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The Sound and Complete *R*-Calculi with Respect to Pseudo-Revision and Pre-Revision*

Wei Li[1], Yuefei Sui[2]
[1]State Key Laboratory of Software Development Environment, Beijing University of Aeronautics and Astronautics, Beijing, China
[2]Key Laboratory of Intelligent Information Processing, Institute of Computing Technology, Chinese Academy of Sciences, Beijing, China
Email: liwei@nlsde.buaa.edu.cn, yfsui@ict.ac.cn

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ABSTRACT

The AGM postulates ([1]) are for the belief revision (revision by a single belief), and the DP postulates ([2]) are for the iterated revision (revision by a finite sequence of beliefs). Li [3] gave an *R*-calculus for *R*-configurations $\Delta|\Gamma$, where Δ is a set of literals, and Γ is a finite set of formulas. We shall give two R-calculi such that for any consistent set Γ and finite consistent set Δ of formulas in the propositional logic, in one calculus, there is a pseudo-revision Θ of Γ by Δ such that $\Delta|\Gamma \Rightarrow \Theta$ is provable and $\Theta \subseteq \Delta \cup \Gamma$; and in another calculus, there is a pre-revision Ξ of Γ by Δ such that $\Delta|\Gamma \Rightarrow \Xi$ is provable, $\Xi \vdash \Delta$ and $\Delta, \Theta \vdash \Xi$ for some pseudo-revision Θ; and prove that the deduction systems for both the R-calculi are sound and complete with the pseudo-revision and the pre-revision, respectively.

Keywords: Belief Revision; *R*-Calculus; Maximal Consistent Set; Pseudo-Revision; Pre-Revision

1. Introduction

The AGM postulates ([1],[4-6]) are for the revision $K \circ \varphi$ of a theory K by a formula φ; and the DP postulates ([2]) are for the iterated revision $(\cdots(K \circ \varphi_1)\circ\cdots)\circ\varphi_n$.

The R-calculus ([3]) gave a Gentzen-type deduction system to deduce a consistent theory $\Gamma' \cup \Delta$ from any theory $\Gamma \cup \Delta$, where $\Gamma' \cup \Delta$ should be a maximal consistent subtheory of $\Gamma \cup \Delta$ which includes Δ as a subset, where $\Delta|\Gamma$ is an R-configuration, Γ is a consistent set of formulas, and Δ is a consistent sets of literals (atomic formulas or the negation of atomic formulas). It was proved that if $\Delta|\Gamma \Rightarrow \Delta|\Gamma'$ is deducible and $\Delta|\Gamma'$ is an R-termination, *i.e.*, there is no R-rule to reduce $\Delta|\Gamma'$ to another R-configuration $\Delta|\Gamma''$, then $\Delta \cup \Gamma'$ is a pseudo-revision of Γ by Δ.

The R-calculus has the following features:

- Δ is a finite set of literals (propositional variables or the negation of propositional variables);
- Γ is a set of formulas;
- $R^{\neg}, R^{\wedge}, R^{\vee}, R^{\rightarrow}$ are not sufficient for pseudo-revision, and R^{cut} is introduced to deduce $\Delta|\Gamma$ into a consistent set Θ of formulas including Δ;
- the soundness theorem holds, that is, if $\Delta|\Gamma \Rightarrow \Theta$ is provable then Θ is a pseudo-revision of Γ by Δ; and
- the completeness theorem holds, that is, if Θ is a pseudo-revision of Γ by Δ then $\Delta|\Gamma \Rightarrow \Theta$ is provable.

Because each rule in the R-calculus consists of the statements of form

$$\Delta|\varphi, \Gamma \Rightarrow \Delta|\Gamma,$$

the R-calculus is based on pseudo-revision, *i.e.*, to contract φ from $\Delta \cup \Gamma \cup \{\varphi\}$ if $\Delta \cup \Gamma \cup \{\varphi\}$ is inconsistent, which makes the R-calculus not preserve the minimal change principle.

Given two theories Δ and Γ, a pseudo-revision Θ of Γ by Δ is a consistent subset of $\Gamma \cup \Delta$ including Δ (if $\Delta \cup \Gamma$ is inconsistent; otherwise, $\Theta = \Delta \cup \Gamma$).

We shall give two R-calculi such that

- in one R-calculus, say R_1, for any consistent formula set Δ and finite formula set Γ, there is a

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consistent formula set $\Theta \subseteq \Delta \cup \Gamma$ such that $\Delta|\Gamma \Rightarrow \Theta$ is provable and Θ is a pseudo-revision of Γ by Δ (the soundness theorem); and conversely, given any pseudo-revision Θ of Γ by $\Delta, \Delta|\Gamma \Rightarrow \Theta$ is provable (the completeness theorem);

- in another R-calculus, say R_2, for any consistent formula set Δ and finite formula set Γ, there are consistent formula sets Θ and Ξ such that
  - ◦ $\Delta|\Gamma \Rightarrow \Xi$ is provable,
  - ◦ Θ is a pseudo-revision of Γ by Δ,
  - ◦ $\Xi \vdash\dashv \Theta$ and
  - ◦ there is no subformula ξ of Ξ contradictory to Δ (the soundness theorem);

and conversely, given any pseudo-revision Θ of Γ by Δ, there is a consistent formula set Ξ such that $\Delta|\Gamma \Rightarrow \Xi$ is provable, $\Theta \vdash\dashv \Xi$ and Ξ is contradictory to no subformula ξ of Ξ (the completeness theorem).

The R-calculi are different from the R-calculus in [3] as follows:

◊ Δ is any set of formulas;

◊ The cut-rule in the R-calculus is eliminated in the R-calculi;

◊ Because $(\wedge)$-rule in the R-calculus is not sufficient for reducing

$$\Delta|\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta|\Gamma$$

to either $\Delta|\varphi_1, \Gamma \Rightarrow \Delta|\Gamma$ or $\Delta|\varphi_2, \Gamma \Rightarrow \Delta|\Gamma$ the R-calculus is not complete with respect to the pseudo-revision of Γ by Δ. In the new R-calculi, we split $(\wedge)$ into two deduction rules $(R_1^{\wedge})$ and $(R_2^{\wedge})$ according to whether φ_1 is consistent with $\Delta \cup \Gamma$ or not. The reason is given as follows.

