Intelligent Information Management, 2010, 2, 295-298
doi:10.4236/iim.2010.23034 Published Online April 2010 (http://www. SciRP.org/journal/iim)

On the Reflected Geometric Brownian Motion with Two Barriers

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Received January 28, 2009; *revised March* 3, 2010; *accepted April* 7, 2010

Abstract

In this paper, we are concerned with Reflected Geometric Brownian Motion (RGBM) with two barriers. And the stationary distribution of RGBM is derived by Markovian infinitesimal Generator method. Consequently the first passage time of RGBM is also discussed.

Keywords: Geometric Brownian Motion, Stationary Distribution, First Passage Time

1. Introduction

We consider a finite-capacity fluid queue, the level of which at time t is denoted by Z_t. And Z_t satisfies the following differential equation:

$$\begin{cases} dZ_t = (\alpha_1 - \alpha_2) Z_t dt + \sigma Z_t dB_t + dL_t - dU_t \\ Z_0 = x \in [a, d] \end{cases} \qquad (1)$$

This model shows fluid arrives into this queue at rate $\alpha_1 Z_t$ and leaves the queue at rate $\alpha_2 Z_t$. This fluid level can be also varied by a local variance function σZ_t and a standard Brownian motion B_t. L_t and U_t are non-decreasing processes, interfering only when Z_t hits *a* or *d* and make Z_t vary between *a* and *d*.

In particular, when $a \to 0^+$ and $d \to +\infty$, L_t and U_t disappear. Then the process Z_t becomes Geometric Brownian Motion. So we call Z_t determined by (1) Reflected Geometric Brownian Motion(RGBM).

Speak precisely, we are concerned with RGBM $Z = \{Z_t, t \ge 0\}$ with two barriers *a* and *d* (*d* > *a* > 0), which is defined by

$$\begin{cases} dZ_t = \mu Z_t dt + \sigma Z_t dB_t + dL_t - dU_t \\ Z_0 = x \in [a, d] \end{cases} \qquad (2)$$

where $B = \{B_t, t \ge 0\}$ is a standard Brownian Motion, $\sigma > 0, \mu$ and σ are constants and satisfy $\mu \ne \frac{\sigma^2}{2}$.

Moreover, the processes *L* and *U* are uniquely determined by the following property [1,2]:

1) Both *L* and *U* are continuous nondecreasing processes with $L_0 = U_0 = 0$;

2) *L* and *U* increase only when $Z_t = a$ and $Z_t = d$; respectively, *i.e.*,

$$\int_0^t \mathrm{I}_{\{Z_s = a\}} dL_s = L_t, \int_0^t \mathrm{I}_{\{Z_s = d\}} dU_s = U_t, \; for \;\; t \ge 0.$$

According to the theory of stochastic differential equation, (2) is equivalent to

$$Z_t = \mathrm{x} + \mu \int_0^t Z_s ds + \sigma \int_0^t Z_s dB_s + L_t - U_t \qquad (3)$$

Such a process is a regenerative Markov process with state space [*a*,*d*] compact. Then it has a unique stationary distribution [1,3,4]. In the coming section, our objective is to derive the stationary distribution and give an expression for the Laplace Transform of the first passage time of RGBM $Z = \{Z_t, t \ge 0\}$ by the method in [5-7].

2. Main Results on RGBM

2.1. On the Stationary Distribution of RGBM

In this section, we firstly give a Lemma on the stationary distribution of the reflected process $Z = \{Z_t\}$ with two-sided barriers and omit its proof.

Lemma 2.1 Let *Z* be the RGBM defined by (2) (or(3)). Then, as a Markov process, the stationary distribution $\pi(\cdot)$ of the process must satisfy the following equation

$$\int_a^d Af(x)\pi(dx)=\int_a^d\{\frac{\sigma^2x^2}{2}f''(x)+\mu xf'(x)+f'(a)\gamma(x)-f'(d)\beta(x)\}\pi(dx)=0 \quad (4)$$

where $\gamma(x)=\lim_{t\to\infty}\frac{dE_xL_t}{dt},\beta(x)=\lim_{t\to\infty}\frac{dE_xU_t}{dt}$ and $f\in C_b^2([a,d])$ which denotes the space of all bounded continuous functions having twicely continuous derivatives on [*a*,*d*].

