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Lexicographic Constant-Weight Equidistant Codes over the Alphabet of Three, Four and Five Elements

Todor Todorov, Galina Bogdanova, Teodora Yorgova
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Bulgaria
E-mail: *todor@math.bas.bg, galina@math.bas.bg, teda_aj@yahoo.com*
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Abstract

In this paper we consider the problem of finding bounds on the size of lexicographic constant-weight equidistant codes over the alphabet of three, four and five elements with $2 \le w < n \le 10$. Computer search of lexicographic constant-weight equidistant codes is performed. Tables with bounds on the size of lexicographic constant-weight equidistant codes are presented.

Keywords: Lexicographic Codes, Equidistant Codes, Constant-Weight Codes, Bounds of Codes

1. Introduction

Consider a finite set of q elements and containing a distinguished element "zero". The choice of a set does not matter in our context and we will use the set Z_q of integers modulo q. Let Z_q^n be the set of n-tuples (or vectors) over Z_q and $Z_q^{n,w}$ be the set of n-tuples over Z_q of Hamming weight w.

A code is called *equidistant* if all the Hamming distances between distinct codewords are d. A code is called *constant-weight* if all the codewords have the same Hamming weight w. Let $B_q(n,d)$ denote the maximum number of codewords in an equidistant code over Z_q of length n and distance d (called an $(n,M,d)_q$ equidistant code or EC) and $B_q(n,d,w)$ denote the maximum number of codewords in an constant-weight equidistant code over Z_q of length n, distance d, and weight w (called an $(n,M,d,w)_q$ *constant-weight equidistant* code or ECWC). Equidistant codes have been investigated by a large number of authors, mainly as examples of designs and other combinatorial objects [1]. Some works published on this topic are [2–5].

Constant-weight codes have been studied by many authors. For some references for the binary case, see Brouwer *et al.* [6], Agrell [7] and for the ternary case, see Bogdanova [8] and Svanström [9]. A few papers study codes which are both equidistant and of constant weight, for example [10,11].

Let B be an ordered base $b_1, b_2, \ldots, b_n$ over Z_q^n and let $x = \lambda b_1 + \lambda b_2 + \ldots + \lambda b_n$ and $y = \mu b_1 + \mu b_2 + \ldots + \mu b_n$ be vectors from Z_q^n.

Definition 1 We say that x precedes y in lexicographical order if $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ precedes $(\mu_1, \mu_2, \ldots, \mu_n)$ in lexicographical order, *i.e.* $\lambda_1 \le \mu_1, \lambda_2 \le \mu_2 \ldots, \lambda_n \le \mu_n$.

Lexicographic codes of length n and Hamming distance d are obtained by considering all q-ary vectors with weight w in lexicographic order, and adding them to the code if they are at a distance exactly d from the words that have been added earlier. Lexicographic codes were introduced by Conway and Sloane in [12,13]. These codes may be regarded as a heuristically good "approximation" to the optimal codes because of their good parameters and coding performance. Brualdi and Pless [14] have examined a similar generalization of lexicographic codes, known as greedy codes, and presented certain bounds on their parameters.

In the present paper is considered the problem of finding bounds on the size of *lexicographic constant-weight equidistant codes* over the alphabet of three, four and five elements. Some general bounds for equidistant and constant-weight equidistant codes are presented in Section 2. In Section 3 are described computer search methods used in our research. Section 4 presents main results in obtained in this paper. In Section 5 are described some aspects of future work in this area. In Section 6 are shown tables with bounds on the size of lexicographic constant-weight equidistant codes over the alphabet of three, four and five elements.

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2. Preliminaries

Some bounds for ECWC and EC are given by the following theorems:

Theorem 1 [11]

$$B_q(n,d) = 1 + B_q(n,d,d)$$

Theorem 2 (Plotkin) [4,15]

$$B_q(n,d) \le \frac{dq}{dq - n(q-1)}$$

if the denominator is positive.

Theorem 3 (Delsarte) [16]

$$B_q(n,d) \le (q-1)n + 1 .$$

Theorem 4

$$B_q(n,n,w) \le q ,$$

$$B_q(n+1,d,w) \ge B_q(n,d,w) ,$$

$$B_q(n+1,d,w+1) \ge B_q(n,d,w) .$$

The proof of Theorem 4 is easy and we omit it here.

Theorem 5 (Trivial values)

$$B_3(n,d,w) = 1 \;\; if \;\; d > 2w ,$$

$$B_q(n,d,n) = B_{q-1}(n,d) .$$

Theorem 6 (the Johnson bounds for ECWC)
The maximum number of codewords in a q-ary ECWC satisfies the inequalities:

$$B_q(n,d,w) \le \frac{n}{n-w} B_q(n-1,d,w) ,$$

$$B_q(n,d,w) \le \frac{n(q-1)}{w} B_q(n-1,d,w-1) .$$

The proof of the Theorem 6 is the same as the proof of Johnson bound for constant-weight codes [9].

