A Fractional Programming Problem for Bank Asset and Liability Management

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Abstract

The purpose of the paper is to present an Asset Liability Management (ALM) technique which uses a fractional programming model to determine the balance sheet of a commercial bank in Mongolia. We examine the balance sheet of a commercial bank introducing some common ratios used in the Bank. Numerical results were done in case of Mongolia in Central Bank.

Keywords

Asset and Liability Management, Banking, A Fractional Programming Problem

1. Introduction

Bank asset and liability management is defined as the simultaneous planning of all asset and liabilitypositions on the bank's balance sheet under consideration of the different bank management objectives and legal, managerial and market constraints, for the purpose of mitigating interest rate risk, providing liquidity and enhancing the value of the bank [1].

The central problem of ALM revolves around the bank's balance sheet and the main question is that what should be the composition of a bank's assets and liabilities on average given the corresponding returns and costs, in order to achieve certain goals, such as maximization of the bank's gross revenue. This need has led banks to determine their optimal balance among profitability, risk, liquidity and other uncertainties. The optimal balance between these factors cannot be found without considering important interactions that exist between the structure of a bank's liability and capital and the composition of its assets [2].

Looking to the past, asset and liability management models can be classified in several approaches [3]. The first, based on Markowitz's [4] theory of portfolio

selection, assumes that returns are normally distributed and that bank managers utilize risk-averse utility functions. However, the model omits trading activity, matching assets and liabilities, transaction costs, and other similar features. The second approach is a linear programming model, proposed by Chambers and Charnes [5]. They were concerned with formulating, exploring and interpreting the use and construction which may be derived from a mathematical programming model which expresses more realistically than past efforts the actual conditions of current operations. However, this approach will not generate an optimal solution to the total problem, but rather acts as a deterministic simulation to observe portfolio behavior under various economic conditions [6].

The third approach is dynamic programming, proposed by Samuelson [7], Merton [8], and Eppen and Fama [9], who modeled of asset problems. These models are dynamic and account for the inherent uncertainty of the problem. However, given the small number of financial instruments that can be analyzed simultaneously, they are of limited use in practice. The fourth approach is a stochastic linear programming with simple recourse, which is also called linear programming under uncertainty. This technique explicitly characterizes each realization of the random variables by a constraint and leads to large problems in realistic situations. Cohen and Thore [10] viewed their model more as a tool for sensitivity analysis (in the aggregate) than a normative decision tool. The computational intractability and the perceptions of the formulation precluded consideration of problems other than those that were limited both in terms of time periods and in a number of variables and realizations.

The fifth, Eatman and Sealey [11] developed a multi-objective linear programming model for commercial bank balance sheet management considering profitability and solvency objectives subject to policy and managerial constraints. Indeed, Kusy and Ziemba [3] employed a multi-period stochastic linear program with simple recourse to model the management of assets and liabilities in banking while maintaining computational feasibility. Their results indicate that the proposed ALM model is theoretically and operationally superior to a corresponding deterministic linear programming model and that the computational effort required for its implementation is comparable to that of the deterministic model.

The sixth, Giokas and Vassiloglou [12] developed a goal-programming model for bank asset and liability management. Conventional linear programming is unable to deal with this kind of problem, as it can only handle a single goal in the objective function. Goal programming is the most widely used approach that solves large-scale multi-criteria decision making problem. Expansion of this approach is Kosmidou and Zopounidis [13] model, which determines the optimal balance among profitable, risk, liquidity and other uncertainties by considering several goals, such as the maximization of returns, the minimization of risk, the maintenance of a desirable level of liquidity and solvency, the expansion of deposits and loans.

The last approach is stochastic integer programming problem proposed by Schultz and Tiedermann [14]. The logical constraints in the portfolio impose upper bounds on the number of classes of assets to be considered in the portfolio and the number of assets to be traded in any time period. They also force conditional lower bounds on the volume to be traded in periods. Then, they claim that the stochastic continuous programming problem is converted to a stochastic integer programming problem. Most authors consider the problem with only continuous variables, see Kouwenberg and Zenios [15], and Ziemba and Mulvey [16].

