

Canonical Treatments for the Two Bodies Problem with Varying Mass Taking into Consideration the Periastron Effect

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Abstract

In this work, the Hamiltonian of the two body problem with varying mass was developed in the extended phase space taking into consideration the periastron effects. The short period solution was obtained through constructing a second order canonical transformation using “Hori’s” method developed by “Kamel”. The elements of the transformation as well as the inverse transformation were obtained too. The final solution of the problem was derived using “Delva-Hanslmeier” method.

Keywords

Two Body with Varying Mass; Perturbation; Hori’s Method

1. Introduction

The problem of the two bodies with varying mass has roots going back in the history since the middle of the 19th century. Since F. Mestschersky’s initial study a great many researchers have dedicated much of their time to this problem, telling it became one of the classics of celestial mechanics, which has become known as the Gylden-Mestschersky.

An ample bibliography can be found in the published works of E. N. Polyakova [1] and C. Prieto [2]. The specific case which results in a slow isotropic mass loss has also been the focus of exhaustive studies carried out by researchers, for instance, J. D. Hadjidemetriou [3] [4], to name but a few. The vast majority of these, in search of the stellar application, have taken the so-called Eddington-Jeans law [5] [6], as a law of the variation of mass,

$$\dot{m} = -\alpha m^n \quad (1)$$

where α and n are real numbers, the first of them is positively proximate to zero and the second is varying between 1.4 and 4.4.

Prieto [7] [8] intended to present an approximate analytic solution of the two-body problem with slowly decreasing mass which is obtained through the integration of the Hamilton equations using Deprit's method of perturbations. The solution, obtained through the Eddington-Jeans law, is put into practice in a specific case and compared with F. Mestschersky's exact equation; $n = 2$; and with that which results from numerically integrating the equations.

In the framework of celestial mechanics, this problem has been exhaustively addressed by Docobo [9], Andrade [10], among others.

In 2002, M. Andrade [11] analyzed the dynamics of binary systems with time-dependent mass loss and periastron effect *i.e.*, a supposed enhanced mass loss during periastron passage by means of analytical and numerical techniques.

M. Andrade and J. A. Docobo [12] studied the dynamics of binary systems with small parameter perturbation model, the time-dependence of the whole set of orbital elements, concluded, could be calculated over long timescales and even for high eccentricities. In these models, they studied the following time- and distance-dependent mass-loss law:

$$\dot{\mu}(t; r; P_\theta) = \dot{\mu}(t) - \beta \frac{P_\theta}{r^2} \quad (2)$$

where the first term represent time-dependent mass loss, and the last one introduces the periastron effect, where "r" is the distance between the two components, P_θ is the total angular momentum and β is another small parameter close to zero.

W. A. Rahoma *et al.* [13] were introduction paper concerned with the two-body problem with varying mass in case of isotropic mass loss from both components of the binary systems. The law of mass variation used gives rise to a perturbed Keplerian problem depending on two small parameters. The problem is treated analytically in the Hamiltonian frame-work and the equations of motion are integrated using the Lie series developed and applied, separately by Delva [14] and Hanslmeier [15]. A second order theory of the two bodies eject mass was also constructed, returning the terms of the rate of change of mass up to second order in the small parameters of the problem.

W. A. Rahoma *et al.* [16] studied the two-body problem with varying mass in case of isotropic mass loss. In that work, the problem was treated analytically in the Hamiltonian frame-work and the equations of motion ware integrated using the Lie operator and Lie series.

Our work is aimed to introduce an analytical solution for the problem of two bodies with varying mass, taking into consideration the periastron effects, using the canonical perturbation technique derived by Hori and developed by Kamel.