Given a consistent theory Δ and formulas $\varphi_1, \varphi_2, \Delta \cup \{\varphi_1 \vee \varphi_2\}$ is inconsistent if and only if $\Delta \cup \{\varphi_1\}$ and $\Delta \cup \{\varphi_2\}$ are inconsistent; and if either $\Delta \cup \{\varphi_1\}$ or $\Delta \cup \{\varphi_1\}$ is inconsistent then $\Delta \cup \{\varphi_1 \wedge \varphi_2\}$ is inconsistent; and if $\Delta \cup \{\varphi_1 \wedge \varphi_2\}$ is inconsistent then we cannot deduce that either $\Delta \cup \{\varphi_1\}$ or $\Delta \cup \{\varphi_2\}$ is inconsistent, and what we have is that $\Delta \cup \{\varphi_1 \wedge \varphi_2\}$ is inconsistent if and only if either $\Delta \cup \{\varphi_1\}$ is inconsistent or $\Delta \cup \{\varphi_1, \varphi_2\}$ is inconsistent. Formally,

$$\frac{\text{incon}(\Delta, \varphi_1) \text{ or incon}(\Delta, \varphi_2)}{\text{incon}(\Delta, \varphi_1 \wedge \varphi_2)} \quad (1)$$

$$\frac{\text{incon}(\Delta, \varphi_1)\ \text{incon}(\Delta, \varphi_2)}{\text{incon}(\Delta, \varphi_1 \vee \varphi_2)} \quad (2)$$

$$\frac{\text{incon}(\Delta, \varphi_1) \text{ or incon}(\Delta \cup \{\varphi_1\}, \varphi_2)}{\text{incon}(\Delta, \varphi_1 \wedge \varphi_2)} \quad (3)$$

where $\text{con}(\Delta, \varphi)$ and $\text{incon}(\Delta, \varphi)$ denote that $\Delta \cup \{\varphi\}$ is consistent and inconsistent, respectively. Therefore, we use

$$(R_1^{\wedge})\ \frac{\Delta|\varphi_1, \Gamma \Rightarrow \Delta|\Gamma}{\Delta|\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta|\Gamma},$$

$$(R_2^{\wedge})\ \frac{\Delta, \varphi_1|\varphi_2, \Gamma \Rightarrow \Delta, \varphi_1|\Gamma}{\Delta|\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta|\Gamma}$$

in R_1 and R_2 instead of

$$\frac{\Delta|\varphi_1, \Gamma \Rightarrow \Delta|\Gamma}{\Delta|\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta|\Gamma} \quad \frac{\Delta|\varphi_2, \Gamma \Rightarrow \Delta|\Gamma}{\Delta|\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta|\Gamma}$$

in the R-calculus.

In R_1 we use a rule

$$(R^{\text{con}})\ \frac{\Delta \cup \Gamma \nvdash \neg\varphi}{\Delta|\varphi, \Gamma \Rightarrow \Delta, \varphi, \Gamma}$$

to deduce $\Delta|\varphi, \Gamma$ to Δ, φ, Γ if Δ, φ, Γ are consistent. In R_2, we shall give a deduction rule to reduce $\Delta|\varphi, \Gamma$ to the atomic cases where

$$\frac{\Delta \cup \Gamma \vdash \neg p}{\Delta|p, \Gamma \Rightarrow \Delta|\Gamma}, \quad \frac{\Delta \cup \Gamma \vdash p}{\Delta|\neg p, \Gamma \Rightarrow \Delta|\Gamma},$$

$$\frac{\Delta \cup \Gamma \nvdash \neg p}{\Delta|p, \Gamma \Rightarrow \Delta, p, \Gamma}, \quad \frac{\Delta \cup \Gamma \nvdash p}{\Delta|\neg p, \Gamma \Rightarrow \Delta, \neg p, \Gamma},$$

with a cost that we cannot prove that if $\Delta|\Gamma \Rightarrow \Xi$ is provable then Ξ is a pseudo-revision of Γ by Δ. Instead we shall prove that if $\Delta|\Gamma \Rightarrow \Xi$ is provable then Ξ is a pre-revision of Γ by Δ, that is, there is a consistent theory $\Theta \subseteq \Delta \cup \Gamma$ such that 1) $\Theta \supseteq \Delta$ is a pseudo-revision of Γ by Δ; 2) $\Theta \vdash\dashv \Xi$; and 3) no subformula ξ of Ξ is contradictory to Δ.

The paper is organized as follows: the next section gives the R-calculus in [3] and basic definitions; the third section defines an R-calculus R_1 for the pseudo-revision and proves that R_1 is sound and complete with respect to the pseudo-revision; the fourth section defines another R-calculus R_2 for the pre-revision and prove that R_2 is sound and complete with respect to the pseudo-revision, and the last section concludes the whole paper.

2. The R-Calculus

The R-calculus is defined on a first-order logical language. Let L' be a logical language of the first-order logic; $\varphi_1, \varphi_2, \varphi_3$ formulas and Γ, Δ sets of formulas (theories), where Δ is a set of atomic formulas or the negations of atomic formulas, and $\Delta|\Gamma$ is called an R-configuration.

The R-calculus consists of the following axiom and inference rules:

$$(A^{\neg})\quad \Delta,\varphi_1|\neg\varphi_1,\Gamma\Rightarrow\varphi_1,\Delta|\Gamma$$

$$(R^{\text{cut}})\quad \frac{\Gamma_1,\varphi_1\vdash\varphi_2\quad \varphi_1\mapsto_T\varphi_2 \quad \Gamma_2,\varphi_2\vdash\varphi_3\quad \Delta|\varphi_3,\Gamma_2\Rightarrow\Delta|\Gamma_2}{\Delta|\varphi_1,\Gamma_1,\Gamma_2\Rightarrow\Delta|\Gamma_1,\Gamma_2}$$

$$(R^{\wedge})\quad \frac{\Delta|\varphi_1,\Gamma\Rightarrow\Delta|\Gamma}{\Delta|\varphi_1\wedge\varphi_2,\Gamma\Rightarrow\Delta|\Gamma}$$

$$(R^{\vee})\quad \frac{\Delta|\varphi_1,\Gamma\Rightarrow\Delta|\Gamma\quad \Delta|\varphi_2,\Gamma\Rightarrow\Delta|\Gamma}{\Delta|\varphi_1\vee\varphi_2,\Gamma\Rightarrow\Delta|\Gamma}$$

$$(R^{\to})\quad \frac{\Delta|\neg\varphi_1,\Gamma\Rightarrow\Delta|\Gamma\quad \Delta|\varphi_2,\Gamma\Rightarrow\Delta|\Gamma}{\Delta|\varphi_1\to\varphi_2,\Gamma\Rightarrow\Delta|\Gamma}$$

$$(R^{\forall})\quad \frac{\Delta|\varphi[t/x],\Gamma\Rightarrow\Delta|\Gamma}{\Delta|\forall x\varphi,\Gamma\Rightarrow\Delta|\Gamma}$$

where in $R^{\text{cut}}, \varphi_1\mapsto_T\varphi_2$ means that φ_1 occurs in the proof tree T of φ_2 from Γ_1 and φ_1; and in $R^{\forall}, t$ is a term, and is free in φ for x.

The R-calculus is in the first-order logic. In the following we discuss the R-calculi in the propositional logic.

Let L be a logical language of the propositional logic which contains the following symbols:

- propositional variables: $p_0, p_1, \cdots$;
- logical connectives: $\neg,\wedge,\vee$.

Formulas are defined as follows:

$$\varphi = p \mid \neg p \mid \varphi_1\wedge\varphi_2 \mid \varphi_1\vee\varphi_2.$$

Definition 2.1. Given a consistent set Δ of formulas and a finite consistent set Γ of formulas, a consistent set Θ of formulas is a pseudo-revision of Γ by Δ if $\Theta=\Delta\cup\Gamma$ (if $\Delta\cup\Gamma$ is consistent), or (if $\Delta\cup\Gamma$ is inconsistent then) Θ satisfies the following conditions:

1) $\Theta\subseteq\Delta\cup\Gamma$,
2) $\Delta\subseteq\Theta$, and
3) there is a $\varphi\in\Gamma$ such that $\Theta\cup\{\varphi\}$ is inconsistent.

