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Collinear Libration Points in the Photogravitational CR3BP with Zonal Harmonics and Potential from a Belt

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Abstract

We have studied a reformed type of the classic restricted three-body problem where the bigger primary is radiating and the smaller primary is oblate; and they are encompassed by a homogeneous circular cluster of material points centered at the mass center of the system (belt). In this dynamical model, we have derived the equations that govern the motion of the infinitesimal mass under the effects of oblateness up to the zonal harmonics J_4 of the smaller primary, radiation of the bigger primary and the gravitational potential generated by the belt. Numerically, we have found that, in addition to the three collinear libration points L_i (i = 1, 2, 3) in the classic restricted three-body problem, there appear four more collinear points L_{ni} (i = 1, 2, 3, 4). L_{n1} and L_{n2} result due to the potential from the belt, while L_{n3} and L_{n4} are consequences of the oblateness up to the zonal harmonics J_4 of the smaller primary. Owing to the mutual effect of all the perturbations, L_1 and L_3 come nearer to the primaries while L_{n3} advances away from the primaries; and L_2 and L_{n1} tend towards the smaller primary whereas L_{n2} and L_{n4} draw closer to the bigger primary. The collinear libration points L_i (i = 1, 2, 3) *and* L_{n2} are linearly unstable whereas the L_{n1}, L_{n3} and L_{n4} are linearly stable. A practical application of this model could be the study of motion of a dust particle near a radiating star and an oblate body surrounded by a belt.

Keywords

Circular Restricted Three-Body Problem, Photogravitational, Zonal Harmonic Effect, Potential from the Belt

1. Introduction

In celestial mechanics, one amidst various inspiring subject is the restricted three-body problem (R3BP). The

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problem entails three bodies: two primary bodies having finite masses moving under their mutual gravitational attraction and the third with a negligible-mass (infinitesimal) body, whose motion is influenced by the primaries. If the primaries move on circular orbits about their common centre of mass, it is termed as the circular R3BP (CR3BP). Then, the objective of this CR3BP is to determine the motion of the infinitesimal mass. [1] and [2] gave a detailed description of the solution of the CR3BP. They showed that if the primary bodies were fixed in a rotating coordinate system, five libration points existed. That is the points where the infinitesimal mass can remain permanent, if placed there with zero velocity. Three of the points L_1, L_2, L_3 are on the line linking the primaries, whereas the other two L_4, L_5 are in equilateral triangular alignment with the primaries. The collinear points L_1, L_2, L_3 are linearly unstable, while the triangular points L_4, L_5 are linearly stable for the mass ratio of the primaries less than 0.03852.

Researches on the sites and stability of the libration points of the CR3BP with perturbations have achieved ample attention in recent times. [3] indicated that small particles were equally influenced by the gravitation and light radiation force as they moved toward luminous celestial bodies. [4] [5] established that the presence of direct solar radiation pressure caused a variation in the sites of the libration points of the CR3BP. He called the CR3BP, photogravitational when one or both of the masses of the primaries were discharges of radiation. Researchers [6]-[10] have examined the existence of libration points and their linear stability in the photogravitational CR3BP.

[11] [12] studied a modified CR3BP by considering the influence from a belt (circular cluster of material points) for planetary systems and found that the likelihood to get libration points around the inner part of the belt was greater than the one nigh the outer part. The impact of the belt makes the configuration of the dynamical system altered such that new libration points emerge under certain condition [13]-[16].

The primaries in CR3BP are generally considered to be spherical in shape, whereas in real situations, numerous celestial bodies are non-spherical (e.g. the Earth, Jupiter, Saturn, Regulus stars are oblate). The oblateness of the planets causes large deviations from a two-body orbit. The most salient instance of disturbance due to oblateness in the solar system is the orbit of the fifth satellite of Jupiter, Amalthea. This planet is extremely oblate and the satellite's orbit is exceptionally small that its line of apsides progresses approximately 900˚ in one year [17]. This vindicates the incorporation of oblateness of the primaries in the study of CR3BP [18]-[25].

The orbital effects of the oblateness up to the quadrupole, *i.e.* J_2, and the octupole, *i.e.* J_4, on the orbital motion of a particle in the field of a non-spherical body have been worked out in the general case of an arbitrarily oriented spin axis [26]. [22] certified that the sites of the triangular libration points and their linear stability were influenced by the oblateness up to J_4 of the bigger primary in the CR3BP. [27] examined the effects of photogravitational force and oblateness in the perturbed restricted three-body problem. [15] analyzed analytically and numerically the effects of oblateness up to J_2 of the smaller primary and gravitational potential from the belt on the linear stability of libration points in the photogravitational CR3BP. [16] explored the combined effect of radiation and oblateness up to J_2 of both primaries, together with additional gravitational potential from the circumbinary belt on the motion of an infinitesimal body in the binary stellar systems within the frame work of CR3BP. [9] studied the effects of oblateness up to J_4 of the smaller primary and gravitational potential from a belt, on the linear stability of triangular libration points in the photogravitational CR3BP. [24] looked at the effects of oblateness of both primaries up to zonal harmonic J_4 and gravitational potential from the belt on the linear stability of the triangular libration points in the CR3BP.