2. The Hamiltonian of the Problem

The Hamiltonian is constructed in terms of Delaunay's variables as described in Deprit [17]:

$$\mathcal{K}^*(\ell, L, G; t) = -\frac{1}{2} \frac{\mu^2}{L^2} + \frac{\dot{\mu}}{\mu} L e \sin E \quad (3)$$

The first term in the Hamiltonian function, described by Equation (3), is the Hamiltonian in the Keplerian case (with constant mass) μ_0 while the second term represents the varying mass effects. In terms of the Delaunay variable, we can develop the Hamiltonian as:

$$\mathcal{K}^* = -\frac{\mu_0^2}{2L_1^2} + \frac{\dot{\mu}}{\mu} (L_1^2 - L_2^2)^{\frac{1}{2}} \sin E \quad (4)$$

That is meant that the Hamiltonian is function of ℓ_1 , L_1 and L_2 as:

$$\mathcal{K}^* = \mathcal{K}^*(\ell_1, \dots, \dots; L_1, L_2, \dots, \dots)$$

The variation of μ may be retained from one of the two masses m_1 or m_2 and this is the case of one body

eject mass. Otherwise the case of the two bodies ejects masses. We will concern with the first case. Then;

$$\dot{\mu} \equiv \dot{m}_1 \quad (5)$$

The Hamiltonian \mathcal{H}^* represented by Equation (4) is implicit depending on time through the variable mass μ and its time derivative $\dot{\mu}$. By modifying Docobo's law for the rate of change of mass assigned by Equation (2) and use Jeans law described by Equation (1) we get:

$$\dot{\mu}(r, t, L_2) = -\alpha m^n - \beta \frac{L_2 \mu^2}{L_1^4} \left(\frac{a}{r} \right)^2 \quad (6)$$

By applying Equation (6) in the Hamiltonian function (4) which can be written as:

$$\mathcal{H}^* = \frac{-\mu_o^2}{2L_1^2} - \alpha \mu^{n-1} (L_1^2 - L_2^2)^{\frac{1}{2}} \sin E - \beta L_2 \frac{\mu}{L_1^4} (L_1^2 - L_2^2)^{\frac{1}{2}} \left(\frac{a}{r} \right)^2 \sin E \quad (7)$$

with n positive number varying between **1.4** and **4.4**.

Since the variable mass can be expressed as a Taylor series expansion as:

$$\mu = \mu_0 + \frac{d\mu}{dt} \Big|_{t=t_0} \frac{(t-t_0)}{1!} + \frac{d^2\mu}{dt^2} \Big|_{t=t_0} \frac{(t-t_0)^2}{2!} + \dots,$$

that will lead to a Hamiltonian with explicit dependence of time. So we must extend the phase space by introducing a new pair of variable (ℓ_4, L_4) . The first is " ℓ_4 " assigned as the variable mass and the second its conjugate momentum " L_4 ".

The new systems of canonical equations of motion are:

$$\dot{\ell}_i = \frac{\partial K}{\partial L_i} \quad \text{and} \quad \dot{L}_i = -\frac{\partial K}{\partial \ell_i} \quad i=1, \dots, 4 \quad (8)$$

where K is the new Hamiltonian in the extended phase space which can be expressed as:

$$K = \frac{-\mu_o^2}{2L_1^2} - \alpha \ell_4^n L_4 - \alpha \mu^{n-1} (L_1^2 - L_2^2)^{\frac{1}{2}} \sin E - \beta L_2 \frac{\mu}{L_1^4} \left(\frac{a}{r} \right)^2 (L_1^2 - L_2^2)^{\frac{1}{2}} \sin E \quad (9)$$

The first term in Equation (9) is contribution of the two body with constant mass with μ_o is total mass of the system at specific time t_o . The small parameter β is close to zero as well as the small parameter α . In this stage, we can handle the problem as two bodies with two small parameters. But, if we assume that the two small parameters α and β has the same order of magnitude the problem will turn into one small parameter problem. The Hamiltonian (9) can be written in the form:

$$K = \sum_{i=0}^1 \alpha^i K_i, \quad (10)$$

where,

$$K_o = -\frac{\mu_o}{2L_1^2} \quad (10.1)$$

with,

$$K_1 = \sum_{i=1}^2 K_{1i} \quad (10.2)$$

and,

$$K_{11} = -L_4 \ell_4^n - (L_1^2 - L_2^2)^{\frac{1}{2}} \ell_4^{n-1} \sin E$$

$$K_{12} = -\frac{\beta L_2}{\alpha L_1^4} (L_1^2 - L_2^2)^{\frac{1}{2}} \ell_4 \left(\frac{a}{r} \right)^2 \sin E.$$