Each pseudo-revision Θ can be generated by the following procedure: given any consistent set Δ and finite consistent set Γ, assume that $\Gamma=\{\varphi_1,\cdots,\varphi_n\}$ is ordered by a linear ordering $\preceq$ (without loss of generality, assume that $\varphi_1\preceq\varphi_2\preceq\cdots\preceq\varphi_n$), define

$$\Theta_0=\Gamma\cup\Delta;$$

$$\Theta_i=\begin{cases}\Theta_{i-1}-\{\varphi_i\} & \text{if } \Theta_{i-1}\vdash\neg\varphi_i\\ \Theta_i & \text{otherwise}\end{cases}$$

Let $\Theta=\Theta_n$. Then, Θ is a subset of $\Delta\cup\Gamma$ such that $\Theta\supseteq\Delta$, and Θ is consistent.

Lemma 2.2. Θ is a pseudo-revision of Γ by Δ. Moreover, Let i_0 be the least i such that $\Theta_{i-1}-\{\varphi_i\}\nvdash\neg\varphi_i$. Then, $\Theta=\Delta\cup\{\varphi_{i_0},\varphi_{i_0+1},\cdots,\varphi_n\}$.

Definition 2.3. Given a consistent set Δ of formulas and a finite consistent set Γ of formulas, a consistent set Ξ of formulas is a pre-revision of Γ by Δ if there is a pseudo-revision Θ of Γ by Δ such that

1) $\Theta\vdash\dashv\Xi$,
2) $\Delta\subseteq\Xi$, and
3) no subformula ξ of Ξ is contradictory to Δ.

Each pre-revision Ξ can be generated by the following procedure: given any consistent set Δ and finite consistent set Γ, assume that $\Gamma=\{\varphi_1,\cdots,\varphi_n\}$, define

$$\Xi_i=\begin{cases}\Xi_{i-1}-\{\varphi_i\} & \text{if } \Xi_{i-1}\vdash\neg\varphi_i\\ (\Xi_{i-1}-\{\varphi_i\})\cup\{\varphi_i'\} & \text{otherwise}\end{cases}$$

where

$$\varphi_i'=\begin{cases}\lambda & \text{if } \varphi_i=l \text{ and } \Xi_i\vdash\neg l\\ l & \text{if } \varphi_i=l \text{ and } \Xi_i\nvdash\neg l\\ \varphi_1'\wedge\varphi_2' & \text{if } \varphi_i=\varphi_1\wedge\varphi_2\\ \varphi_1'\vee\varphi_2' & \text{if } \varphi_i=\varphi_1\vee\varphi_2 \text{ and } \operatorname{con}(\Xi_i,\varphi_1),\ \operatorname{con}(\Xi_i,\varphi_2)\\ \varphi_2' & \text{if } \varphi_i=\varphi_1\vee\varphi_2 \text{ and } \operatorname{incon}(\Xi_i,\varphi_1),\ \operatorname{con}(\Xi_i,\varphi_2)\\ \varphi_1' & \text{if } \varphi_i=\varphi_1\vee\varphi_2 \text{ and } \operatorname{con}(\Xi_i,\varphi_1),\ \operatorname{incon}(\Xi_i,\varphi_2)\end{cases}$$

where λ is the empty string.

Let $\Xi=\Xi_n$, and Θ be the pseudo-revision of Γ by Δ in the same ordering as Ξ. Then, we have the following

Lemma 2.4. Let i_0 be the least i such that $\Theta_{i-1}\nvdash\neg\varphi_i$. Then, for any $j<i_0, \Theta_j=\Xi_j$; and for any $j\geq i_0, \varphi_j'$ is a subformula of φ_j.

Lemma 2.5. Ξ is a pre-revision of Γ by Δ such that $\Xi\vdash\dashv\Theta$, and no subformula of Ξ is contradictory to Δ.

Proof. Let i_0 be the least i such that $\Theta_{i-1}\nvdash\neg\varphi_i$. Then,

$$\Xi=\Delta\cup\{\varphi_{i_0}',\varphi_{i_0+1}',\cdots,\varphi_n'\}.$$

We prove that for any i with $i_0\leq i\leq n, \Xi_i\vdash\Theta_i$ and $\Theta_i\vdash\Xi_i$ by induction on i.

Let $\Omega=\Theta_{i_0-1}-\{\varphi_{i_0}\}$ and $\psi=\varphi_{i_0}$. Then, $\Xi_{i_0}=\Omega\cup\{\psi'\}$. We prove by induction on the structure of ψ that $\Omega,\psi\vdash\psi'$ and $\Omega,\psi'\vdash\psi$.

If $\psi=l$ and $\Omega\vdash\neg l$ then $\Omega\cup\{\psi\}$ is inconsistent, a contradiction to the choice of i_0;

If $\psi=l$ and $\Omega\nvdash\neg l$ then $\psi'=\psi$, and

$\Omega,\psi \vdash\dashv \Omega,\psi';$

If $\psi=\psi_1\wedge\psi_2$ and $\Omega\cup\psi$ is consistent then $\Omega\cup\{\psi_1\}$ and $\Omega\cup\{\psi_2\}$ are consistent, and by the induction assumption,

$$\Omega,\psi_1 \vdash\dashv \Omega,\psi_1';$$

$$\Omega,\psi_2 \vdash\dashv \Omega,\psi_2',$$

and hence,

$$\Omega,\psi_1\wedge\psi_2 \vdash\dashv \Omega,\psi_1'\wedge\psi_2';$$

If $\psi=\psi_1\vee\psi_2$ and $\Omega\cup\psi$ is consistent then either $\Omega\cup\{\psi_1\}$ or $\Omega\cup\{\psi_2\}$ is consistent.

If $\Omega\cup\{\psi_1\}$ and $\Omega\cup\{\psi_2\}$ are consistent. then by the induction assumption,

$$\Omega,\psi_1 \vdash\dashv \Omega,\psi_1';$$

$$\Omega,\psi_2 \vdash\dashv \Omega,\psi_2',$$

and hence, $\Omega,\psi_1\vee\psi_2 \vdash\dashv \Omega,\psi_1'\vee\psi_2';$

If $\Omega\cup\{\psi_1\}$ is inconsistent and $\Omega\cup\{\psi_2\}$ is consistent. then $\Omega\,|\,\psi_1\Rightarrow\Omega,[\lambda]$ and by the induction assumption, $\Omega,\psi_2 \vdash\dashv \Omega,\psi_2',$ and hence, $\Omega,\psi_1\vee\psi_2 \vdash\dashv \Omega,\psi_2',$ because $\Omega,\psi_2\vdash\psi_2',$ and $\Omega,\psi_1\vdash\psi_2'$ ($\Omega\cup\{\psi_1\}$ is inconsistent, and hence, for any formula $\theta,\Omega,\psi_1\vdash\theta$).

Similar for the case that $\Omega\cup\{\psi_1\}$ is consistent and $\Omega\cup\{\psi_2\}$ is inconsistent.

