.1016/S0362-546X(01)00575-2

[2] I. Ozturk, F. Bozkurt and S. Ozen, "On the Difference Equation $y_{n+1}=\dfrac{\alpha+\beta e^{-y_n}}{\gamma+y_{n-1}}$," *Applied Mathematics and Computations*, Vol. 181, No. 2, 2006, pp. 1387-1393. doi:10.1016/j.amc.2006.03.007

[3] I. Ozturk, F. Bozkurt and S. Ozen, "Global Asymptotic Behavior of the Difference Equation $y_{n+1}=\dfrac{\alpha\cdot e^{-(ny_n+(n-k)y_{n-k})}}{\beta+ny_n+(n-k)y_{n-k}}$," *Applied Mathematic Letters*, Vol. 22, No. 4, 2009, pp. 595-599. doi:10.1016/j.aml.2008.06.037

[4] C. H. Gibbons, M. R. S. Kulenovic, G. Ladas and H. D. Voulov, "On the Trichotomy Character of $x_{n+1}=(\alpha+\beta x_n+\gamma x_{n-1})/(A+x_n)$," *Journal of Difference Equations and Applications*, Vol. 8, No. 1, 2002, pp. 75-92. doi:10.1080/10236190211940

[5] M. M. El-Afifi and A. M. Ahmed, "On the Difference Equation $x_{n+1}=\dfrac{\alpha+\alpha x_n+\alpha x_{n-1}+\cdots+\alpha x_{n-k+2}}{x_{n-k+1}}$," *Applied Mathematics and Computations*, Vol. 144, No. 2-3, 2003, pp. 537-542. doi:10.1016/S0096-3003(02)00429-0

[6] W. S. He and W. T. Li, "Attractivity ın a Nonlinear Delay Difference Equation," *Applied Mathematics E—Notes*, Vol. 4, 2004, pp. 48-53.

[7] S. Ozen, I. Ozturk and F. Bozkurt, "On the Recursive Sequence $y_{n+1}=\dfrac{\alpha+y_{n-1}}{\beta+y_n}+\dfrac{y_{n-1}}{y_n}$," *Applied Mathematics and Computation*, Vol. 188, No. 1, 2007, pp. 180-188. doi:10.1016/j.amc.2006.09.106

[8] K. Cunningham, *et al.*, "On the Recursive Sequence $x_{n+1}=\dfrac{\alpha+\beta x_n}{Bx_n+Cx_{n-1}}$," *Nonlinear Analysis*, Vol. 47, No. 7, 2001, pp. 4603-4614.

doi:10.1016/S0362-546X(01)00573-9

[9] C. Gibbons, M. R. S. Kulenovic and G. Ladas, "On the Recursive Sequence $x_{n+1}=\frac{\alpha+\beta x_{n-1}}{\gamma+x_n}$," *Mathematical Sciences Research Hot-Line*, Vol. 4, No. 2, 2000, pp. 1-11.

[10] C. Celik, H. Merdan, O. Duman and O. Akın, "Allee Effects on Population Dynamics with Delay," *Chaos, Solitons and Fractals*, Vol. 37, No. 1, 2008, pp. 65-74.