The canonical equations of motion for the Hamiltonian K are:

$$\dot{\ell}_i = \frac{\partial K}{\partial L_i} \quad \text{and} \quad \dot{L}_i = -\frac{\partial K}{\partial \ell_i} \quad i=1, \dots, 4 \quad (11)$$

Applying the last Equation for the Hamiltonian function represented by Equation (10) yields:

$$\dot{\ell}_1 = \frac{\mu_o^2}{L_1^3} - \alpha \left(\ell_4^{n-1} \frac{L_1^2}{e} \sin E \right) + \frac{\beta L_2}{\ell_4 r^2 e} \left((1-4e^2) \sin E + \frac{2}{r} \left(\frac{L_2}{L_1} \right)^2 \cos E \right) \quad (11.1)$$

$$\dot{\ell}_2 = -\alpha \left(\ell_4^{n-1} \left(\frac{L_2}{L_1 e} \right) \sin E \right) + \frac{\beta}{\ell_4 L_1 r^2} \sin E \left[L_1^2 e - \frac{L_2^2}{e} - \frac{2L_2^2 a}{r} \cos E \right]. \quad (11.2)$$

$$\dot{\ell}_3 = \frac{\partial K}{\partial L_3} = 0 \quad (11.3)$$

$$\dot{\ell}_4 = \frac{\partial K}{\partial L_4} = -\alpha \ell_4^n \quad (11.4)$$

$$\dot{L}_1 = -\alpha \left(\ell_4^{n-1} L_1 \frac{ea}{r} \right) + \beta \frac{L_2 L_1 a e}{\ell_4 r^3} \cdot \left(\cos E - 2 \frac{e}{r} \sin^2 E \right) \quad (11.5)$$

$$\dot{L}_2 = \frac{\partial K}{\partial L_2} = 0 \quad (11.6)$$

$$\dot{L}_3 = \frac{\partial K}{\partial L_3} = 0 \quad (11.7)$$

$$\dot{L}_4 = \frac{\partial K}{\partial L_4} = -\alpha \left[L_4 n \ell_4^{n-1} + L_1 e \sin E (n-1) \ell_4^{n-2} \right] + \beta \frac{L_2}{L_1^3} e \left(\frac{a}{r} \right)^2 \sin E \quad (11.8)$$

3. Short Period Solution Using Hori Method's

The Hameltonian function K is function of ℓ_1, ℓ_4, L_1, L_2 , and L_4 . The integrable part of the Hameltonian, K_0 , is function of L_1 . So the variable ℓ_1 can be considered as the fast variable.

In the next stage we will use Hori's method [18], developed by Kamel [19] to eliminate the short period terms from the Hameltonian. And the transformed Hameltonian, K^* , can be written, up to second orders, as follows:-

3.1. Zero Order

By applying the Hori's Perturbation technique developed by Kamel yields:

$$K_o^* = K_o(-; L_1) = \frac{-\mu_o^2}{2L_1^2} \quad (12)$$

3.2. First Order

The first order transformed Hamiltonian derived by Horis' can be outlined as:-

$$K_1^* = \tilde{K}_1 + (K_o; S_1) \quad (13.1)$$

where;

$$\tilde{K}_1 = K_1. \quad (13.2)$$

And by choosing;

$$K_1^* = \left\langle \tilde{K}_1 \right\rangle_{\ell_1}. \quad (13.3)$$

then

$$K_{1p}^* = \tilde{K}_1 - K_1^* = (S_1; K_o) \quad (13.4)$$

So the first order generating function, S_1 , is therefore obtained by integral:

$$S_1 = \left(\frac{\partial K_o}{\partial L_1} \right)^{-1} \int K_{1p}^* \partial \ell_1. \quad (13.5)$$