Similarly we can prove that for any i with $i_0<i\le n,\Xi_i \vdash\dashv \Theta_i.$

3. The R-Calculus R_1

In this section we give an R-calculus R_1 which is sound and complete with respect to the pseudo-revision, where the decision of whether $\Delta\cup\{\varphi\}\cup\Gamma$ is consistent is needed so that if $\Delta\cup\{\varphi\}\cup\Gamma$ is consistent then $\Delta|\varphi,\Gamma\Rightarrow\Delta,\varphi,\Gamma$ is provable; otherwise, $\Delta|\varphi,\Gamma\Rightarrow\Delta|\Gamma$ is provable.

Let Δ,Γ be any consistent sets of formulas.

Definition 3.1. $t=\Delta|\Gamma$ is a term; and $t\Rightarrow t'$ is a statement, where $t=\Delta|\Gamma$ and $t'=\Delta|\Gamma'$; and $\frac{S_1,\cdots,S_n}{S}$ is a deduction rule, where $S_1,\cdots,S_n,S$ are statements.

R_1 has the following deduction rules:

$$\left(R^{\text{con}}\right)\ \frac{\Delta\cup\Gamma\nvdash\neg\varphi}{\Delta|\varphi,\Gamma\Rightarrow\Delta,\varphi,\Gamma}$$

$$\left(R_1^{\neg}\right)\ \frac{\Delta\cup\Gamma\vdash p}{\Delta|\neg p,\Gamma\Rightarrow\Delta|\Gamma}$$

$$\left(R_2^{\neg}\right)\ \frac{\Delta\cup\Gamma\vdash\neg p}{\Delta|p,\Gamma\Rightarrow\Delta|\Gamma}$$

$$\left(R_1^{\wedge}\right)\ \frac{\Delta|\varphi_1,\Gamma\Rightarrow\Delta|\Gamma}{\Delta|\varphi_1\wedge\varphi_2,\Gamma\Rightarrow\Delta|\Gamma}$$

$$\left(R_2^{\wedge}\right)\ \frac{\Delta,\varphi_1|\varphi_2,\Gamma\Rightarrow\Delta,\varphi_1|\Gamma}{\Delta|\varphi_1\wedge\varphi_2,\Gamma\Rightarrow\Delta|\Gamma}$$

$$\left(R^{\vee}\right)\ \frac{\Delta|\varphi_1,\Gamma\Rightarrow\Delta|\Gamma\ \ \Delta|\varphi_2,\Gamma\Rightarrow\Delta|\Gamma}{\Delta|\varphi_1\vee\varphi_2,\Gamma\Rightarrow\Delta|\Gamma}$$

Definition 3.2. $\Delta|\Gamma\Rightarrow\Theta$ is provable if there is a sequence

$$\{\Delta_1|\Gamma_1\Rightarrow\Delta_1'|\Gamma_1',\cdots,\Delta_n|\Gamma_n\Rightarrow\Delta_n'|\Gamma_n'\}$$

of statements such that

1) $\Delta_1|\Gamma_1=\Delta|\Gamma;$

2) $\Delta_n'|\Gamma_n'=\Theta,$ and

3) for each $i\le n,\Delta_i|\Gamma_i\Rightarrow\Delta_i'|\Gamma_i'$ is either an axiom or deduced from the previous statements by the deduction rules.

For example, the following

$$(1)\ \neg p\vee\neg q,p\vdash\neg q$$

$$(2)\ \neg p\vee\neg q,p\,|\,q\Rightarrow\neg p\vee\neg q,p\,|\qquad\left((1),R_2^{\neg}\right)$$

$$(3)\ \neg p\vee\neg q\,|\,p\wedge q\Rightarrow\neg p\vee\neg q\,|\qquad\left((2),R_2^{\wedge}\right)$$

is a proof and so $\neg p\vee\neg q\,|\,p\wedge q\Rightarrow\neg p\vee\neg q$ is provable.

Also, the following

$$(1)\ \neg p\wedge\neg q\vdash\neg p$$

$$(2)\ \neg p\wedge\neg q\,|\,p\Rightarrow\neg p\wedge\neg q\,|\qquad\left(R_1^{\neg}\right)$$

$$(3)\ \neg p\wedge\neg q\vdash\neg q$$

$$(4)\ \neg p\wedge\neg q\,|\,q\Rightarrow\neg p\wedge\neg q\,|\qquad\left(R_1^{\neg}\right)$$

$$(5)\ \neg p\wedge\neg q\,|\,p\vee q\Rightarrow\neg p\wedge\neg q\,|\qquad\left((2),(4),R_1^{\wedge}\right)$$

is a proof and so $\neg p\wedge\neg q\,|\,p\vee q\Rightarrow\neg p\wedge\neg q$ is provable.

Theorem 3.3. For any consistent sets Γ,Δ of formulas and formula φ, if $\Delta\,|\,\varphi,\Gamma\Rightarrow\Delta\,|\,\Gamma$ is provable then $\Delta\cup\{\varphi\}\cup\Gamma$ is inconsistent; and if $\Delta\,|\,\varphi,\Gamma\Rightarrow\Delta,\varphi,\Gamma$ is provable then $\Delta\cup\{\varphi\}\cup\Gamma$ is consistent.

Proof. If $\Delta\,|\,\varphi,\Gamma\Rightarrow\Delta,\varphi,\Gamma$ is provable then $\left(R^{\text{con}}\right)$ is used and $\Delta\cup\{\varphi\}\cup\Gamma$ is consistent.

If $\Delta\,|\,\varphi,\Gamma\Rightarrow\Delta\,|\,\Gamma$ is provable then we prove that $\Delta\cup\Gamma\vdash\neg\varphi,$ *i.e.*, $\Delta\cup\Gamma\cup\{\varphi\}$ is inconsistent, by the induction on the length of a proof of $\Delta\,|\,\varphi,\Gamma\Rightarrow\Delta\,|\,\Gamma$ and the cases that the last inference rule is used.

If the last rule used is $R_1^{\neg}$ then $\varphi=\neg p,$ and $\Delta\cup\Gamma\vdash p,$ *i.e.*, $\Delta\cup\Gamma\vdash\neg\varphi;$

If the last rule used is $R_2^{\neg}$ then $\varphi=p,$ and $\Delta\cup\Gamma\vdash\neg p,$ *i.e.*, $\Delta\cup\Gamma\vdash\neg\varphi;$

If the last rule used is $R_1^{\wedge}$ then $\varphi=\varphi_1\wedge\varphi_2$, and $\Delta\,|\,\varphi_1,\Gamma\Rightarrow\Delta\,|\,\Gamma$. By the induction assumption, $\Delta\cup\Gamma\vdash\neg\varphi_1$, and hence, $\Delta\cup\Gamma\vdash\neg\varphi_1\vee\neg\varphi_2,$ *i.e.*, $\Delta\cup\Gamma\vdash\neg\varphi;$

If the last rule used is $R_2^{\wedge}$ then $\varphi=\varphi_1\wedge\varphi_2,$ and

$\Delta,\varphi_1 \mid \varphi_2,\Gamma \Rightarrow \Delta,\varphi_1 \mid \Gamma$. By the induction assumption, $\Delta\cup\Gamma\cup\{\varphi_1\}\vdash\neg\varphi_2$, and hence, $\Delta\cup\Gamma\vdash\neg\varphi_1\vee\neg\varphi_2$, *i.e.*, $\Delta\cup\Gamma\vdash\neg\varphi$;

If the last rule used is $R_1^{\vee}$ then $\varphi=\varphi_1\vee\varphi_2$, and

$$\Delta \mid \varphi_1,\Gamma \Rightarrow \Delta \mid \Gamma,$$

$$\Delta \mid \varphi_2,\Gamma \Rightarrow \Delta \mid \Gamma.$$

By the induction assumption, $\Delta\cup\Gamma\vdash\neg\varphi_1$, $\Delta\cup\Gamma\vdash\neg\varphi_2$, and hence, $\Delta\cup\Gamma\vdash\neg\varphi_1\wedge\neg\varphi_2$, *i.e.*, $\Delta\cup\Gamma\vdash\neg\varphi$.