Here, our intention is to look into the resultant effect of radiation of the bigger primary, oblateness up to the zonal harmonic J_4 of the smaller primary and gravitational potential from the belt on the sites and stability of collinear libration points in the CR3BP.

The manuscript is structured in five units. Unit 2 deals with the mathematical formulation of the problem, while Unit 3 is dedicated to the determination of the sites of the collinear libration points. The linear stability of collinear points and the conclusion are presented in Units 4 and 5 respectively.

2. Mathematical Formulation of Model

2.1. The Problem

Let m_1 and m_2 be the masses of the primaries with $m_1 > m_2$, and let m be the mass of the infinitesimal body moving in the plane of motion of the primaries. The positions of the primaries are defined with respect to a rotating coordinate frame *oxyz* whose *x*-axis overlaps with the line connecting them and whose origin coincides with the center of mass of m_1 and m_2. The *y*-axis is perpendicular to the *x*-axis and the *z*-axis is normal to the orbital plane of the primaries. Let r_1 be the distance between *m* and m_1, r_2 *the* distance between *m* and m_2; and *R* the distance between m_1 and m_2. The coordinates of m_1, m_2 and *m* are $(x_1, 0)$, $(x_2, 0)$ and (x, y) correspondingly. Our aim is to find the equations of motion of m under the influence of radiation of m_1, oblateness up to J_4 of the smaller primary, and a circumbinary belt centred at the origin of the coordinate system *oxyz* (see **Figure 1**).

2.2. The Kinetic Energy

The kinetic energy (K.E) of the infinitesimal body in the barycentric coordinate system *oxyz* rotating about *z*-axis with uniform angular velocity n **Figure 1**, is given as

$$K = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + mn\left(x\dot{y} - \dot{x}y\right) + \frac{1}{2} mn^2\left(x^2 + y^2\right), \tag{1}$$

where over dot represents differentiation with respect to time *t*.

2.3. Force Due to Radiation Pressure

Now, since the radiation pressure force F_p varies with distance by the same law as the gravitational attraction force F_g and works opposite to it, it is likely that this force will lead to a decrease of the effective mass of the bigger primary. Furthermore this decrease relies on the properties of the particle; it is therefore tolerable to talk about a reduced mass. Hence, the consequential force on the particle is [4]

$$F = F_g - F_p = F_g\left(1 - \frac{F_p}{F_g}\right) = qF_g; \tag{2}$$

where $q = \left(1 - \frac{F_p}{F_g}\right)$, a constant for a particular particle, is the mass reduction factor. We represent the radiation factor for the bigger primary as $q_1 = 1 - p_1$, $0 < p_1 = \frac{F_{p_1}}{F_{g_1}} \ll 1$.

2.4. Potential Due to an Oblate Body

In free space the gravitational potential exterior to an oblate body with its mass distributed symmetrically about its equator, can be expanded in terms of Legendre polynomials in the form

$$V_o\left(r_o, \phi, \theta\right) = -\frac{Gm_o}{r_o}\left[1 - \sum_{n=1}^{\infty} J_{2n} P_{2n}\left(\cos\theta\right)\left(\frac{R_o}{r_o}\right)^{2n}\right] \tag{3}$$

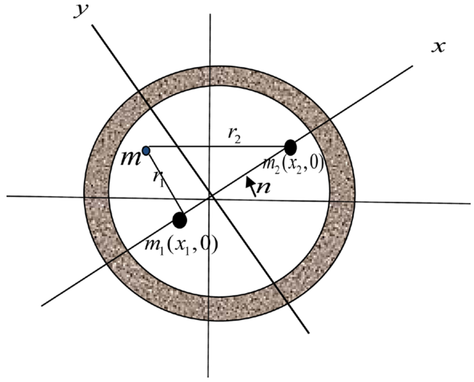

Figure 1. The planar configuration of the problem.

[28]. Equation (3) is expressed in standard spherical coordinates, with ϕ the longitude and θ representing the angle between the body's symmetry axis and the vector to a particle r_o (*i.e.*, the colatitudes). R_o is the mean radius of the oblate body. The terms $P_{2n}(\cos\theta)$ are the Legendre polynomials, given by

$$P_{2n}(x) = \frac{1}{2^{2n}(2n)!}\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left(x^2 - 1\right)^{2n} \tag{4}$$

J_{2n} are dimensionless coefficients that characterize the size of non spherical components of the potential, called the zonal harmonic coefficients. Since the present study is concerned with planar problem, assuming the equatorial plane of the smaller primary coincides with the plane of motion, then with $\theta = 90°$, Equation (3) becomes

$$V_o(r_o,\phi,\theta) = -Gm_o\left[\frac{1}{r_o} + \frac{J_2R_o^2}{2r_o^3} - \frac{3J_4R_o^4}{8r_o^5} + \frac{5J_6R_o^6}{16r_o^7} + \cdots\right] \tag{5}$$

We denote the oblateness coefficient for the smaller primary as B_i, $0 < B_i = J_{2i}R_0^{2i} \ll 1, i = 1, 2$.