By averaging the Hameltonian, K_1 , over the mean anomaly ℓ_1 we can determine the transformed first order Hamiltonian, K_1^* , as:-

$$K_1^* = \frac{1}{2\pi} \int_0^{2\pi} \left[-\ell_4^n L_4 - \ell_4^{n-1} L_1 e \sin E - \frac{\beta L_2}{\alpha L_1^3} e \ell_4 \left(\frac{a}{r} \right)^2 \sin E \right] \partial \ell_1.$$

After calculating the required integrals needed in the last equation we get:

$$K_1^* = -\ell_4^n L_4. \quad (14)$$

By applying Equation (13.4) to calculate the periodic part of the Hamiltonian as:

$$K_{1p}^* = \tilde{K}_1 - K_1^* = K_1 - K_1^* = -\ell_4^{n-1} L_1 e \times \sin E - \frac{\beta L_2}{\alpha L_1^3} e \ell_4 \left(\frac{a}{r} \right)^2 \sin E \quad (15)$$

To calculate the first order generating function, S_1 , we must use Equation (13.5) (with the help of Equations (12) and (15)) which lead, after calculate the required derevativs and products, to the results:-

$$\therefore S_1 = \sum_{i=1}^2 B_i^c \cos iE + B_o \ln \left(\frac{r}{a} \right) \quad (16)$$

where B 's are function of $(\ell_4; L_1, L_2)$ and given by:

$$B_o = -\frac{\beta L_2}{\alpha \mu_o^2} \ell_4 \quad (16.1)$$

$$B_1^c = \frac{L_1^4}{\mu_o^2} e \ell_4^{n-1} \quad (16.2)$$

$$B_2^c = \frac{-L_1^4}{4\mu_o^2} e^2 \ell_4^{n-1} \quad (16.3)$$

3.3. Second Order

The seconded order transformed Hamiltonian derived by Horis' can be outlined as:-

$$K_2^* = \tilde{K}_2 + (K_o; S_2), \quad (17)$$

where;

$$\tilde{K}_2 = K_2 + (K_1 + K_1^*; S_1). \quad (17.1)$$

Choosing;

$$K_2^* = \langle \tilde{K}_2 \rangle_{\ell_1}, \quad (17.2)$$

and;

$$K_{2p}^* = \tilde{K}_2 - K_2^* = (S_1; K_o). \quad (17.3)$$

Hence S_2 therefore can be obtained by;

$$S_2 = \left(\frac{\partial K_o}{\partial L_1} \right)^{-1} \int K_{2p}^* d\ell_1, \quad (17.4)$$

To construct the function \tilde{K}_2 , we need, first, to calculate the Poisson bracket $(K_1 + K_1^*; S_1)$. By the help of

Equations (10.2), (14) and (16) with $K_2 = 0$ yields:

$$\tilde{K}_2 = \sum_{g=2}^5 \sum_{f=-1}^4 C_{g,f}^c \left(\frac{a}{r}\right)^g \cos fE - \sum_{t=2}^6 \sum_{s=-1}^4 C_{t,s}^c \left(\frac{a}{r}\right)^t \cos tE + 2\ell_4^n \sum_{i=1}^2 B_i^{*c} \cos iE + 2\ell_4^n B_0^* \ln\left(\frac{r}{a}\right) \quad (18)$$

where the coefficients $C_{g,f}^c$, $C_{t,s}^c$ and B_i^{*c} are function of $(\ell_4; L_1, L_2)$ defined as:-