Theorem 3.4. For any consistent sets Γ,Δ of formulas and formula φ, if $\Delta\cup\{\varphi\}\cup\Gamma$ is inconsistent then $\Delta \mid \varphi,\Gamma \Rightarrow \Delta \mid \Gamma$ is provable; and if $\Delta\cup\{\varphi\}\cup\Gamma$ is consistent then $\Delta \mid \varphi,\Gamma \Rightarrow \Delta,\varphi,\Gamma$ is provable.

Proof. If φ is consistent with $\Delta\cup\Gamma$ then by $\left(R^{\text{con}}\right)$, $\Delta \mid \varphi,\Gamma \Rightarrow \Delta,\varphi,\Gamma$ is provable;

Assume that φ is inconsistent with Δ. We prove by the induction on the structure of φ that $\Delta \mid \varphi,\Gamma \Rightarrow \Delta \mid \Gamma$ is provable.

If $\varphi = p$ then $\Delta\cup\Gamma\vdash\neg p$ and by $\left(R_1^{\neg}\right)$, $\Delta \mid p, \Gamma \Rightarrow \Delta \mid \Gamma$ is provable.

If $\varphi=\neg p$ then $\Delta\cup\Gamma\vdash p$ and by $\left(R_2^{\neg}\right)$, $\Delta \mid \neg p, \Gamma \Rightarrow \Delta \mid \Gamma$ is provable.

If $\varphi=\varphi_1\wedge\varphi_2$ then there are two subcases: φ_1 is inconsistent with $\Delta\cup\Gamma$, or φ_2 is consistent with $\Delta\cup\{\varphi_1\}\cup\Gamma$. In the first subcase, by the induction assumption, $\Delta \mid \varphi_1,\Gamma \Rightarrow \Delta \mid \Gamma$ is provable, and by $\left(R_1^{\wedge}\right)$, $\Delta \mid \varphi_1\wedge\varphi_2,\Gamma \Rightarrow \Delta \mid \Gamma$ is provable; and in the second subcase, $\Delta\cup\{\varphi_1\}\cup\Gamma$ is consistent and $\Delta\cup\{\varphi_1,\varphi_2\}\cup\Gamma$ is inconsistent. By the induction assumption, $\Delta,\varphi_1 \mid \varphi_2, \Gamma \Rightarrow \Delta,\varphi_1 \mid \Gamma$ is provable, and by $\left(R_2^{\wedge}\right), \Delta \mid \varphi_1\wedge\varphi_2,\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta,\Gamma.$

If $\varphi=\varphi_1\vee\varphi_2$ then both $\Delta\cup\{\varphi_1\}\cup\Gamma$ and $\Delta\cup\{\varphi_2\}\cup\Gamma$ are inconsistent. By the induction assumption, both $\Delta \mid \varphi_1,\Gamma \Rightarrow \Delta \mid \Gamma$ and $\Delta \mid \varphi_2,\Gamma \Rightarrow \Delta \mid \Gamma$ are provable, and by $\left(R^{\vee}\right), \Delta \mid \varphi_1\vee\varphi_2,\Gamma \Rightarrow \Delta \mid \Gamma$ is provable.

Theorem 3.5. For any consistent sets Γ,Δ of formulas, if Γ is finite then there is a set $\Theta\subseteq\Gamma$ of formulas such that $\Delta \mid \Gamma \Rightarrow \Theta$ is provable

Proof. Let $\Gamma=\{\varphi_1,\cdots,\varphi_n\}$.

We prove the theorem by the induction on n.

If $n=1$ then by theorem 3.3, let

$$\Theta=\begin{cases}\Delta & \text{if } \Delta\cup\{\varphi_1\} \text{ is inconsistent}\\ \Delta\cup\{\varphi_1\} & \text{otherwise}\end{cases}$$

and Θ satisfies the theorem.

Assume that the theorem holds for $n=k$, that is, there is a set Θ such that $\Delta \mid \Gamma \Rightarrow \Theta$ is provable. Let $n=k+1$.

If φ_{k+1} is consistent with Θ then $\Delta \mid \Gamma \Rightarrow \Theta'$ is provable, where $\Theta'=\Theta\cup\{\varphi_{k+1}\}$;

If φ_{k+1} is inconsistent with Θ then $\Delta \mid \Gamma\cup\{\varphi_{k+1}\} \Rightarrow \Theta$, because the last formula φ_{k+1} is inconsistent with Θ.

Theorem 3.6 (The soundness theorem for Γ). If $\Delta \mid \Gamma \Rightarrow \Theta$ is provable then Θ is a pseudo-revision of Γ by Δ.

Proof. Firstly we prove that if $\Delta \mid \varphi \Rightarrow \Theta$ is provable then Θ is a pseudo-revision of φ by Δ.

Assume that $\Delta \mid \varphi \Rightarrow \Theta$ is provable.

If $\Theta=\Delta\cup\{\varphi\}$ then Δ is consistent with φ, and Θ is a pseudo-revision of φ by Δ.

If $\Theta=\Delta$ then Δ is inconsistent with φ, $\Delta \mid \varphi \Rightarrow \Delta$ is provable, and Θ is a pseudo-revision of φ by Δ.

Similarly, by the induction on the number of formulas in Γ, we can prove that if $\Delta \mid \Gamma \Rightarrow \Theta$ then Θ is a pseudo-revision of Γ by Δ.

Theorem 3.7 (The completeness theorem for Γ). If Θ is a pseudo-revision of Γ by Δ then $\Delta \mid \Gamma \Rightarrow \Theta$ is provable.

Proof. Let Θ be a pseudo-revision of Γ by Δ under the ordering $\varphi_1,\cdots,\varphi_n$ of Γ.

We prove by induction on $i<n$ that there is a formula set Θ_i such that $\Theta_i \mid \varphi_i,\Gamma_{i+1} \Rightarrow \Theta_{i+1} \mid \Gamma_{i+1}$ is provable, where $\Theta_0=\Delta$, and $\Gamma_{i+1}=\{\varphi_{i+1},\cdots,\varphi_n\}$.

If $\Theta_i\cup\{\varphi_i\}\cup\Gamma_{i+1}$ is consistent then let $\Theta_{i+1}=\Delta\cup\{\varphi_i\}$, and $\Theta_i \mid \varphi_i,\Gamma_{i+1} \Rightarrow \Theta_{i+1} \mid \Gamma_{i+1} \Rightarrow \Theta_{i+1},\Gamma_{i+1}\equiv\Theta$ is provable, where $\Theta'=\Theta_{i+1}\cup\Gamma_{i+1}$.