2.5. Potential Due to the Belt

The gravitational potential from belt (circular cluster of material points) centered at the origin of a coordinates system *oxyz*, **Figure 1** as specified by [29] is

$$V_b(r,z) = -\frac{M_b}{\sqrt{r^2 + \left(a + \sqrt{z^2 + b^2}\right)^2}}, \tag{6}$$

where M_b is the total mass of the belt, $r^2 = x^2 + y^2$, a and b are parameters which determine the density profile of the belt. The parameter a controls the flatness of the profile and is known as the *flatness parameter*. The parameter b controls the size of the core of the density profile and is called the *core parameter*. When $a = b = 0$, the potential reduces to the one by a point mass. Restricting ourselves to the xy-plane, Equation (6) becomes

$$V_b(r,0) = \frac{M_b}{\left(r^2 + T^2\right)^{1/2}}, \quad \text{where } T = a + b. \tag{7}$$

2.6. The Potential Energy of the Infinitesimal Body

The potential energy of the infinitesimal body, under the influence of the oblateness up to J_4 of smaller primary, radiation of the bigger primary and the circumbinary belt, now takes the form

$$V = -Gm\left\{m_1\frac{q_1}{r_1} + m_2\left(\frac{1}{r_2} + \frac{B_1}{2r_2^3} - \frac{3B_2}{8r_2^5}\right) + \frac{M_b}{\left(r^2 + T^2\right)^{1/2}}\right\}, \tag{8}$$

with $r_1^2 = (x - x_1)^2 + y^2$, $r_2^2 = (x - x_2)^2 + y^2$, G is the gravitational constant.

2.7. The Equations of Motion

We start from Lagrangian (L) of the problem which is the kinetic energy minus the potential energy of the infinitesimal body. That is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mn\left(x\dot{y} - \dot{x}y\right) + \frac{1}{2}mn^2\left(x^2 + y^2\right) - V.$$

or

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mn\left(x\dot{y} - \dot{x}y\right) - U, \tag{9}$$

where $U = V - \frac{1}{2} m n^2 \left(x^2 + y^2 \right)$.

Subsequently, we obtain the equations of motion of the infinitesimal body as

$$\ddot{x} - 2n\dot{y} = -\frac{1}{m}\frac{\partial U}{\partial x}, \qquad \ddot{y} + 2n\dot{x} = -\frac{1}{m}\frac{\partial U}{\partial y}. \tag{10}$$

To covert the variables to non dimensional, we choose unit for the mass as the sum of the masses of the primaries, the unit of length as the distance between the primaries and unit of time is such that the gravitational constant is unit. Consequently, $m_1 = 1 - \mu$, $m_2 = \mu$ where $0 < \mu = \frac{m_2}{m_1 + m_2} \le \frac{1}{2}$ is the mass ratio. Thus, in the dimensionless synodic coordinate system, the equations of motion (10) reduce to

$$\ddot{x} - 2n\dot{y} = \Omega_x, \quad \ddot{y} + 2n\dot{x} = \Omega_y, \tag{11}$$

with

$$\Omega = \frac{n^2\left(x^2 + y^2\right)}{2} + \frac{(1-\mu) q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu B_1}{2 r_2^3} - \frac{3\mu B_2}{8 r_2^5} + \frac{M_b}{\left(r^2 + T^2\right)^{1/2}},$$

$$r_1^2 = (x+\mu)^2 + y^2, \quad r_2^2 = (x+\mu-1)^2 + y^2, \tag{12}$$

and n *is* the mean motion, given by [24] as

$$n^2 = 1 + \frac{3}{2}\left(B_1 - \frac{5}{4} B_2\right) + \frac{2 M_b r_c}{\left(r_c^2 + T^2\right)^{3/2}}, \tag{13}$$

r_c is the radial distance of the infinitesimal body in the classical restricted three-body problem.

3. Locations of Collinear Libration Points

We now search for possible collinear libration points of the infinitesimal mass in the rotating reference frame. The libration points are positions of gravitational balance between the primaries. At these points the two finite masses would exert zero net force on the infinitesimal mass, in effect, allowing the infinitesimal mass to have zero velocity in the rotating frame of reference. That is the libration points satisfy $\ddot{x} = \ddot{y} = \dot{x} = \dot{y} = 0$. It thus follows, from Equation (11), that the libration points are the solutions of

$$n^2 x - \frac{(1-\mu)(x+\mu) q_1}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3} - \frac{3\mu(x+\mu-1) B_1}{2 r_2^5} + \frac{15\mu(x+\mu-1) B_2}{8 r_2^7} - \frac{M_b x}{\left(r^2 + T^2\right)^{3/2}} = 0, \tag{14}$$

and

$$n^2 y - \frac{(1-\mu) q_1 y}{r_1^3} - \frac{\mu y}{r_2^3} - \frac{3\mu B_1 y}{2 r_2^5} + \frac{15\mu B_2 y}{8 r_2^7} - \frac{M_b y}{\left(r^2 + T^2\right)^{3/2}} = 0. \tag{15}$$