$$C_{g,f}^c = \sum_{m=1}^2 \sum_{u=0}^2 \left(\frac{1}{2}\right) A_{m,u}^c \left[B_{g-2m,u-f}^c + B_{g-2m,f-u}^c \right].$$

$$C_{t,s}^c = \sum_{m'=0}^2 \sum_{u'=1}^2 \left(\frac{1}{2}\right) A_{m',u'}^{si} \left[B_{\frac{1}{2}t-(m'+1),u'-s}^{si} - B_{\frac{1}{2}t-(m'+1),u'-s}^{si} \right].$$

$$B_i^{*c} = \frac{\partial B_i^c}{\partial \ell_4}$$

Then, applying Equation (17.2) by averaging over the mean anomaly ℓ_1 for the last Equation (18), we can determine the transformed second order Hamiltonian, K_2^* , as:-

$$K_2^* = \langle \tilde{K}_2 \rangle_{\ell_1} = \left\langle \left(K_1 + K_1^*; S_1 \right) \right\rangle_{\ell_1} = \frac{1}{2\pi} \int_0^{2\pi} \left(K_1 + K_1^*; S_1 \right) \partial \ell_1. \quad (19)$$

Substituting from Equation (18) in the last Equation yields:-

$$K_2^* = \frac{1}{2\pi} \left\{ \int_0^{2\pi} \left[\sum_{g=2}^5 \sum_{f=-1}^4 C_{g,f}^c \left(\frac{a}{r}\right)^g \cos fE \right] \partial \ell_1 - \int_0^{2\pi} \left[\sum_{t=2}^6 \sum_{s=-1}^4 C_{t,s}^c \left(\frac{a}{r}\right)^t \cdot \cos tE \right] \partial \ell_1 \right. \\ \left. + \int_0^{2\pi} 2\ell_4^n \left[\sum_{i=1}^2 B_i^{*c} \cos iE \right] \partial \ell_1 + 2\ell_4^n \int_0^{2\pi} B_0^* \ln\left(\frac{r}{a}\right) \partial \ell_1 \right\} \quad (20)$$

A useful paper for Ahmed (1994) [20], was used to calculate the required integral. While the last integral was solved except for the case of hyperbolic orbits.

$$K_2^* = \ell_4^n \left\{ e B_1^{*c} - 2B_0^* \left(-1 + \sqrt{1-e^2} + \ln[2] - \ln[1 + \sqrt{1-e^2}] \right) \right\}$$

where B_0^* and B_1^{*c} are coefficients, were derived during the calculation of the Poisson brackets $(K_1 + K_1^*; S_1)$, defined as:

$$B_0^* = \frac{\partial}{\partial \ell_4} (B_0) = -\frac{\beta L_2}{\alpha \mu_o^2}.$$

$$B_1^{*c} = \frac{\partial}{\partial \ell_4} (B_1^c) = \frac{e L_1^4}{\mu_o^2} (n-1) \ell_4^{n-2}.$$

Finally, the required transformed second order Hamiltonian can be written as:-

$$K_2^* = \frac{1}{\mu_o^2} \left[e^2 L_1^4 (n-1) \ell_4^{2n-2} + 2\ell_4^n \frac{\beta}{\alpha} \Psi \right] \quad (21)$$

where Ψ is function of L_1 and L_2 and defined by:

$$\Psi = L_2 \left[-0.3068528194 + \sqrt{1-e^2} - \ln(1 + \sqrt{1-e^2}) \right]$$

The last, second orders Hamiltonian, is valid in all cases of orbits except in the case of hyperbolic orbits ($e > 1$).