Assume that $\Theta_i\cup\{\varphi_i\}\cup\Gamma_{i+1}$ is inconsistent. Then, $\Theta_i\cup\Gamma_{i+1}\vdash\neg\varphi_i$, and let $\Theta_{i+1}=\Theta_i$, by theorem 3.4, $\Theta \mid \varphi_i,\Gamma_{i+1} \Rightarrow \Theta_{i+1} \mid \Gamma_{i+1}$ is provable.

Let $\Theta=\Theta_n$. Then, $\Delta,\Gamma \Rightarrow \Theta$ is provable.

4. The R-Calculus R_2

In this section we give an R-calculus R_2 which is sound and complete with respect to the pre-revision, where the decision of whether $\Delta\cup\{\varphi\}\cup\Gamma$ is consistent is deduced by a set of λ-rules.

R_1 is used to reduce $\Delta \mid \varphi,\Gamma$ to $\Delta \mid \Gamma$ when $\Delta\cup\{\varphi\}\cup\Gamma$ is inconsistent. When $\Delta\cup\{\varphi\}\cup\Gamma$ is consistent, there are subformulas in φ which is inconsistent with Δ, we hope to reduce those subformulas into the empty string. For example, let

$$\Delta=\{\neg p,\neg q\},$$

$$\Gamma=\{p,(q\wedge r)\vee s\}.$$

Then, by R_1 we have the following reduction:

$$\Delta \mid \Gamma \Rightarrow \neg p,\neg q \mid (q\wedge r)\vee s$$

$$\Rightarrow \neg p,\neg q,(q\wedge r)\vee s;$$

and by R_2 we shall have the following one:

$$\Delta \mid \Gamma \Rightarrow \neg p,\neg q \mid (q\wedge r)\vee s$$

$$\Rightarrow \neg p, \neg q, r \vee s.$$

For the two reductions, we have

$$\neg p, \neg q, r \vee s \vdash\dashv \neg p, \neg q, (q \wedge r) \vee s.$$

Let Δ be a consistent set of formulas and Γ a finite consistent set of formulas.

R_2 consists of two parts: R_1, which we use to decompose formula φ in Γ if $\Delta \cup \{\varphi\} \cup \Gamma$ is inconsistent; and λ-deduction rules, which we use to decompose φ if $\Delta \cup \{\varphi\} \cup \Gamma$ is consistent.

R_2 has the following λ-deduction rules to reduce $\Delta | \varphi, \Gamma$ when $\Delta \cup \{\varphi\} \cup \Gamma$ is consistent:

$$(\lambda_1^{\text{con}}) \quad \frac{\Delta \cup \Gamma \nvdash \neg p}{\Delta | p, \Gamma \Rightarrow \Delta, [p], \Gamma}$$

$$(\lambda_2^{\text{con}}) \quad \frac{\Delta \cup \Gamma \nvdash p}{\Delta | \neg p, \Gamma \Rightarrow \Delta, [\neg p], \Gamma}$$

$$(\lambda^{\wedge}) \quad \frac{\Delta | \varphi_1, \Gamma \Rightarrow \Delta, [\theta_1] | \Gamma \quad \Delta, [\theta_1] | \varphi_2, \Gamma \Rightarrow \Delta, [\theta_1], [\theta_2] | \Gamma}{\Delta | \varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta, [\theta_1 \wedge \theta_2] | \Gamma}$$

$$(\lambda^{\vee}) \quad \frac{\Delta | \varphi_1, \Gamma \Rightarrow \Delta, [\theta_1] | \Gamma \quad \Delta | \varphi_2, \Gamma \Rightarrow \Delta, [\theta_2] | \Gamma}{\Delta | \varphi_1 \vee \varphi_2, \Gamma \Rightarrow \Delta, [\theta_1 \vee \theta_2] | \Gamma}$$

where if θ is consistent then

$$\lambda \vee \theta \equiv \theta \vee \lambda \equiv \theta, \lambda \wedge \theta \equiv \theta \wedge \lambda \equiv \theta, \Delta, \lambda \equiv \Delta;$$

and if θ is inconsistent then

$$\lambda \vee \theta \equiv \theta \vee \lambda \equiv \lambda$$
$$\lambda \wedge \theta \equiv \theta \wedge \lambda \equiv \lambda.$$

The deductions for the inconsistent $\Delta \cup \{\varphi\} \cup \Gamma$ are the same as in R_1 minus (R^{con}).

Definition 4.1. $\Delta | \Gamma \Rightarrow \Xi$ is provable if there is a sequence

$$\{\Delta_1 | \Gamma_1 \Rightarrow \Delta_1' | \Gamma_1', \cdots, \Delta_n | \Gamma_n \Rightarrow \Delta_n' | \Gamma_n'\}$$

of statements such that

1) $\Delta_1 | \Gamma_1 = \Delta | \Gamma$;
2) $\Delta_n' | \Gamma_n' = \Xi$, and
3) for each $i \leq n, \Delta_i | \Gamma_i \Rightarrow \Delta_i' | \Gamma_i'$ is either an axiom or deduced from the previous statements by the deduction rules.

We call the sequence a proof of statement $\Delta | \Gamma \Rightarrow \Xi$.

For example, the following

(1) $\neg p \wedge \neg q \vdash \neg p$

(2) $\neg p \wedge \neg q | p \Rightarrow \neg p \vee \neg q, [\lambda]$ $\quad ((1), (\lambda_1^{\text{con}}))$

(3) $\neg p \wedge \neg q \nvdash \neg r$

(4) $\neg p \wedge \neg q | r \Rightarrow \neg p \vee \neg q, [r]$ $\quad ((3), (\lambda_2^{\neg}))$

(5) $\neg p \wedge \neg q | p \vee r \Rightarrow \neg p \vee \neg q, [\lambda \vee r]$ $\quad ((4), (\lambda^{\vee}))$

is a proof and $\neg p \wedge \neg q | p \vee r \Rightarrow \neg p \wedge \neg q, r$ is provable.

Theorem 4.2. For any consistent sets Γ, Δ of formulas and formula φ, if $\Delta | \varphi, \Gamma \Rightarrow \Delta | \Gamma$ is provable then $\Delta \cup \{\varphi\} \cup \Gamma$ is inconsistent; and if there is a formula $\theta \not\equiv \lambda$ such that $\Delta | \varphi, \Gamma \Rightarrow \Delta, [\theta] | \Gamma$ is provable then $\Delta \cup \{\varphi\} \cup \Gamma$ is consistent.

Proof. If $\Delta | \varphi, \Gamma \Rightarrow \Delta | \Gamma$ is provable then similar to the proof of theorem 3.3, $\Delta \cup \{\varphi\} \cup \Gamma$ is inconsistent.

Assume that there is a formula $\theta \not\equiv \lambda$ such that $\Delta | \varphi, \Gamma \Rightarrow \Delta, [\theta] | \Gamma$ is provable. We prove by the induction on the length of a proof of $\Delta | \varphi, \Gamma \Rightarrow \Delta, [\theta] | \Gamma$ and the cases that the last inference rule is used that $\Delta \cup \{\varphi\} \cup \Gamma$ is consistent.