Proof. See similar argument in [1].

Suppose $\pi(dx)=p(x)dx$ be a probability distribution on [*a*,*d*] and satisfies that

$$\int_a^d Af(x)p(x)dx=0 \quad (5)$$

for $\forall f\in C_b^2([a,d])$.

Define $A_1f(x)=\frac{\sigma^2x^2}{2}f''(x)+\mu xf'(x)$, then by (4) and (5) it is equivalent to the following equation (Note that $\int_a^d p(x)dx=1$)

$$\int_a^d A_1f(x)p(x)dx+f'(a)\gamma-f'(d)\beta=0 \quad (6)$$

where $\gamma=\int_a^d\gamma(x)p(x)dx$ and $\beta=\int_a^d\beta(x)p(x)dx.$

On one hand, γ and β can be computed by the same method in [5].

Proposition 2.1 Choose $f_1=x^{-\frac{2\mu}{\sigma^2}}$ and $f_1=x^{-1}$, then they respectively satisfy the following equations,

$$A_1f_1(x)=0,x\in[a,d]$$
$$A_1f_2(x)=1,x\in[a,d]$$

Then we have

$$\beta=\frac{\mu-\frac{\sigma^2}{2}}{d^{\frac{2\mu}{\sigma^2}-1}-a^{\frac{2\mu}{\sigma^2}-1}}d^{\frac{2\mu}{\sigma^2}},\gamma=\frac{\mu-\frac{\sigma^2}{2}}{d^{\frac{2\mu}{\sigma^2}-1}-a^{\frac{2\mu}{\sigma^2}-1}}a^{\frac{2\mu}{\sigma^2}}.$$

Proof. A straightforward calculation.

On the other hand, since $p(x)$ satisfies that for all $f\in C_b^2([a,d])$,

$$\int_a^d[\frac{\sigma^2x^2}{2}f''(x)+\mu xf'(x)]p(x)dx+f'(a)\gamma-f'(d)\beta=0$$

By twice integral changes, the above equation becomes that

$$\int_a^d[\frac{\sigma^2x^2}{2}p''(x)+(2\sigma^2-\mu)xp'(x)+(\sigma^2-\mu)p(x)]f(x)dx+[\frac{\sigma^2x^2}{2}f'(x)p(x)+\mu xp(x)f(x)]_a^d$$
$$-[(\sigma^2xp(x)+\frac{\sigma^2x^2}{2}p'(x))f(x)]_a^d+f'(a)\gamma-f'(d)\beta=0$$

i.e.

$$\int_a^d[\frac{\sigma^2x^2}{2}p''(x)+(2\sigma^2-\mu)xp'(x)+(\sigma^2-\mu)p(x)]f(x)dx+f(d)[\mu dp(d)-\sigma^2dp(d)+\frac{\sigma^2d^2}{2}p'(d)]$$
$$-f(a)[\mu ap(a)-\sigma^2ap(a)+\frac{\sigma^2a^2}{2}p'(a)] \quad (7)$$
$$+f'(d)[\frac{\sigma^2d^2}{2}p(d)-\beta]-f'(a)[\frac{\sigma^2a^2}{2}p(a)-\gamma]=0$$

Assume that $f\in C_b^2([a,d])$, satisfying that $f(d)=0$, $f'(d)=0$, and $P(x)$ satisfies $\frac{\sigma^2a^2}{2}p(a)=\gamma$ and $\mu ap(a)-\sigma^2ap(a)+\frac{\sigma^2a^2}{2}p'(a)=0$, then it follows from (7) that

$$\int_a^d[\frac{\sigma^2x^2}{2}p''(x)+(2\sigma^2-\mu)xp'(x)+(\sigma^2-\mu)p(x)]f(x)dx=0 \quad (8)$$

Summarizing the discussion, we get the following theorem.