3.4. The Transformed Hamiltonian

The second order transformed Hamiltonian; K^* can be constructed in the form:

$$K^* = K_0^* + \alpha K_1^* + \alpha^2 K_2^*.$$

Which can be written by using Equations (12), (14) and (21) as:

$$K^* = \frac{-\mu_o^2}{2L_1^2} + \alpha[-\ell_4^n L_4] + \frac{\alpha^2}{2\mu_o^2} \left[e^2 L_1^4 (n-1) \ell_4^{2n-2} + 2\ell_4^n \frac{\beta}{\alpha} \Psi \right]. \quad (22)$$

The equations of motion in the transformed augmented phase space can be derived as follows:

$$\dot{\ell}_1 = \frac{\mu_o^2}{L_1^3} + \frac{\alpha^2}{2\mu_o^2} \left[(4L_1^3 + 2L_1 L_2^2)(n-1) \ell_4^{2n-2} - 2\ell_4^n \frac{\beta}{\alpha} \left(\frac{L_2}{L_1} \right)^3 \left(\frac{1}{1+\sqrt{1-e^2}} \right) \right] \quad (23-1)$$

$$\dot{\ell}_2 = \frac{\alpha^2}{2\mu_o^2} \left[-2L_1^4 (n-1) \ell_4^{2n-2} \frac{L_2}{L_1^2} + 2\ell_4^n \frac{\beta}{\alpha} \Psi_{:L_2} \right]. \quad (23-2)$$

$$\dot{\ell}_3 = 0. \quad (23-3)$$

$$\dot{\ell}_4 = -\alpha \ell_4^n. \quad (23-4)$$

$$\dot{L}_1 = 0. \quad (23-5)$$

$$\dot{L}_2 = 0. \quad (23-6)$$

$$\dot{L}_3 = 0. \quad (23-7)$$

$$\dot{L}_4 = \alpha n \ell_4^{n-1} L_4 - \frac{\alpha^2}{\mu_o^2} \left[L_1^4 e^2 (n-1)(2n-2) \ell_4^{2n-3} + 2n \frac{\beta}{\alpha} \ell_4^{n-1} \Psi \right] \quad (23-8)$$

where $\Psi_{:L_2}$ is the partial differentiation for the function Ψ with respect to the momentum L_2 and given by:

$$\Psi_{:L_2} = \frac{1}{L_1^2 (1+\sqrt{1-e^2})} \left\{ 2L_2^2 + L_1^2 \cdot \left[-0.3068528194 + 0.693147\sqrt{1-e^2} + (1+\sqrt{1-e^2}) \ln(1+\sqrt{1-e^2}) \right] \right\}$$

The Hamiltonian and the system of differential equation, described by Equations (23), has one degree of freedom since the variable ℓ_4 is explicitly appears in them.

3.5. The Element of Short Period Transformation and Its Inverse

The elements of short period transformation can be calculated using equations:

$$\ell_i = \ell'_i + \alpha \frac{\partial}{\partial L'_i} S_1 + O(\alpha^2). \quad (24.1)$$

$$L_i = L'_i + \alpha \frac{\partial}{\partial \ell'_i} S_1 + O(\alpha^2). \quad (24.2)$$

The inverse transformation equations can be calculated as follows:

$$\ell'_i = \ell_i - \alpha \frac{\partial}{\partial L'_i} S_1 - O(\alpha^2). \quad (25.1)$$

$$L'_i = L_i + \alpha \frac{\partial}{\partial \ell'_i} S_1 + O(\alpha^2). \quad (25.2)$$

where $\frac{\partial}{\partial L'_1} S_1$, $\frac{\partial}{\partial \ell'_1} S_1$, $\frac{\partial}{\partial \ell'_4} S_1$ and $\frac{\partial}{\partial L'_2} S_1$ can be calculated using Equation (13.5).

The prime over the variables means that variables in the transformed phase space which will be omitted for the sake of simplicity.

The variable mass ℓ_4 is implicitly function of time through the relation (7'), so we will complete the solution using the method which was introduced by Delva [14] and Hanslmeier [15], separately.

4. The Final Solution

A special linear differential operator, the Lie operator, produces a Lie series to construct an approximate solution. The convergence of the Lie series is similar to that of Taylor series, since the Lie is only another analytic form of the Taylor series. The algorithm for the method was outlined in Delva [14] and Hanslmeier [15].