If the last rule used is λ_1^{con} then $\varphi = p, \Delta \cup \Gamma \nvdash \neg p$, and $\Delta | p, \Gamma \Rightarrow \Delta, [p], \Gamma$ is provable, where $\theta = p \neq \lambda$. Hence, $\Delta \cup \Gamma \cup \{p\}$ is consistent.

If the last rule used is λ_2^{con} then $\varphi = p, \Delta \cup \Gamma \nvdash p$, and $\Delta | \neg p, \Gamma \Rightarrow \Delta, [\neg p], \Gamma$ is provable, where $\theta = \neg p = \varphi$. Hence, $\Delta \cup \Gamma \cup \{\neg p\}$ is consistent.

If the last rule used is $\lambda^{\wedge}$ then $\varphi = \varphi_1 \wedge \varphi_2$, and there are formulas θ_1, θ_2 such that

$$\Delta | \varphi_1, \Gamma \Rightarrow \Delta, [\theta_1] | \Gamma,$$

and

$$\Delta, [\theta_1] | \varphi_2, \Gamma \Rightarrow \Delta, [\theta_1], [\theta_2] | \Gamma.$$

By the induction assumption, if $\theta_1 \neq \lambda$ and $\theta_2 \neq \lambda$ then $\Delta \cup \{\varphi_1\} \cup \Gamma$ is consistent and $\Delta \cup \{\theta_1, \varphi_2\} \cup \Gamma$ is consistent, and therefore, $\Delta \cup \{\varphi_1 \wedge \varphi_2\} \cup \Gamma$ is consistent.

If the last rule used is $\lambda^{\vee}$ then $\varphi = \varphi_1 \vee \varphi_2$, and

$$\Delta | \varphi_1, \Gamma \Rightarrow \Delta, [\theta_1] | \Gamma,$$

$$\Delta | \varphi_2, \Gamma \Rightarrow \Delta, [\theta_2] | \Gamma,$$

where either $\theta_1 \neq \lambda$ or $\theta_2 \neq \lambda$.

If $\theta_1 \neq \lambda$ and $\theta_2 \neq \lambda$ then by the induction assumption, $\Delta \cup \{\varphi_1\} \cup \Gamma$ and $\Delta \cup \{\varphi_2\} \cup \Gamma$ are consistent, and so is $\Delta \cup \{\varphi_1 \vee \varphi_2\} \cup \Gamma$.

If $\theta_1 \neq \lambda$ and $\theta_2 \neq \lambda$ then by the induction assumption, $\Delta \cup \{\varphi_1\} \cup \Gamma$ is consistent, and so is $\Delta \cup \{\varphi_1 \vee \varphi_2\} \cup \Gamma$.

If $\theta_1 \neq \lambda$ and $\theta_2 \neq \lambda$ then by the induction assumption, $\Delta \cup \{\varphi_2\} \cup \Gamma$ is consistent, and so is $\Delta \cup \{\varphi_1 \vee \varphi_2\} \cup \Gamma$.

By the proof of the theorem, we have

$$\theta = \begin{cases} \lambda & \text{if } \varphi = l \text{ and } \Delta, \Gamma \vdash \neg l \\ l & \text{if } \varphi = l \text{ and } \Delta, \Gamma \nvdash \neg l \\ \theta_1 \wedge \theta_2 & \text{if } \varphi = \varphi_1 \wedge \varphi_2 \\ \theta_1 \vee \theta_2 & \text{if } \varphi = \varphi_1 \vee \varphi_2 \text{ and } \operatorname{con}(\Delta, \Gamma, \varphi_1), \\ & \qquad \operatorname{con}(\Delta, \Gamma, \varphi_2) \\ \theta_2 & \text{if } \varphi = \varphi_1 \vee \varphi_2 \text{ and } \operatorname{incon}(\Delta, \Gamma, \varphi_1), \\ & \qquad \operatorname{con}(\Delta, \Gamma, \varphi_2) \\ \theta_1 & \text{if } \varphi = \varphi_1 \vee \varphi_2 \text{ and } \operatorname{con}(\Delta, \Gamma, \varphi_1), \\ & \qquad \operatorname{incon}(\Delta, \Gamma, \varphi_2) \end{cases}$$

Theorem 4.3. For any formula sets Γ,Δ and formula φ, if $\Gamma\cup\Delta\cup\{\varphi\}$ is consistent then $\Delta,\varphi,\Gamma\vdash\dashv\Delta,\theta,\Gamma.$

Proof. We prove the theorem by the induction on the structure of φ, Assume that $\Gamma\cup\Delta\nvdash\neg\varphi.$

If $\varphi=l$ then $\Gamma\cup\Delta\nvdash l,$ and $\theta=l.$ Hence,

$$\Delta,\varphi,\Gamma\vdash\dashv\Delta,\theta,\Gamma.$$

If $\varphi=\varphi_1\wedge\varphi_2$ then $\Gamma\cup\Delta\nvdash\neg\varphi_1,$ and $\Gamma\cup\Delta\nvdash\neg\varphi_2.$ By the induction assumption,

$$\Delta,\varphi_1,\Gamma\vdash\dashv\Delta,\theta_1,\Gamma,$$

$$\Delta,\varphi_2,\Gamma\vdash\dashv\Delta,\theta_1,\Gamma.$$

Hence, we have

$$\Delta,\varphi_1\wedge\varphi_2,\Gamma\vdash\dashv\Delta,\theta_1\wedge\theta_2,\Gamma.$$

If $\varphi=\varphi_1\vee\varphi_2$ then either $\Gamma\cup\Delta\cup\{\varphi_1\}$ is consistent or $\Gamma\cup\Delta\cup\{\varphi_2\}$ is consistent.

If $\Gamma\cup\Delta\cup\{\varphi_1\}$ and $\Gamma\cup\Delta\cup\{\varphi_2\}$ are consistent then $\Gamma\cup\Delta\nvdash\neg\varphi_1,$ and $\Gamma\cup\Delta\nvdash\neg\varphi_2.$ By the induction assumption,

$$\Delta,\varphi_1,\Gamma\vdash\dashv\Delta,\theta_1,\Gamma,$$

$$\Delta,\varphi_2,\Gamma\vdash\dashv\Delta,\theta_1,\Gamma.$$

Hence, we have

$$\Delta,\varphi_1\vee\varphi_2,\Gamma\vdash\dashv\Delta,\theta_1\vee\theta_2,\Gamma.$$

If $\Gamma\cup\Delta\cup\{\varphi_1\}$ is inconsistent and $\Gamma\cup\Delta\cup\{\varphi_2\}$ is consistent then $\Gamma\cup\Delta\nvdash\neg\varphi_2.$ By the induction assumption, $\Delta,\varphi_2,\Gamma\vdash\dashv\Delta,\theta_2,\Gamma.$ Hence, by Lemma 2.5, we have

$$\begin{aligned}\Delta,\varphi_1\vee\varphi_2,\Gamma&\vdash\dashv\Delta,\lambda\vee\theta_2,\Gamma\\&\vdash\dashv\Delta,\theta_2,\Gamma\\&\vdash\dashv\Delta,\theta,\Gamma.\end{aligned}$$

$$\Delta,\varphi_1\vee\varphi_2,\Gamma\vdash\dashv\Delta,\lambda\vee\theta_2,\Gamma.$$