Theorem 2.1 $p(x)=\frac{\frac{2\mu}{\sigma^2}-1}{d^{\frac{2\mu}{\sigma^2}-1}-a^{\frac{2\mu}{\sigma^2}-1}}x^{\frac{2\mu}{\sigma^2}-2}$ is the solution of

$$\begin{cases}\frac{\sigma^2x^2}{2}p''(x)+(2\sigma^2-\mu)xp'(x)+(\sigma^2-\mu)p(x)=0\\ \frac{\sigma^2a^2}{2}p(a)=\gamma\\ \int_a^d p(x)dx=1\end{cases} \quad (9)$$

Then for all $f\in C_b^2([a,d])$ satisfying $f(d)=0$, $f'(d)=0$, (5) holds, *i.e.* $\int_a^d Af(x)p(x)dx=0$.

Furthermore (5) holds for all $f\in C_b^2([a,d])$. This implies that $\pi(dx)=p(x)dx$ is a stationary distribution of the corresponding Markov process $Z=\{Z_t,t\geq 0\}$.

Remark 2.1 This theorem is a standard application of renewal theorems, so we sketch its proof.

Thus $p(x)=\dfrac{\frac{2\mu}{\sigma^2}-1}{d^{\frac{2\mu}{\sigma^2}-1}-a^{\frac{2\mu}{\sigma^2}-1}}x^{\frac{2\mu}{\sigma^2}-2}, x\in[a,d]$ is the density of the stationary distribution of RGBM. Finally we will give an expression for the Laplace transform of the first passage time of RGBM.

2.2. On the First Passage Time of RGBM

In this section, we consider Equation (2). Let $y\in[a,d]$, define the first passage time by $T(y):=\inf\{t\geq 0: Z_t=y\}$, with the usual convention $\inf\varnothing=\infty$. On the other hand, suppose $\lambda>0$, for $f\in C_b^2([a,d])$, define a operator

$$A^{(\lambda)}f(x)=\frac{\sigma^2x^2}{2}f''(x)+\mu x f'(x)-\lambda f(x), x\in[a,d]$$

Finally we are going to give the expression of the Laplace transform of $T(y)$.

Theorem 2.2. For $x\in[a,d]$ and $\lambda>0$, then

$$E_x(e^{-\lambda T(y)})=\frac{f_1^\lambda(x)}{f_1^\lambda(y)}, x\leq y\leq d \qquad (10)$$

$$E_x(e^{-\lambda T(y)})=\frac{f_2^\lambda(x)}{f_2^\lambda(y)}, a<y\leq x \qquad (11)$$

where

$$f_1^\lambda(x)=x^{\frac{\sigma^2-2\mu+\sqrt{(\sigma^2-2\mu)^2+8\lambda\sigma^2}}{2\sigma^2}}+c_1x^{\frac{\sigma^2-2\mu-\sqrt{(\sigma^2-2\mu)^2+8\lambda\sigma^2}}{2\sigma^2}},$$

$$f_2^\lambda(x)=x^{\frac{\sigma^2-2\mu+\sqrt{(\sigma^2-2\mu)^2+8\lambda\sigma^2}}{2\sigma^2}}+c_2x^{\frac{\sigma^2-2\mu-\sqrt{(\sigma^2-2\mu)^2+8\lambda\sigma^2}}{2\sigma^2}},$$

and

$$C_1=\frac{(\sqrt{(\sigma^2-2\mu)^2+8\lambda\sigma^2}+(\sigma^2-2\mu))^2}{8\lambda\sigma^2}a^{2\sqrt{(\sigma^2-2\mu)^2+8\lambda\sigma^2}},$$

$$C_2=\frac{(\sqrt{(\sigma^2-2\mu)^2+8\lambda\sigma^2}+(\sigma^2-2\mu))^2}{8\lambda\sigma^2}d^{2\sqrt{(\sigma^2-2\mu)^2+8\lambda\sigma^2}}.$$