Now, an evident solution of Equation (15) is $y = 0$, corresponding to the collinear libration points (the libration points which lie on the x-axis). This deciphers to

$$n^2 x - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - \frac{\mu(x+\mu-1)}{|x+\mu-1|^3} - \frac{3\mu(x+\mu-1) B_1}{2|x+\mu-1|^5} + \frac{15\mu(x+\mu-1) B_2}{8|x+\mu-1|^7} - \frac{M_b x}{\left(x^2 + T^2\right)^{3/2}} = 0. \tag{16}$$

Equation (16) reduces to those of [1], in the absence of the perturbations. That is when $q_1 = 1, B_1 = B_2 = M_b = 0$), we have

$$x - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - \frac{\mu(x+\mu-1)}{|x+\mu-1|^3} = 0, \tag{17}$$

with three collinear points L_1, L_2 and L_3. Only the collinear point L_2 is located between the primaries (**Figure 2**).

If we consider the effects of the potential from the belt only (*i.e.* $A_1 = A_2 = B_1 = B_2 = 0$), the Equation (17) reduces to

$$n^2 x - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - \frac{\mu(x+\mu-1)}{|x+\mu-1|^3} - \frac{M_b x}{\left(x^2+T^2\right)^{3/2}} = 0. \tag{18}$$

[16] showed that whenever $T < \sqrt{2}\mu$ and

$$n^2\left(-\frac{T}{\sqrt{2}}\right) - \frac{(1-\mu)\left(-\frac{T}{\sqrt{2}}+\mu\right)}{\left|-\frac{T}{\sqrt{2}}+\mu\right|^3} - \frac{\mu\left(-\frac{T}{\sqrt{2}}+\mu-1\right)}{\left|-\frac{T}{\sqrt{2}}+\mu-1\right|^3} - \frac{M_b\left(-\frac{T}{\sqrt{2}}\right)}{\left(\left(-\frac{T}{\sqrt{2}}\right)^2+T^2\right)^{3/2}} > 0$$

in the interval $(-\mu, 0)$, Equation (18) will have five collinear points (**Figure 3**).

Now, using Equation (16) and with the help of the MATLAB (R2007b) software package, we obtain the coordinates of the collinear libration points for different cases as classified in the following order which are portrayed in **Table 1**:

1) Absence of radiation, oblateness and potential from the belt (classical case).
2) Radiation of the bigger primary only.
3) Potential from the belt only.
4) Oblateness of the smaller primary up to J_2 only.
5) Oblateness of the smaller primary up to J_4 only.
6) Radiation of the bigger primary, oblateness of the smaller primary up to J_4 and potential from the belt.

The combined effect of these perturbations on the collinear points is given in **Table 2**.

In the absence of the perturbations (*i.e.* $q_1 = 1, B_1 = B_2 = M_b = 0$) **Table 1** Case 1, it is observed that there are three collinear libration points (L_i, $i = 1, 2, 3$) which correspond to the classical case of [1]. Owing to the effect of the radiation of the bigger primary only (*i.e.* $q_1 = 0.98, B_1 = B_2 = M_b = 0$) Case 2, L_1 and L_3 stepped closer to the primaries while L_2 moved towards the bigger primary. Nevertheless, on taking into account the effect of the potential from the belt only (*i.e.* $q_1 = 1, B_1 = B_2 = 0, M_b = 0.01$) Case 3, there surface five collinear libration points (L_{n1}, L_{n2} *and* L_i, $i = 1, 2, 3$), this confirms those of [14]-[16]. The collinear points L_1 and L_3 shifted nearer to the primaries while L_2 moved away from the bigger primary, due to the potential from the belt. In the presence of the oblateness of the smaller primary up to J$_2$ only (*i.e.* $q_1 = 1, B_1 = 0.01, B_2 = M_b = 0$) Case 4, the collinear point L_1 sifted away from the primaries while L_2 and L_3 stepped closer to the bigger primary. In Case 5, due oblateness of the smaller primary up to J_4 only (*i.e.* $q_1 = 1, B_1 = 0.01, B_2 = 0.005, M_b = 0$), L_{n1} moved away from the bigger primary while L_{n2} stepped towards it. Similarly, owing to the oblateness of the smaller primary up to J_2 with

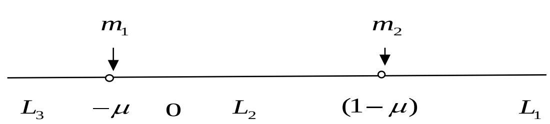

Figure 2. Disposition of the collinear points in the classical case.

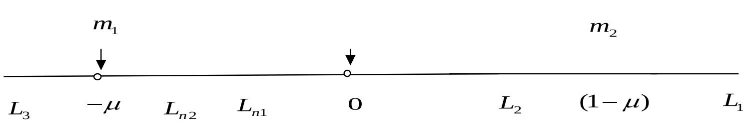

Figure 3. Disposition of the collinear points under the effects of the belt.