The Lie operator, D , can be defined as:

$$DX(Q_j, P_j; t) = \frac{d}{dt} X(Q_j, P_j; t) = \sum_{j=1}^4 \left(\frac{\partial K^*}{\partial L_j} \frac{\partial}{\partial \ell_j} - \frac{\partial K^*}{\partial \ell_j} \frac{\partial}{\partial L_j} \right) + \frac{\partial}{\partial t} \quad (26.1)$$

and the second order "D" operator, which can be noted as D^2 , constructed as:

$$D^2 X = \frac{d}{dt} DX \quad (26.2)$$

The coefficients of the operator D , $\frac{\partial K^*}{\partial \ell_j}$ and $\frac{\partial K^*}{\partial L_j}$, can be replaced by the canonical equations of motion represented by Equations (23). Then, Lie operator for our case can be written as:

$$\begin{aligned} D = & \left\{ \frac{\mu_0^2}{L_1^3} + \frac{\alpha^2}{2\mu_0^2} \left[(4L_1^3 + 2L_1L_2^2)(n-1)\ell_4^{2n-2} - 2\ell_4^n \frac{\beta}{\alpha} \left(\frac{L_2}{L_1} \right)^3 \left(\frac{1}{1+\sqrt{1-e^2}} \right) \right] \right\} \frac{\partial}{\partial \ell_1} \\ & + \frac{\alpha^2}{2\mu_0^2} \left[-2L_1^4(n-1)\ell_4^{2n-2} \frac{L_2}{L_1^2} + 2\ell_4^n \frac{\beta}{\alpha} \Psi_{:L_2} \right] \frac{\partial}{\partial \ell_2} \\ & - \alpha \ell_4^n \frac{\partial}{\partial \ell_4} + \left\{ \alpha n \ell_4^{n-1} L_4 - \frac{\alpha^2}{\mu_0^2} \left[L_1^4 e^2 (n-1) \cdot (2n-2) \ell_4^{2n-3} + 2n \frac{\beta}{\alpha} \ell_4^{n-1} \Psi \right] \right\} \cdot \frac{\partial}{\partial L_4} + \frac{\partial}{\partial t}. \end{aligned} \quad (27)$$

Now, we can calculate the solution by applying the D operator, represented by Equation (27), up to second orders, to the variables ℓ_i, L_i and the results are as follows:

$$\begin{aligned} \ell_1 = & \ell_{1,o} \\ & + \left\{ \frac{\mu_0^2}{L_1^3} + \frac{\alpha^2}{2\mu_0^2} \left[(4e^2 L_1^3 + 2L_1 L_2^2) \cdot (n-1) \ell_4^{2n-2} + \frac{2\ell_4^n}{\sqrt{1-e^2}} \frac{\beta}{\alpha} \left(\frac{L_2}{L_1} \right)^3 \cdot \left(-1 + \frac{1}{1+\sqrt{1-e^2}} \right) \right] \right\} \Big|_{\ell_1=\ell_{1,o}} \cdot \frac{(t-t_o)}{1!} \\ & - \frac{\alpha^3}{2\mu_0^2} \left[(4e^2 L_1^3 + 2L_1 L_2^2) (n-1)(2n-2) \cdot \ell_4^{2n-3} + \frac{2n\ell_4^{n-1}}{\sqrt{1-e^2}} \frac{\beta}{\alpha} \left(\frac{L_2}{L_1} \right)^3 \cdot \left(-1 + \frac{1}{1+\sqrt{1-e^2}} \right) \right] \Big|_{\ell_1=\ell_{1,o}} \cdot \frac{(t-t_o)^2}{2!} + \dots \end{aligned} \quad (28)$$

