

A New Approach for a Class of Optimal Control Problems of Volterra Integral Equations

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Abstract

In this paper, we propose a new approach for a class of optimal control problems governed by Volterra integral equations which is based on linear combination property of intervals. We convert the nonlinear terms in constraints of problem to the corresponding linear terms. Discretization method is also applied to convert the new problems to the discrete-time problem. In addition, some numerical examples are presented to illustrate the effectiveness of the proposed approach.

Keywords: Volterra Integral Equations, Optimal Control, Linear Programming

1. Introduction

Consider the following optimal control problem governed by Volterra integral equation (OCV):

$$\text{minimize } G(y(T)) + \int_0^T F(t, y(t), u(t)) dt \quad (1)$$

$$\text{subject to } y(t) = p(t) + \int_0^t K(t, s, y(s), u(s)) ds, \quad (2) \\ 0 \leq t \leq T$$

where $y(\cdot)$ and $u(\cdot)$ are the state and control functions respectively on $[0, T]$. The integral Equation (2) is applied in a natural way in the study of economic problems, population dynamics and etc., see for instance Hritonenko and Yatsenco [1], and Kamien and Schwartz [2]. The OCV problem (1)-(2) has been studied by many authors, including Neustadt [3-5], Bakke [6], Carlson [7], Vinokurov [8], Medhim [9], Schmidt [10-13], Wolfansdorf [14], Elnagar, Kazemi and Kim [15], Pan and Teo [16], Angell [17,18], Belbas [19,20], Carlier and Tahraoui [21], and Burnap, and Kazemi [22]. The method usually employed for OCV problem (1)-(2) is method of necessary conditions of the type of Pontryagin maximum principle. In the Recent works, Vega [23] gives the necessary condition for optimal terminal time of OCV problem (1)-(2) and verifies the terminal time T and final state $y(T)$ by conditions $\varphi(T, y(T)) = 0$ and $\psi(T, y(T)) \leq 0$. Bonnens and Vega [24] discuss problem (1)-(2) with running state on the initial and final states. Also, Belbas

[25] applied the ideas of dynamic programming to OCV problem (1)-(2).

In this paper, we are interested to solving the following class of the OCV problem (1)-(2) which we called it COCV problem:

$$\text{minimize } \int_0^T c(t) y(t) dt \quad (3)$$

$$\text{subject to } y(t) = p(t) + \int_0^t [f(s, u(s)) + d(t, s) y(s)] ds, \\ 0 \leq t \leq T \quad (4)$$

where function $f(\cdot, \cdot)$ is a continuous function. A controlled Volterra integral equation similar to equation (4) is discussed in [16]. We suppose that $u(t) \in U$, $t \in [0, T]$ where U a compact and connected set. In addition, we let the final state $y(T)$ is a given known number. Here, the linear combination property of intervals is used to convert nonlinear controlled Volterra integral Equation (4) to the equivalent linear equation. The new optimal control problem with this linear Volterra integral equation is transformed to a discrete-time problem that could be solved by linear programming methods. This paper organized as follows. Section 2, transforms the nonlinear function $f(\cdot, \cdot)$ to a corresponding function that is linear respect to a new control function. Section 3, converts the new problem to the discrete-time problem via discretization. In Section 4, numerical examples are presented to illustrated effectiveness of this approach.

Finally, the conclusion of this paper is given in Section 5.

2. Metamorphosis of the COCV Problem

In this section, COCV problem (1) is transformed to the new equivalent problem. First, we state and prove the following two theorems:

Theorem 2.1: Let $f(.,.)$ be a continuous function on $[0, T] \times U$ where U is a compact and connected subset of \mathbb{R}^m , then for any arbitrary (but fixed) $s \in [0, T]$ the set $\{f(s, u) : u \in U\}$ is a closed interval in \mathbb{R} .

Proof: Assume that $s \in [0, T]$ be given. Let $\phi(u) = f(s, u)$. Obviously, $\phi(.)$ is a continuous function on U . Since, continuous function preserve compactness and connectedness, the set $\{\phi(u) : u \in U\}$ is compact and connected. Therefore, $\{\phi(u) : u \in U\}$ is a closed interval in \mathbb{R} . □

For any $s \in [0, T]$, we may suppose the lower and upper bounds of interval $\{F(s, u) : u \in U\}$ are $g(s)$ and $w(s)$ respectively. Thus we have:

$$g(s) \leq f(s, u) \leq w(s), \quad s \in [0, T] \tag{5}$$

In other words

$$g(s) = \min_u \{f(s, u) : u \in U\}, \quad s \in [0, T] \tag{6}$$

$$w(s) = \max_u \{f(s, u) : u \in U\}, \quad s \in [0, T] \tag{7}$$

Theorem 2.2: Let functions $g(.)$ and $w(.)$ be defined by relations (6) and (7). Then they are uniformly continuous on $[0, T]$.