Table 1. Positions of the collinear points when $\mu = 0.35$, $q_1 = 0.98$, $B_1 = 0.01$, $B_2 = 0.005$ and $M_b = T = 0.01$, $r_c = 0.8789$.

Case	L_1	L_2	L_3	L_{n1}	L_{n2}	L_{n3}	L_{n4}
1	1.244813	0.213295	−1.142867				
2	1.243714	0.210813	−1.137286				
3	1.239362	0.224700	−1.137090	−0.000451	−0.038855		
4	1.249564	0.205046	−1.138453				
5	1.235582	0.245494	−1.141267			0.961931	0.319350
6	1.228444	0.259431	−1.129916	−0.000441	−0.039247	0.962537	0.314837

Table 2. Combined effects of the perturbations on the collinear points when $\mu = 0.35$, $T = 0.01$, $r_c = 0.8789$.

q_1	B_1	B_2	M_b	L_1	L_2	L_3	L_{n1}	L_{n2}	L_{n3}	L_{n4}
1	0	0	0	1.24481	0.21329	−1.14287	–	–	–	–
0.99	0.001	0.0005	0.01	1.23795	0.22579	−1.13420	−0.000445	−0.03905	0.82421	0.47525
0.98	0.002	0.0006	0.02	1.23240	0.23420	−1.12557	−0.000219	−0.05365	0.83068	0.46863
0.97	0.003	0.0007	0.03	1.22707	0.24145	−1.11719	−0.000144	−0.06391	0.83634	0.46280
0.96	0.004	0.0008	0.04	1.22194	0.24787	−1.10906	−0.000107	−0.07207	0.84138	0.45758

potential from the belt only (*i.e.* $A_1 = A_2 = B_2 = 0, B_1 = M_b = 0.01$) Case 5, collinear points L_1 and L_3 moved nigh to the primaries while L_2 stepped away from the bigger primary; and there emerge additional two new collinear points L_{n3}, L_{n4}. In the presence of all these perturbations (*i.e.* $q_1 = 0.98, B_1 = M_b = 0.01, B_2 = 0.005$) Case 6, there appeared seven collinear points: L_1, L_2, L_3, L_{n1}, L_{n2}, L_{n3}, L_{n4} as shown in **Figure 4**. With increase in these perturbations **Table 2**, the collinear points L_1, L_3 draw closer to the primaries while L_{n3} moves away from the them; L_2, L_{n1} move away from the bigger primary while L_{n2}, L_{n4} tend towards it.

4. Linear Stability of the Collinear Points

To study the stability of a libration point (x_0, y_0), we employ small displacement η, ξ to the coordinates (x_0, y_0). So, the variations η and ξ can take the form: $\eta = x - x_0$ and $\xi = y - y_0$ and the equations of the motion (5) become

$$\ddot{\eta} - 2n\dot{\xi} = \left(\Omega_{xx}^0\right)\eta + \left(\Omega_{xy}^0\right)\xi,$$
$$\ddot{\xi} + 2n\dot{\eta} = \left(\Omega_{yx}^0\right)\eta + \left(\Omega_{yy}^0\right)\xi. \tag{19}$$

The superscript "0" indicates that the partial derivatives have been evaluated at the libration point under consideration (x_0, y_0).

Let solutions of the equations of (19) be $\eta = A\exp(\lambda t)$, $\xi = B\exp(\lambda t)$ where A, B and λ are constants. Then, Equation (19) will have a non –trivial solution for A and B when

$$\begin{vmatrix} \lambda^2 - \Omega_{xx}^0 & -2n\lambda - \Omega_{xy}^0 \\ 2n\lambda - \Omega_{yx}^0 & \lambda^2 - \Omega_{yy}^0 \end{vmatrix} = 0. \tag{20}$$

On expanding the determinant we obtain the characteristic equation equivalent to the variational equations of (19) as

$$\lambda^4 + \left(4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0\right)\lambda^2 + \Omega_{xx}^0\Omega_{yy}^0 - \left(\Omega_{xy}^0\right)^2 = 0 \tag{21}.$$

Now, we obtain the second partial derivatives as:

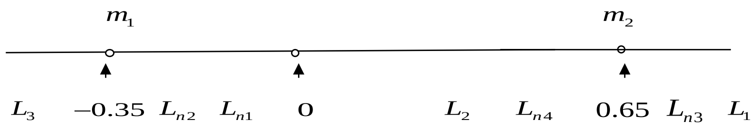

Figure 4. Disposition of the collinear points under the combined effects of the perturbations.