Theorem 7 [11] For $k=1,2,...,n$, if $P_k^2(w) \ge P_k(d)P_k(0)$, then

$$B_q(n,d,w) \le \frac{P_k^2(0) - P_k(d)P_k(0)}{P_k^2(w) - P_k(d)P_k(0)}$$

Here $P_k(x)$ is the Krawtchouk polynomial defined by

$$P_k(x) = \sum_{i=0}^{k} \binom{x}{i}\binom{n-x}{k-i}(-1)^i (q-1)^{k-i}$$

and

$$P_k(0) = \binom{n}{k}(q-1)^k$$

Definition 2 [5] A balanced incomplete block design with parameters (*v,b,k,r,λ*) (BIB design (*v,b,k,r,λ*) is defined as an array of *v* different symbols or elements in *b* subsets or blocks such that every block contains *k*<*v* different elements, each element occurs in *r* blocks, and each pair of elements occurs in *λ* blocks.

Definition 3 [5] A BIB design is called resolvable (an RBIB design), if its *b* blocks can be separated into *r* groups or repetitions of *q* blocks in such a way that each of the *v* elements occurs exactly once in each repetition.

Theorem 8 [5] The optimal equidistant $(n,qt,d)_q$ codes and RBIB designs (*v=qk,b,k,r,λ*) are equivalent to one another and their parameters are connected by the conditions *v=M*, *b=nq*, *k=t*, *r=n*, *λ =n-d*.

Theorem 9 If there exists an $(n,M,d,w)_q$ code, then there exists a $(\lambda n,M,\lambda d,\lambda w)_q$ code for all integers $\lambda \ge 1$.

3. Computer Search of Lexicographic Constant-Weight Equidistant Codes

In the present paper is considered the problem of finding bounds on the size of lexicographic constant-weight equidistant codes using computer search methods. We apply greedy search beginning with an empty array and while looping through all possible codewords, we add one if it has weight *w* and is on distance *d* from every member of the current code. In order to restrict our search and to compare founded lexico-graphic codes with optimal constant-weight equidistant codes we use theorems from Section 2 and results from [17,18].

For improving the results we use lexicographic codes with seed. Lexicographic codes with a seed are obtained in a similar way as the standard lexicographic codes. The difference is that we use an initial set of vectors (called a seed) instead of the empty set. In order to find lexicographic codes we can start the search in the following ways:

Without seeds: As a result, we construct lexicographic code where the first codeword is the first word in the set of vectors Z_q^n;

With seeds: In this case, it is important to choose a proper seed from one or more vectors. We apply the following methods: exhaustive search, consecutively choosing the possible seeds from restricted area of Z_q^n and search with randomly selected seed.

Remark: In some cases of searching with seed with one codeword we are applying cyclic shift of the space before start searching (see **Figure 1**). Codewords from first to the index of the codeword included in the seed are moved at the end of the space. Thus we have two parts of lexicographically ordered spaces. Then we obtain greedy search. This search produces better results in some cases.

Example: We consider searching of $(9,8,3,3)_4$ lexicographic constant-weight equidistant code.

If we apply search without seed the best code that we can obtain is with three codewords (see **Figure 2**).

But if we use search with a seed we can obtain code with eight codewords (which is number of codewords of optimal code with such parameters) (see **Figure 3**).

4. Results

We obtain bounds for constant-weight equidistant codes using standard lexicographic codes or lexicographic codes with seeds. We use searching methods described in the previous section. These methods are included in coding theory computer package *QPlus* [19]. This software includes options for searching with or without a seed and seed size choosing. Also it offers searching with manually selected seed or automatically trying all possible seeds to find best code with given parameters. *QPlus* includes option for cyclic shift of the searching space. We perform search with all possible seeds with size one in order to find the best possible code. If the size of the space is not too large we try with seeds with bigger size. Results shown on the next tables are obtained using lexicographic code unit of *QPlus*. Tables with bounds on the size of lexicographic constant-weight equidistant codes for $q=3$, $q=4$ and $q=5$ and for $2 \le w < n \le 10$ are presented in tables, in the Appendix.

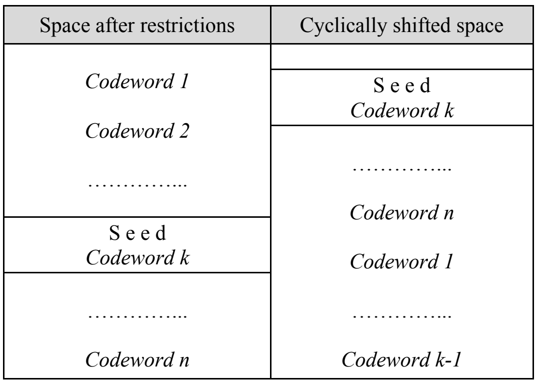

Figure 1. Cyclic shift of space.