ALM models have been developed by a mixed approach since the 2006. Escudero and Garin [17] developed on multistage stochastic integer programming for incorporating logical constraints in asset and liability management under uncertainty, where the 0 - 1 variables and the continuous variables appear at any stage. Zeng and Li [18] investigated continuous-time asset liability management under benchmark and mean-variance criteria in a jump diffusion market. The benchmark model was solved by employing the stochastic dynamic programming and its results are extended to the mean-variance model by adopting the duality theory. Mohammand and Sherafati [19] did an attempt to propose a model for optimizing liquidity management with goal programming perspective and integrating goal programming with fuzzy AHP.

The purpose of this paper is to develop a new type of a fractional programming problem for the bank asset liability management into a deterministic environment based on some common ratios.

2. Problem Formulation

The paper presents a new ALM methodology in a deterministic environment in order to choose strategic directions in a commercial bank's financial plan. The formulation is based on a balance sheet information and 14 structural variables. 7 variables correspond to assets and another 7 variables to liabilities. In Table 1, we show these variables.

In order to formulate our model, we need to introduce the following ratio variables as follows:

$$v_1 = \frac{A_1 + A_2 + A_3 + A_4}{L_1 + L_2 + L_3 + L_4 + L_5}, \quad v_2 = \frac{A_5}{A}, \quad v_3 = \frac{L_2}{L_1 + L_2 + L_3 + L_4 + L_5},$$

$$v_4 = \frac{A_1 + A_2 + A_3 + A_4}{A}, \quad v_5 = \frac{L_4}{A}, \quad v_6 = \frac{A_5 + A_4}{L_1 + L_2 + L_3}, \quad v_7 = \frac{A_4}{A},$$

$$v_8 = \frac{L_1 + L_3}{L}, \quad v_9 = \frac{L_4}{L}, \quad v_{10} = \frac{E}{L}.$$

where v_1: liquidity ratio, v_2: total loans to total assets ratio, v_3 and v_8: stability and instability ratio of deposit, respectively, v_4: liquidity asset ratio, v_5: due to banks and financial sector to total assets ratio, v_6: earning assets to total assets ratio, v_7: financial investment ratio, v_9: due to banks and financial sector to total liabilities ratio, v_{10}: capital sufficiency.

Table 1. The decision variables.

Assets	Liabilities
A_1: Cash and cash equivalents	L_1: Current account
A_2: Deposits to the Bank of Mongolia	L_2: Time deposit
A_3: Deposits at other banks and financial sector	L_3: Demand deposit
A_4: Financial investments	L_4: Due to banks and financial sector
A_5: Loans and advances to customers	L_5: Other deposits
A_6: Other financial assets	L_6: Other liabilities
A_7: Fixed assets	E: Equity
A: Total assets	$L + E = A$: Total assets

Remark 1. If asset and liability management committee approve, then the following inequalities hold:

$$L_2 \leq L_1 + L_3, \ L_1 + L_2 \geq L_4, \ A_1 + A_2 + A_3 + A_4 \geq L_4, \ L_2 \leq A_1 + A_2 + A_3 + A_4,$$

$$A_5 \geq A_1 + A_2 + A_3 + 2A_4, \ A_6 + A_7 \leq E, \ A_6 + A_7 \leq A_4, \ L_5 + L_6 \geq E.$$

Remark 2. Based on remark 1, the main ratio variables satisfy the following inequalities:

$$v_1 \geq v_4, \ v_3 \leq v_8, \ v_6 \geq v_2 + v_7, \ v_9 \geq v_5, \ v_8 \geq v_9,$$

$$v_4 \geq v_5, \ v_3 \leq v_4, \ v_2 \geq v_4 + v_7.$$

Remark 3. It can be checked that if $A_6 + A_7 \leq A_4$ then $v_2 + v_4 + v_7 \geq 1$.
Proof: Indeed, this implies from Remark 1 and Remark 2;

$$\frac{A_5}{A} + \frac{A_1 + A_2 + A_3 + A_4}{A} + \frac{A_4}{A} = 1 - \left(\frac{A_6 + A_7 - A_4}{A} \right) \geq 1.$$