$$
\begin{aligned}
\Omega_{xx} &= n^2 - \frac{(1-\mu)q_1}{r_1^3} + \frac{3(1-\mu)(x+\mu)^2 q_1}{r_1^5} - \frac{\mu}{r_2^3} + \frac{3\mu(x+\mu-1)^2}{r_2^5} - \frac{3\mu B_1}{2r_2^5} + \frac{15\mu(x+\mu-1)^2 B_1}{2r_2^7} \\
&\quad + \frac{15\mu B_2}{8r_2^7} - \frac{105\mu(x+\mu-1)^2 B_2}{8r_2^9} - \frac{M_b}{(r^2+T^2)^{3/2}} + \frac{3M_b x^2}{(r^2+T^2)^{5/2}}, \\
\Omega_{yy} &= n^2 - \frac{(1-\mu)q_1}{r_1^3} + \frac{3(1-\mu)q_1 y^2}{r_1^5} - \frac{\mu}{r_2^3} + \frac{3\mu y^2}{r_2^5} - \frac{3\mu B_1}{2r_2^5} + \frac{15\mu y^2 B_1}{2r_2^7} \\
&\quad + \frac{15\mu B_2}{8r_2^7} - \frac{105\mu y^2 B_2}{8r_2^9} - \frac{M_b}{(r^2+T^2)^{3/2}} + \frac{3M_b y^2}{(r^2+T^2)^{5/2}}, \\
\Omega_{xy} &= \Omega_{yx} = \frac{3(1-\mu)(x+\mu)q_1 y}{r_1^5} + \frac{3\mu(x+\mu-1)y}{r_2^5} + \frac{15\mu(x+\mu-1)yB_1}{2r_2^7} \\
&\quad - \frac{105\mu(x+\mu-1)yB_2}{8r_2^9} + \frac{3M_b xy}{(r^2+T^2)^{5/2}}.
\end{aligned} \tag{22}
$$

The partial derivatives computed at any collinear libration points (x_0, 0), are

$$\Omega_{xx}^0 = n^2 + \frac{2(1-\mu)q_1}{|x_0+\mu|^3} + \frac{2\mu}{|x_0+\mu-1|^3} + \frac{6\mu B_1}{|x_0+\mu-1|^5} - \frac{45\mu B_2}{4|x_0+\mu-1|^7} + \frac{3M_b x_0^2}{(x_0^2+T^2)^{5/2}} - \frac{M_b}{(x_0^2+T^2)^{3/2}}, \tag{23}$$

$$\Omega_{yy}^0 = n^2 - \frac{(1-\mu)q_1}{|x_0+\mu|^3} - \frac{\mu}{|x_0+\mu-1|^3} - \frac{3\mu B_1}{2|x_0+\mu-1|^5} + \frac{15\mu B_2}{8|x_0+\mu-1|^7} - \frac{M_b}{(x_0^2+T^2)^{3/2}}, \tag{24}$$

$$\Omega_{xy}^0 = \Omega_{yx}^0 = 0. \tag{25}$$

Substituting these values in Equation (21), the characteristic equation reduces to

$$\lambda^4 + b\lambda^2 + c = 0 \tag{26}$$

where $b = 4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0$, $c = \Omega_{xx}^0 \Omega_{yy}^0$.

The libration point is stable if all the roots of the characteristic equation (26) are either negative real numbers or distinct pure imaginary numbers or real parts of the complex numbers are negative.

The roots of the characteristic equation (26) for the libration points L_i (i = 1, 2, 3), L_{nj} (j = 1, 2, 3, 4) of **Table 1** are presented in **Tables 3-9** correspondingly.

Studying **Tables 3-9**, we find that all the collinear libration points L_i (i = 1, 2, 3) *and* L_{n2} are unstable (**Table 3**, **Table 4**, **Table 5**, **Table 7**), whereas the additional new collinear points L_{n1}, L_{n3} and L_{n4} are stable (**Table 6**, **Table 8**, **Table 9**).

5. Conclusion

The collinear libration points are investigated in a modified CR3BP when the bigger primary is a source of radiation, the smaller primary is an oblate spheroid; and the bodies are surrounded by a belt (circular cluster of material points). We have established the equations that govern the motion of the infinitesimal body under the

Table 3. Stability of L_1.

Case	L_1	Ω^0_{xx}	Ω^0_{yy}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
1	1.244813	4.6468	−0.8234	±1.3674	±1.4305i	Unstable
2	1.243714	4.6595	−0.8297	±1.3722	±1.4329i	Unstable
3	1.239362	4.7796	−0.8510	±1.3897	±1.4512i	Unstable
4	1.249564	4.8515	−0.8355	±1.4112	±1.4267i	Unstable
5	1.235582	4.2890	−0.8377	±1.2772	±1.4841i	Unstable
6	1.228444	4.3987	−0.8739	±1.2973	±1.5113i	Unstable

Table 4. Stability of L_2.

Case	L_2	Ω^0_{xx}	Ω^0_{yy}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
1	0.213295	16.6783	−6.8391	±3.7405	±2.8552i	Unstable
2	0.210813	16.4862	−6.7431	±3.7147	±2.8383i	Unstable
3	0.224700	18.7266	−7.8271	±3.9966	±3.0293i	Unstable
4	0.205046	17.7676	−7.0603	±3.8738	±2.8912i	Unstable
5	0.245494	8.5669	−5.9936	±2.5451	±2.8155i	Unstable
6	0.259431	7.6595	−6.4396	±2.3914	±2.9368i	Unstable

Table 5. Stability of L_3.