Proof. Let $h(t,x)=e^{-\lambda t}f(x)$ for $f\in C_b^2([a,d])$. Then applying $It\hat{o}$ formula for $h(t,x)$, we have

$$h(t,Z_t)=h(0,Z_0)+\int_0^t\frac{\partial h}{\partial s}(s,Z_s)ds+\int_0^t\frac{\partial h}{\partial Z_s}(s,Z_s)dZ_s$$
$$+\frac{1}{2}\int_0^t\frac{\partial^2 h}{\partial Z_s^2}(s,Z_s)d<Z,Z>_s$$
$$=\int_0^t e^{-\lambda s}\left[\frac{\sigma^2 {Z_s}^2}{2}f''(Z_s)+\mu Z_s f'(Z_s)-\lambda f(Z_s)\right]ds$$
$$+f(Z_0)+\int_0^t e^{-\lambda s}f'(Z_s)dL_s-\int_0^t e^{-\lambda s}f'(Z_s)dU_s$$
$$+\sigma\int_0^t e^{-\lambda s}Z_s f'(Z_s)dB_s$$
$$=\int_0^t e^{-\lambda s}A^{(\lambda)}f(Z_s)ds+f(Z_0)+f'(a)\int_0^t e^{-\lambda s}dL_s$$
$$-f'(d)\int_0^t e^{-\lambda s}dU_s+\sigma\int_0^t e^{-\lambda s}Z_s f'(Z_s)dB_s \qquad (12)$$

The last equation holds, for L_t and U_t increase only when $Z_t=a$ and $Z_t=d$. Let $T<\infty$ be a stopping time and $x\in[a,d]$. It follows from martingale optional theorem, that

$$E_x\left[e^{-\lambda T}f(Z_T)\right]$$
$$=f(x)+f'(a)E_x\left[\int_0^T e^{-\lambda s}dL_s\right] \qquad (13)$$
$$-f'(d)E_x\left[\int_0^T e^{-\lambda s}dU_s\right]+E_x\left[\int_0^T e^{-\lambda s}A^{(\lambda)}f(Z_s)ds\right]$$

In particular, take $T=T(y)$ for $y\in[a,d]$, and note that

$$\int_0^{T(y)}e^{-\lambda s}dU_s=0, \textit{ for } \quad x\leq y\leq d,$$

and

$$\int_0^{T(y)}e^{-\lambda s}dL_s=0, \textit{ for } \quad a\leq y\leq x,$$

Then

$$E_x(e^{-\lambda T(y)}f(Z_{T(y)}))$$
$$=f(x)+E_x(\int_0^{T(y)}e^{-\lambda s}A^{(\lambda)}f(Z_s)ds) \qquad (14)$$
$$+f'(a)E_x(\int_0^{T(y)}e^{-\lambda s}dL_s), \textit{ for } \quad x\leq y\leq d,$$

and

$$E_x(e^{-\lambda T(y)}f(Z_{T(y)}))$$
$$=f(x)+E_x(\int_0^{T(y)}e^{-\lambda s}A^{(\lambda)}f(Z_s)ds) \qquad (15)$$
$$-f'(d)E_x(\int_0^{T(y)}e^{-\lambda s}dU_s), \textit{ for } \quad a\leq y\leq x,$$

Replace f by f_1^λ in (14) and by f_2^λ in (15), we immediately get (10) and (11) by $Z_{T(y)}=y$, $f_1^\lambda{}'(a)=0$ and $f_2^\lambda{}'(d)=0$. Thus the Proof of the theorem is completed.

3. Conclusions

This paper studies Reflected Geometric Brownian Motion (RGBM) with two barriers. Both the stationary distribution and Laplace transform of the first passage time of RGBM are derived. The studies for RGBM have not only practical significance, but also give an important result in theory of stochastic process.

4. Acknowledgements

This research is supported by the National Natural Science foundation of China (Grant No.70671074) and the Research Foundation of Tianjin university of Science and technology (Grant No.20080207). The authors would like to thank an anonymous referee for his constructive comments and suggestions on the first version of the manuscript.

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