Proof: We will show that $g(.)$ is a uniformly continuous function. It is sufficient that we show that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that if $s_1 \in N_\delta(s_2)$ then $|g(s_1) - g(s_2)| < \varepsilon$ where $N_\delta(z)$ is a δ -neighborhood of z . Since, any continuous function on compact set is a uniformly continuous. The function $f(.,.)$ on compact set $[0, T] \times U$ is a uniformly continuous, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$, such that if $(s_1, u) \in N_\delta(s_2, u)$ then $|f(s_1, u) - f(s_2, u)| < \varepsilon$. Thus $f(s_1, u) < f(s_2, u) + \varepsilon$. In addition, by (5), $g(s_1) \leq f(s_1, u)$ and so $g(s_1) \leq f(s_2, u) + \varepsilon$. Now, by taking infimum on the right hand side of the last inequality $g(s_1) \leq g(s_2) + \varepsilon$. By a similar procedure, we have $g(s_2) - g(s_1) \leq \varepsilon$. Thus $|g(s_1) - g(s_2)| \leq \varepsilon$. The proof of uniformly continuity of $w(.)$ is similar. □

By linear combination property of intervals and relation (5), we have for any $s \in [0, T]$:

$$f(s, u(s)) = (w(s) - g(s))\lambda(s) + g(s), \lambda(s) \in [0, 1] \tag{8}$$

Thus, we transform COCV problem (3)-(4) by relation

(6) as the following continuous-time problem:

$$\text{minimize } \int_0^T c(t) y(t) dt \tag{9}$$

$$\text{subject to } y(t) = q(t) + \int_0^t [h(s)\lambda(s) + d(t, s)y(s)] ds, \\ 0 \leq \lambda(t) \leq 1, t \in [0, T]$$

where $q(t) = p(t) + \int_0^t g(s) ds$ and $h(t) = w(t) - g(t)$ for any $t \in [0, T]$. Note that in the new problem (9), which is a optimal control of linear Volterra integral equation, $\lambda(.)$ is the new control function. Next section, converts the problem (9) to the corresponding linear programming problem.

3. Discrete-Time Problem

Now, discretization method enables us transforming continuous problem (9) to the corresponding discrete form. Consider equidistance points $s_0 = 0 < s_1 < s_2 < \dots < s_N = T$ of interval $[0, T]$ which defined as

$$s_j = \frac{T}{N} j, \quad j = 0, 1, \dots, N$$

where N is a given big number. Also, we set $t_j = s_j$ for $j = 0, 1, \dots, N$. By trapezoidal approximation in numerical integration, problem (9) is converted to the following discrete-time problem:

$$\text{minimize } \left(\frac{T}{2N} c_0 y_0 + \frac{T}{N} \sum_{j=1}^{N-1} c_j y_j + \frac{T}{2N} c_N y_N \right) \tag{10}$$

subject to

$$\left(1 - \frac{T}{2j} d_{jj} \right) y_j - \frac{T}{2j} h_0 \lambda_0 - \frac{T}{j} \sum_{i=1}^{j-1} (h_0 \lambda_i + d_{ji} y_i) - \frac{T}{2j} h_j \lambda_j \\ = q_j + \frac{T}{2j} d_{j0} y_0,$$

$$0 \leq \lambda_j \leq 1, y_0 = q_0, y_N = \eta \quad j = 0, 1, 2, \dots, N$$

where

$y_j = y(t_j)$, $h_j = h(s_j)$, $d_{ij} = d(t_i, s_j)$, $c_j = c(s_j)$, $\lambda_j = \lambda(t_j)$ and $q_j = q(t_j)$ for all $i, j = 0, 1, 2, \dots, N$. In problem (10), final state is η that is a known number. By solving problem (10), which is a linear programming problem, we are able to obtain the optimal solution λ_j^* and y_j^* for all $j = 0, 1, 2, \dots, N$. Note that, for evaluating optimal control variable $u^*(.)$, we must use the following equation:

$$f(s, u^*) = h(s)\lambda^* + g(s). \tag{11}$$

4. Numerical Examples

Here, we use our approach to obtain approximate optimal

solution of the following two COCV problems by solving linear programming (LP) problem (10) via simplex method [26] in MATLAB software.