Case	L_3	Ω^0_{xx}	Ω^0_{yy}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
1	−1.142867	3.7297	−0.3648	±0.9441	±1.2355i	Unstable
2	−1.137286	3.7334	−0.3667	±0.9463	±1.2364i	Unstable
3	−1.137090	3.8282	−0.3753	±0.9574	±1.2519i	Unstable
4	−1.138453	3.7908	−0.3726	±0.9540	±1.2458i	Unstable
5	−1.141267	3.7523	−0.3675	±0.9476	±1.2392i	Unstable
6	−1.129916	3.8558	−0.3805	±0.9638	±1.2568i	Unstable

Table 6. Stability of L_{n1}.

Case	L_{n1}	Ω^0_{xx}	Ω^0_{yy}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
3	−0.000451	−9874.8	−9985.0	±98.6059i	±100.7019i	Stable
6	−0.000441	−9879.7	−9986.0	±98.6019i	±100.7356i	Stable

Table 7. Stability of L_{n2}.

Case	L_{n2}	Ω^0_{xx}	Ω^0_{yy}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
3	−0.038855	327.1441	−176.4617	±18.0135	±13.3382i	Unstable
6	−0.039247	319.0101	−171.7775	±17.7859	±13.1617i	Unstable

Table 8. Stability of L_{n3}.

Case	L_{n3}	Ω^0_{xx}	Ω^0_{yy}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
5	0.961931	−36.7586	−1.1726	±1.0266i	±6.3953i	Stable
6	0.962537	−36.0006	−1.2218	±1.0453i	±6.3447i	Stable

Table 9. Stability of L_{n4}.

Case	L_{n2}	Ω^0_{xx}	Ω^0_{yy}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
5	0.319350	−15.5449	−4.5783	±1.8538i	±4.5507i	Stable
6	0.314837	−11.8748	−5.0872	±1.8490i	±4.2035i	Stable

influence of radiation of the bigger primary, oblateness up to the zonal harmonics J_4 of the smaller primary and gravitational potential from the belt. The equations are affected by the aforementioned perturbations. Numerically, we have determined the positions of the collinear libration points and investigated the resultant effect of the aforesaid perturbations on them. It is found that in count to the three libration points L_1, L_2, L_3 in the classical problem, there emerge four new collinear points which we call L_{n1}, L_{n2}, L_{n3} and L_{n4}. L_{n1} and L_{n2} arise from the effect of the potential from the belt, whereas L_{n3} and L_{n4} stem from the influence of the oblateness up to the zonal harmonics J_4 of the smaller primary. Due to the pooled impact of the aforesaid perturbations, the collinear points L_1 and L_3 advance toward the primaries while L_{n3} moves away from the primaries; and L_2 and L_{n1} tend towards the smaller primary as L_{n2} and L_{n4} come closer to the bigger primary. Despite the influence of radiation of the bigger primary, oblateness up to the zonal harmonics J_4 of the smaller primary and gravitational potential from the belt, the collinear libration points L_i (i = 1, 2, 3) as in the classical case, remain unstable. However, all the additional new collinear points are stable except L_{n2}. The existence of stable new collinear points can be utilized as stations for artificial satellites.

References

[1] Szebehely, V. (1967) Theory of Orbits: The Restricted Problem of Three Bodies. Academic Press, New York.

[2] Valtonen, M. and Karttunen, H. (2006) The Three-Body Problem. Cambridge University Press, Cambridge. http://dx.doi.org/10.1017/CBO9780511616006

[3] Poynting, J.H. (1903) Radiation in the Solar System: Its Effect on Temperature and Its Pressure on Small Bodies. *Philosophical Transactions of the Royal Society of London A*, **202**, 525-552. http://dx.doi.org/10.1098/rsta.1904.0012

[4] Radzievskii, V.V. (1950) The Restricted Problem of Three-Body Taking Account of Light Pressure. *Astronomicheskii-Zhurnal*, **27**, 250-256.

[5] Radzievskii, V.V. (1953) The Space Photogravitational Restricted Three-Body Problem. *Astronomicheskii-Zhurnal*, **30**, 225.

[6] Bhatnagar, K.B. and Chawla, J.M. (1979) A Study of the Lagrangian Points in the Photogravitational Restricted Three-Body Problem. *Indian Journal of Pure and Applied Mathematics*, **10**, 1443-1451.

[7] Simmons, J.F.L., McDonald, J.C. and Brown, J.C. (1985) The Three-Body Problem with Radiation Pressure. *Celestial Mechanics*, **35**, 145-187. http://dx.doi.org/10.1007/BF01227667

[8] Das, M.K., Narang, P., Mahajan, S. and Yuasa, M. (2008) Effect of Radiation on the Stability of Equilibrium Points in the Binary Stellar Systems: RW-Monocerotis, Krüger 60. *Astrophysics Space Science*, **314**, 261. http://dx.doi.org/10.1007/s10509-008-9765-z

[9] Singh, J. and Taura, J.J. (2014) Stability of Triangular Libration Points in the Photogravitational Restricted Three-Body Problem with Oblateness and Potential from a Belt. *Journal of Astrophysics and Astronomy*, **35**, 107-119. http://dx.doi.org/10.1007/s12036-014-9299-4