$$\begin{aligned} \ell_2 = & \ell_{2,o} \\ & + \frac{\alpha^2}{2\mu_0^2} \left[2 \frac{\beta}{\alpha} \frac{L_2^2}{L_1^2} \frac{\ell_4^n}{\sqrt{1-e^2}} \cdot \left(1 - \frac{1}{1+\sqrt{1-e^2}} \right) - 2L_1^2 L_2 (n-1) \ell_4^{2n-2} + 2 \frac{\ell_4^n}{L_2} \frac{\beta}{\alpha} \Psi \right] \Big|_{\ell_2=\ell_{2,o}} \cdot \frac{(t-t_o)}{1!} \\ & - \frac{\alpha^3 \ell_4}{2\mu_0^2} \left[2 \frac{\beta}{\alpha} \frac{L_2^2}{L_1^2} \frac{\ell_4^{n-1}}{\sqrt{1-e^2}} \cdot \left(1 - \frac{1}{1+\sqrt{1-e^2}} \right) - 2L_1^2 L_2 (n-1)(2n-2) \ell_4^{2n-3} + 2n \frac{\ell_4^{n-1}}{L_2} \cdot \frac{\beta}{\alpha} \Psi \right] \Big|_{\ell_2=\ell_{2,o}} \cdot \frac{(t-t_o)^2}{2!} + \dots \end{aligned} \quad (29)$$

$$\ell_3 = \ell_{3,o} \quad (30)$$

$$\ell_4 = \ell_{4,o} - \alpha \ell_4^n \Big|_{\ell_4=\ell_{4,o}} \cdot \frac{(t-t_o)}{1!} + n \alpha^2 \ell_4^{2n-1} \Big|_{\ell_4=\ell_{4,o}} \cdot \frac{(t-t_o)^2}{2!} + \dots \quad (31)$$

$$L_1 = L_{1,o} \quad (32)$$

$$L_2 = L_{2,o} \quad (33)$$

$$L_3 = L_{3,o} \quad (34)$$

$$L_4 = L_{4,o} + \left\{ \alpha n \ell_4^{n-1} L_4 - \frac{\alpha^2}{2\mu_0^2} \left[L_1^4 e^2 (n-1) \cdot (2n-2) \ell_4^{2n-3} + 2 \frac{\beta}{\alpha} n \ell_4^{n-1} \Psi \right] \right\} \Big|_{L_4=L_{4,o}}$$

$$\cdot \frac{(t-t_o)}{1!} + \left\{ \alpha^2 \ell_4^{2n-2} L_4 n - \frac{\alpha^3}{2\mu_0^2} \cdot \left[2 \frac{\beta}{\alpha} n \ell_4^{2n-2} \Psi - \alpha L_1^4 e^2 (n-1) (2n-3) \cdot (2n-2) \ell_4^{3n-4} \right] \right\} \Big|_{L_4=L_{4,o}} \frac{(t-t_o)^2}{2!} + \dots \quad (35)$$

where the quantities $\ell_{1,o}$, $\ell_{2,o}$, $\ell_{3,o}$, $\ell_{4,o}$, $L_{1,o}$, $L_{2,o}$, $L_{3,o}$ and $L_{4,o}$ are the orbital elements at time t_0 . Using the initial values for the orbital elements, with the help of Equations (28) - (35), we can calculate the new perturbed orbital elements due to the varying mass.

5. Discussion and Conclusions

The last group of equations is a second order solution for the two body problem with m_1 varying mass. In those equations, the following results are included.

1. The varying mass gives rise to the change of mean anomaly ℓ_1 and the argument of periside ℓ_2 . That rises influenced by both the varying mass and the periastron effect.
2. A new varying, second order, mass formula was derived and represented by Equation (31).
3. The varying mass, as well as the periastron effects, does not affect the argument of ascending node ℓ_3 and the momenta L_1 , L_2 and L_3 in the extended transformed phase space.
4. A second order equation for calculation of the momentum associated to the varying mass, L_4 , as well as the periastron effect, was derived and represented by Equation (35).

Future work: We plane to apply these equations for deferent systems as well as comparing the results with others and with observations.

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