Example 4.1: Consider the following COCV problem:

$$\text{minimize } \int_0^1 \cos(3\pi t) y(t) dt \tag{12}$$

subject to

$$y(t) = \sin(3\pi t) + \int_0^t \left[\sin\left(\frac{\pi}{4} u(s) + s\right) + \cos(3(t-s)\pi) y(s) \right] ds$$

$$0 \leq u(t) \leq 0.5, t \in [0,1]$$

$$y(1) = 1.$$

Here, $f(s, u) = \sin\left(\frac{\pi}{4}u + s\right)$, $d(t, s) = \cos 3(t-s)\pi$
 $c(t) = \cos(3\pi t)$ and $p(t) = \sin(3\pi t)$ for all $t \in [0,1]$, $s \in [0,1]$ and $u \in [0,0.5]$. Thus by (6), (7)

$$g(s) = \min_{u \in [0,0.5]} \left\{ \sin\left(\frac{\pi}{4}u + s\right) \right\} = \sin(s), s \in [0,1]$$

$$w(s) = \max_{u \in [0,0.5]} \left\{ \sin\left(\frac{\pi}{4}u + s\right) \right\} = \sin\left(\frac{\pi}{8} + s\right), s \in [0,1].$$

hence

$$q(t) = p(t) + \int_0^t g(s) ds = \sin(3\pi t) - \cos(t) + 1, t \in [0,1]$$

$$h(s) = w(s) - g(s) = \sin\left(\frac{\pi}{8} + s\right) - \sin(s), s \in [0,1]$$

Assume that $N = 100$ and $s_j = \frac{j}{100}$ for all $j = 0, 1, 2, \dots, 100$. The optimal solutions y_j^* and λ_j^* , $j = 0, 1, 2, \dots, 100$ of problem (12) is obtained by solving problem (10) which is illustrated in **Figures 1** and **2** respectively. Here, the value of optimal solution of objective function is -0.470 . The corresponding Equation (11) for this example is

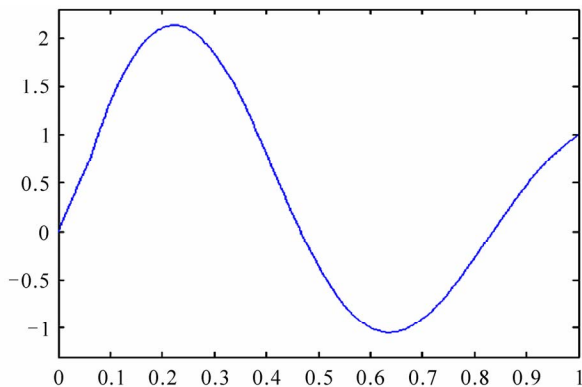


Figure 1. Optimal trajectory $y^*(.)$ of Ex.12.

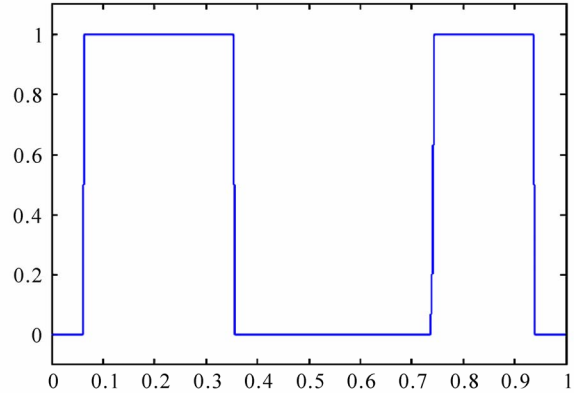


Figure 2. Corresponding optimal control $\lambda^*(.)$ of Ex.12.

$$\sin\left(\left(\frac{\pi}{4}\right)u_j^* + s_j\right) = h(s_j)\lambda_j^* + g(s_j), j = 0,1,2,\dots,100$$

therefore

$$u_j^* = \frac{4}{\pi} \left(\sin^{-1} \left(h(s_j)\lambda_j^* + g(s_j) \right) - s_j \right), j = 0,1,2,\dots,100$$

The optimal control u_j^* , $j = 0,1,2,\dots,100$ of this example is showed in **Figure 3**.

Example 4.2: Consider the following COCV problem:

$$\text{minimize } \int_0^1 (0.5-t) y(t) dt \tag{13}$$

$$\text{subject to } y(t) = t - \int_0^t \left(3 \ln(u^3(s) + s + 2) + te^{ts} y(s) \right) ds$$

$$0 \leq u(t) \leq 1, t \in [0,1]$$

$$y(1) = -1.$$

Here $f(s, u) = -3 \ln(u^3(s) + s + 2)$, $d(t, s) = -te^{ts}$, $c(t) = 0.5 - t$ and $p(t) = t$ for all $t \in [0,1]$, $s \in [0,1]$ and $u \in [0,1]$. In this example for all $s \in [0,1]$

$$g(s) = \min_{u \in [0,1]} \left\{ -3 \ln(u^3(s) + s + 2) \right\} = -3 \ln(s + 3),$$

$$w(s) = \max_{u \in [0,1]} \left\{ -3 \ln(u^3(s) + s + 2) \right\} = -3 \ln(s + 2),$$

$$h(s) = w(s) - g(s) = 3(\ln(s + 3) - \ln(s + 2)).$$

Moreover for all $t \in [0,1]$

$$q(t) = p(t) + \int_0^t g(s) ds = t - 3((t+3)\ln(t+3) - (t+3) - 3\log(3) + 3),$$

Let $N = 100$ and $t_j = \frac{j}{100}$ for all $j = 0, 1, 2, \dots, 100$.