[10] Singh, J. and Taura, J.J. (2015) Triangular Libration Points in the CR3BP with Radiation, Triaxiality and Potential from a Belt. *Differential Equations and Dynamical System*. http://dx.doi.org/10.1007/s12591-015-0243-0

[11] Jiang, I.G. and Yeh, L.C. (2004) The Drag-Induced Resonant Capture for Kuiper Belt Objects. *Monthly Notices of the Royal Astronomical Society*, **355**, L29-L32. http://dx.doi.org/10.1111/j.1365-2966.2004.08504.x

[12] Jiang, I.G. and Yeh, L.C. (2004) On the Chaotic Orbits of Disk-Star-Planet Systems. *The Astronomical Journal*, **128**, 923-932. http://dx.doi.org/10.1086/422018

[13] Jiang, I.G. and Yeh, L.C. (2003) Bifurcation for Dynamical Systems of Planet-Belt Interaction. *International Journal of Bifurcation and Chaos*, **13**, 617-630. http://dx.doi.org/10.1142/s0218127403006807

[14] Yeh, L.C. and Jiang, I.G. (2006) On the Chermnykh-Like Problems: II. The Equilibrium Points. *Astrophysics and Space Science*, **306**, 189-200. http://dx.doi.org/10.1007/s10509-006-9170-4

[15] Kushvah, B.S. (2008) Linear Stability of Equilibrium Points in the Generalized Photogravitational Chermnykh's Problem. *Astrophysics and Space Science*, **318**, 41-50.

[16] Singh, J. and Taura, J.J. (2013) Motion in the Generalized Restricted Three-Body Problem. *Astrophysics and Space Science*, **343**, 95-106. http://dx.doi.org/10.1007/s10509-012-1225-0

[17] Moulton, F.R. (1914) An Introduction to Celestial Mechanics. 2nd Edition, Dover, New York.

[18] Sharma, R.K. (1987) The Linear Stability of Libration Points of the Photogravitational Restricted Three-Body Problem When the Smaller Primary Is an Oblate Spheroid. *Astrophysics and Space Science*, **135**, 271-281. http://dx.doi.org/10.1007/BF00641562

[19] Kalvouridis, T.J. (1997) The Oblate Spheroids Version of the Photo-Gravitational 2+2 Body Problem. *Astrophysics and Space Science*, **246**, 219-227. http://dx.doi.org/10.1007/BF00645642

[20] Singh, J. and Umar, A. (2013) Application of Binary Pulsars to Axisymmetric Bodies in the Elliptic R3BP. *Astrophysics and Space Science*, **348**, 393-402. http://dx.doi.org/10.1007/s10509-013-1585-0

[21] Abdul Raheem, A. and Singh, J. (2006) Combined Effects of Perturbations, Radiation and Oblateness on the Stability of Equilibrium Points in the Restricted Three-Body Problem. *Astronomical Journal*, **131**, 1880-1885. http://dx.doi.org/10.1086/499300

[22] Abouelmagd, E.I. (2012) Existence and Stability of Triangular Points in the Restricted Three-Body Problem with Numerical Applications. *Astrophysics and Space Science*, **342**, 45-53. http://dx.doi.org/10.1007/s10509-012-1162-y

[23] Singh, J. and Taura, J.J. (2014) Effects of Triaxiality, Oblateness and Gravitational Potential from a Belt on the Linear Stability of $L_{4,5}$ in the Restricted Three-Body Problem. *Journal of Astrophysics and Astronomy*, **35**, 729-743. http://dx.doi.org/10.1007/s12036-014-9308-7

[24] Singh, J. and Taura, J.J. (2014) Effects of Zonal Harmonics and a Circular Cluster of Material Points on the Stability of Triangular Equilibrium Points in the R3BP. *Astrophysics and Space Science*, **350**, 127-132. http://dx.doi.org/10.1007/s10509-013-1719-4

[25] Singh, J. and Taura, J.J. (2014) Combined Effect of Oblateness, Radiation and a Circular Cluster of Material Points on the Stability of Triangular Libration Points in the R3BP. *Astrophysics and Space Science*, **351**, 499-506. http://dx.doi.org/10.1007/s10509-014-1860-8

[26] Renzetti, G. (2013) Satellite Orbital Precessions Caused by the Octupolar Mass Moment of a Non-Spherical Body Arbitrarily Oriented in Space. *Journal of Astrophysics and Astronomy*, **34**, 341-348. http://dx.doi.org/10.1007/s12036-013-9186-4

[27] Abouelmagd, E.I. (2013) The Effect of Photogravitational Force and Oblateness in the Perturbed Restricted Three-Body Problem. *Astrophysics and Space Science*, **346**, 51-69. http://dx.doi.org/10.1007/s10509-013-1439-9

[28] Peter, I.D. and Lissauer, J.J. (2007) Planetary Science. Cambridge University Press, New York.

[29] Miyamoto, M. and Nagai, R. (1975) Three-Dimensional Models for the Distribution of Mass in Galaxies. *Publications of the Astronomical Society of Japan*, **27**, 533-543.