$(9,3,3,3)_4$ No seed
000000111
000000222
000000333

Figure 2. $(9,3,3,3)_4$ **- No seed.**

$(9,8,3,3)_4$ Seed 000001011
000001011
000000112
000000221
000001120
000001202
000002022
000002101
000002210

Figure 3. $(9,8,3,3)_4$ **- Seed 000001011.**

5. Conclusions

We use computer search methods in order to solve the problem of finding bounds on the size of lexicographic constant-weight equidistant codes over the alphabet of three, four and five elements with $2 \le w < n \le 10$. Interesting fact that could be used for future work is that in most of the cases lexicographic codes coincide with optimal such codes [17,18].

6. Appendix

In **Tables 1, 2** and **3** are shown bounds on the size of lexicographic constant-weight equidistant codes over the alphabet of three, four and five elements.

Table 1. Lexicographic bounds for $B_3(n,d,w)$ **.**

n	w	d							
		3	4	5	6	7	8	9	10
4	2	3	2						
	3	8	2						
5	2	3	2						
	3	8	5	2					
	4	8	5	2					
6	2	3	3						
	3	8	5	4	2				
	4	8	6	4	3				
	5	8	6	3	2				
7	2	3	3						
	3	8	7	4	2				
	4	8	7	7	3	2			
	5	8	6	6	3	2			
	6	8	6	7	2	2			
8	2	3	4						
	3	8	7	4	2				
	4	8	7	7	5	2	2		
	5	8	8	7	8	3	2		
	6	8	6	7	8	3	2		
	7	8	8	8	4	2	2		
9	2	3	4						
	3	8	7	4	3				
	4	8	7	7	9	3	2		
	5	8	8	7	9	5	3	2	
	6	8	8	7	11	6	3	3	
	7	8	8	8	12	5	3	2	
	8	8	8	8	9	3	2	2	
10	2	3	5						
	3	8	7	4	3				
	4	8	7	7	15	5	2		
	5	8	8	7	11	8	4	2	2
	6	8	8	7	14	8	5	3	2
	7	8	8	8	12	9	5	3	2
	8	8	8	8	12	10	5	2	2
	9	8	8	8	10	5	2	2	2

Table 2. Lexicographic bounds for $B_4(n,d,w)$.

n	w	d							
		3	4	5	6	7	8	9	10
4	2	3	2						
	3	8	4						
	4	9	3						
5	2	3	2						
	3	8	10	2					
	4	9	15	3					
	5	9	6	3					
6	2	3	3						
	3	8	10	4	2				
	4	9	15	9	3				
	5	9	15	8	3				
	6	9	7	4	3				
7	2	3	3						
	3	8	10	7	2				
	4	9	15	9	5	2			
	5	9	15	9	7	3			
	6	9	15	9	7	3			
	7	9	8	7	3	3			
8	2	3	4						
	3	8	10	7	2				
	4	9	15	9	8	2	2		
	5	9	15	11	10	5	2		
	6	9	15	9	12	5	4		
	7	9	15	11	12	4	3		
	8	9	8	8	9	3	3		
9	2	3	4						
	3	8	10	7	3				
	4	9	15	9	9	3	2		
	5	9	15	11	10	9	3	2	
	6	9	15	11	12	11	5	3	
	7	9	15	11	12	11	5	3	
	8	9	15	11	12	11	4	3	
	9	9	8	8	12	6	3	3	
10	2	3	5						
	3	8	10	7	3				
	4	9	15	9	15	5	2		
	5	9	15	11	10	10	6	2	2
	6	9	15	11	14	11	10	5	2
	7	9	15	11	12	11	14	5	3
	8	9	15	11	12	11	15	5	3
	9	9	15	11	12	11	10	4	3
	10	9	8	8	9	10	6	3	3

Table 3. Lexicographic bounds for $B_5(n,d,w)$.

n	w	d							
		3	4	5	6	7	8	9	10
4	2	3	2						
	3	8	4						
	4	9	4						
5	2	4	2						
	3	8	10	2					
	4	9	15	5					
	5	9	16	4					
6	2	3	3						
	3	8	10	4	2				
	4	9	15	9	3				
	5	9	16	24	4				
	6	9	16	9	4				
7	2	4	3						
	3	8	10	7	2				
	4	9	15	13	6	2			
	5	9	16	24	12	3			
	6	9	16	24	14	4			
	7	9	16	10	8	4			
8	2	4	4						
	3	8	10	7	2				
	4	9	15	13	9	2	2		
	5	9	16	24	9	6	2		
	6	9	16	24	12	7	4		
	7	9	16	24	10	4	4		
	8	9	16	10	12	5	4		
9	2	4	4						
	3	8	10	7	3				
	4	9	15	13	11	3	2		
	5	9	16	24	9	8	3	2	
	6	9	16	24	11	10	6	3	
	7	9	16	24	10	4	6	4	
	8	9	16	24	12	4	4	4	
	9	9	16	10	12	4	4	4	
10	2	4	5						
	3	8	10	7	3				
	4	9	15	13	15	5	2		
	5	9	16	24	9	9	6	2	2
	6	9	16	24	11	10	10	3	2
	7	9	16	24	10	4	9	6	3
	8	9	16	24	12	4	4	5	5
	9	9	16	24	12	4	4	4	4
	10	9	16	10	12	4	10	4	4

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