We obtain the optimal solution y_j^* and λ_j^* , $j = 0, 1, 2, \dots, 100$ of problem (13) by solving problem (10) which is illustrated in **Figures 4** and **5** respectively. In

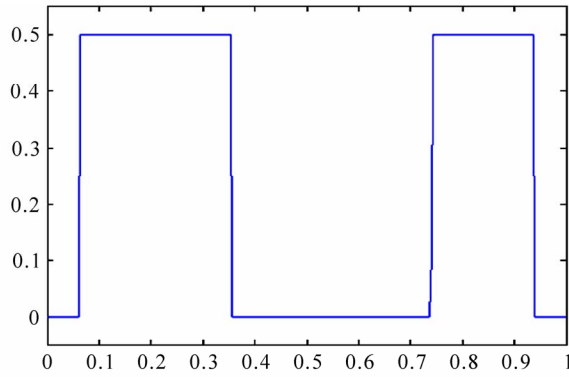


Figure 3. Optimal control $u^*(.)$ of Ex.12.

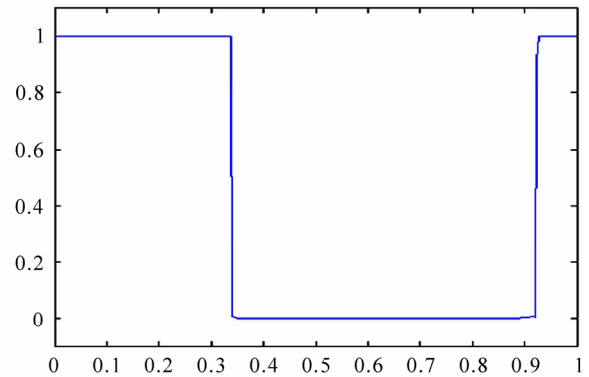


Figure 6. Optimal control $u^*(.)$ of Ex.13.

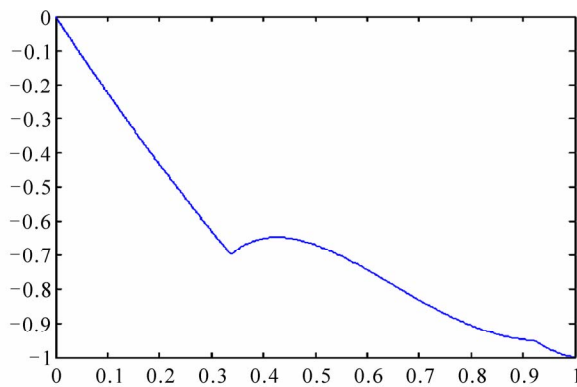


Figure 4. Optimal trajectory $y^*(.)$ of Ex.13.

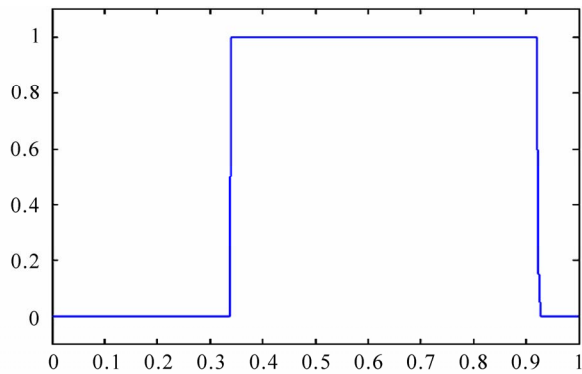


Figure 5. Corresponding optimal control $\lambda^*(.)$ of Ex.13.

addition, by (11) the corresponding $u^*(.)$ for this example is

$$u_j^* = \left(e^{-h(s_j)\lambda_j^* + g(s_j)} / 3 - s_j - 2 \right)^{1/3}, \quad j = 0, 1, 2, \dots, 100$$

The optimal control $u_j^*, j = 0, 1, 2, \dots, 100$ of problem (10) is showed in **Figure 6**. Here, the value of optimal solution of objective function is 0.071.

5. Conclusions

In this paper, we posed a different approach for a class of nonlinear optimal control problem including Volterra integral equations. In our approach, the linear combination property of intervals is used to obtain the new corresponding problem which is a linear problem. The new problem can be converted to a LP problem by discretezation method. Finally, we obtain an approximate solution for the main problem. In next works, we are going to use our approach for subclasses of problem (1)-(2) which Volterra integral equation is similar to Equation (4), but objective functional is quadratic or nonlinear with respect to state variables.

6. References

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