If $\Gamma\cup\Delta\cup\{\varphi_1\}$ is consistent and $\Gamma\cup\Delta\cup\{\varphi_2\}$ is inconsistent then $\Gamma\cup\Delta\nvdash\neg\varphi_1.$ By the induction assumption, $\Delta,\varphi_1,\Gamma\vdash\dashv\Delta,\theta_1,\Gamma.$ Hence, by Lemma 2.5, we have

$$\begin{aligned}\Delta,\varphi_1\vee\varphi_2,\Gamma&\vdash\dashv\Delta,\theta_1\vee\lambda,\Gamma\\&\vdash\dashv\Delta,\theta_1,\Gamma\\&\vdash\dashv\Delta,\theta,\Gamma.\end{aligned}$$

Theorem 4.4. For any consistent sets Γ,Δ of formulas and formula φ, if $\Delta\cup\{\varphi\}\cup\Gamma$ is inconsistent then $\Delta\mid\varphi,\Gamma\Rightarrow\Delta\mid\Gamma$ is provable; and if $\Delta\cup\{\varphi\}\cup\Gamma$ is consistent then there is a formula $\theta\neq\lambda$ such that $\Delta\mid\varphi,\Gamma\Rightarrow\Delta,[\theta]\mid\Gamma$ is provable.

Proof. If φ is inconsistent with $\Delta\cup\Gamma$ then similar to theorem 3.5, $\Delta\mid\varphi,\Gamma\Rightarrow\Delta\mid\Gamma$ is provable.

Assume that φ is consistent with $\Delta\cup\Gamma.$ We prove the theorem by the induction on the structure of $\varphi.$

If $\varphi=p$ then $\Delta\cup\Gamma\nvdash\neg p$ and by $(\lambda_1^{\rm con}),\Delta\mid p,$ $\Gamma\Rightarrow\Delta,[p],\Gamma$ is provable, where $\theta=p.$

If $\varphi=\neg p$ then $\Delta\cup\Gamma\nvdash p$ and by $(\lambda_2^{\rm con}),\Delta\mid\neg p,$ $\Gamma\Rightarrow\Delta,[\neg p],\Gamma$ is provable, where $\theta=\neg p.$

If $\varphi=\varphi_1\wedge\varphi_2$ then φ_1 is consistent with $\Delta\cup\Gamma,$ and φ_2 is consistent with $\Delta\cup\{\varphi_1\}\cup\Gamma.$ By the induction assumption, there are formulas θ_1,θ_2 such that $\Delta\mid\varphi_1,$ $\Gamma\Rightarrow\Delta,[\theta_1]\mid\Gamma$ and $\Delta,[\theta_1]\mid\varphi_2,\Gamma\Rightarrow\Delta,[\theta_1],[\theta_2]\mid\Gamma$ are provable. By $(\lambda^\wedge),$ we have

$$\Delta\mid\varphi_1\wedge\varphi_2,\Gamma\Rightarrow\Delta,[\theta_1\wedge\theta_2]\mid\Gamma$$

is provable, where $\theta=\theta_1\wedge\theta_2.$

If $\varphi=\varphi_1\vee\varphi_2$ then either $\Delta\cup\{\varphi_1\}\cup\Gamma$ or $\Delta\cup\{\varphi_2\}\cup\Gamma$ is consistent. By the induction assumption, if $\Delta\cup\{\varphi_1\}\cup\Gamma$ is consistent then there is a formula $\theta_1\not\equiv\lambda$ such that $\Delta\mid\varphi_1,\Gamma\Rightarrow\Delta,[\theta_1]\mid\Gamma;$ and if $\Delta\cup\{\varphi_2\}\cup\Gamma$ is consistent then there is a formula $\theta_2\not\equiv\lambda$ such that $\Delta\mid\varphi_2,$ $\Gamma\Rightarrow\Delta,[\theta_2]\mid\Gamma.$ Then, by $(\lambda^\vee),\Delta\mid\varphi_1\vee\varphi_2,\Gamma\Rightarrow\Delta,[\theta_1'\vee\theta_2'],\Gamma$ is provable, where

$$\theta_1'\vee\theta_2'=\begin{cases}\theta_1\vee\theta_2 & \text{if both } \varphi_1,\varphi_2 \text{ are consistent with } \Delta\cup\Gamma\\ \theta_1\vee\lambda & \text{if only } \varphi_1 \text{ is consistent with } \Delta\cup\Gamma\\ \lambda\vee\theta_2 & \text{if only } \varphi_2 \text{ is consistent with } \Delta\cup\Gamma\end{cases}$$

Remark. In fact, in theorem 4.3, if $\Delta\cup\{\varphi\}\cup\Gamma$ is consistent then there is a formula $\theta\neq\lambda$ such that $\Delta\mid\phi,\Gamma\Rightarrow\Delta,[\theta],\Gamma$ is provable.

By Theorem 4.3, we have the following

Theorem 4.5. (The soundness theorem for Γ). If $\Delta\mid\Gamma\Rightarrow\Xi$ is provable then Ξ is a pre-revision of Γ by $\Delta.$

Proof. We only prove that no subformula ξ of Ξ is contradictory to $\Delta.$

Assume that there is a subformula ξ of some formula θ in Ξ such that $\Delta\vdash\neg\xi.$ Let $\Gamma'=\{\varphi_{i+1},\cdots,\varphi_n\}\subseteq\Gamma$ such that $\varphi_i=\varphi.$

If $\Delta\cup\Gamma'\cup\{\varphi\}$ is inconsistent then $\theta=\lambda,$ a contradiction.

If $\Delta\cup\Gamma'\cup\{\varphi\}$ is consistent then by Lemma 3.5,

$$\Delta,\varphi,\Gamma'\vdash\dashv\Delta,\theta,\Gamma',$$

and for any subformula ξ of $\theta,$ if $\Delta,\Gamma'\vdash\neg\xi$ then, by the definition of $\theta,$ ξ is replaced by λ in $\theta,$ a contradiction to the assumption that ξ is a subformula of $\theta.$

Theorem 4.6. (The completeness theorem for Γ). If Ξ is a pre-revision of Γ by Δ then $\Delta\mid\Gamma\Rightarrow\Xi$ is provable.

Proof. The proof is similar to theorem 3.7 and omitted.

5. Conclusion

This paper gave two R-calculi which are sound and complete with respect to the pseudo-revision and pre-revision, respectively. The calculi are of Gentzen-type, in which each statement is of form $\Delta | \varphi, \Gamma \Rightarrow \Delta | \Gamma'$. Different orderings of Γ give different results of revision $\Delta | \Gamma$. Correspondingly, if $\Delta | \Gamma'$ is irreducible, that is, no deduction rule can be used to reduce $\Delta | \Gamma'$, then Γ' may be a minimal change of Γ by Δ. A further work is to give an R-calculus such that if $\Delta | \Gamma \Rightarrow \Delta | \Gamma'$ is irreducible then $\Delta \cup \Gamma'$ is consistent and Γ' is a minimal change of Γ by Δ, that is, for any Γ'' with $\Gamma' \subset \Gamma'' \subseteq \Gamma, \Delta \cup \Gamma''$ is inconsistent.

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