Remark 4. If $L_5 + L_6 \geq E$ then $v_3 + v_8 + v_9 + v_{10} \leq 1$.
Proof: Taking into account that $L = L_1 + L_2 + L_3 + L_4 + L_5 + L_6$, we have

$$v_3 + v_8 + v_9 + v_{10} = \frac{L_2}{L_1 + L_2 + L_3 + L_4 + L_5} + \frac{L_1 + L_3}{L} + \frac{L_4}{L} + \frac{E}{L}$$

$$\approx \frac{L_2 + L_1 + L_3 + L_4 + E}{L} = \frac{L - L_5 - L_6 + E}{L}$$

$$= 1 + \frac{E - L_5 - L_6}{L} \leq 1.$$

Remark 5. If $A_6 + A_7 \leq E$ then

$$v_2 + v_4 \geq \frac{v_5}{v_9}.$$

Proof: We have the following obvious equalities and inequalities:

$$v_2 + v_4 = \frac{A_5}{A} + \frac{A_1 + A_2 + A_3 + A_4}{A} = \frac{A - (A_6 + A_7)}{A}$$

$$= 1 - \frac{A_6 + A_7}{A} \geq 1 - \frac{E}{A} = \frac{L}{A} = \frac{\frac{L_4}{A}}{\frac{L_4}{L}} = \frac{v_5}{v_9}.$$

3. Auxiliary Results

Lemma1. For $v_1, v_2, \cdots, v_9$, we have

$$v_{10} = \frac{v_1}{v_3 v_4}\left(\frac{v_9}{v_5}\left(\frac{v_2 + v_7}{v_6} \right) - v_8 \right) - 1.$$

Proof: For the variables v_1, v_3, v_4, v_5, v_6, we can write the following expression

$$\frac{v_1}{v_3 v_4} = \frac{\dfrac{A_1 + A_2 + A_3 + A_4}{L_1 + L_2 + L_3 + L_4 + L_5}}{\dfrac{L_2}{L_1 + L_2 + L_3 + L_4 + L_5}\dfrac{A_1 + A_2 + A_3 + A_4}{A}} = \frac{A}{L_2},$$

and

$$\frac{v_9}{v_5} = \frac{\dfrac{L_4}{L}}{\dfrac{L_4}{A}} = \frac{A}{L}.$$

Indeed, denoting by $D = L_1 + L_2 + L_3$, we get

$$\frac{v_2 + v_7}{v_6} = \frac{\dfrac{A_5}{A} + \dfrac{A_4}{A}}{\dfrac{A_5 + A_4}{D}} = \frac{D}{A}.$$

Substituting all above expressions into formula of v_{10}, we have

$$v_{10} = \frac{A}{L_2}\left(\frac{A}{L}\frac{D}{A} - \frac{L_1 + L_3}{L} \right) - 1$$

$$= \frac{A}{L_2}\left(\frac{L_1 + L_2 + L_3}{L} - \frac{L_1 + L_3}{L} \right) - 1 = \frac{A}{L} - 1 = \frac{E}{L}.$$

Lemma 2. For $v_1, v_2, \cdots, v_9$, we have the connection:

$$v_1 = \frac{v_4 v_3}{\dfrac{v_2 + v_7}{v_6} - \dfrac{v_8 v_5}{v_9}}.$$

Proof: From the definition of v_1, we have

$$v_1 = \frac{A_1 + A_2 + A_3 + A_4}{L_1 + L_2 + L_3 + L_4 + L_5}\frac{A}{A}\frac{L_2}{L_2} = v_4 v_3 \frac{A}{L_2}.$$

If we use $D = L_1 + L_2 + L_3$, then L_2 is expressed as

$$L_2 = D - L_3 - L_1 = \frac{A_5 + A_4}{v_6} - L_3 - L_1$$

$$= \frac{A}{v_6}\frac{A_5 + A_4}{A} - L\frac{L_3 + L_1}{L} = \frac{A}{v_6}(v_2 + v_7) - L v_8.$$

Consequently, we have

$$v_1 = v_4 v_3 \frac{A}{\dfrac{A}{v_6}(v_2 + v_7) - L v_8} = \frac{v_4 v_3}{\dfrac{1}{v_6}(v_2 + v_7) - \dfrac{L}{A}\dfrac{L_3}{L_3}v_8} = \frac{v_4 v_3}{\dfrac{v_2 + v_7}{v_6} - \dfrac{v_8 v_5}{v_9}}.$$

which proves the lemma.

Lemma 3. For v_5, v_9, we have

$$\frac{v_5}{v_9} = \frac{1}{1 + v_{10}}.$$

Proof: Substituting to v_5, v_9,

$$\frac{v_9}{v_5} = \frac{A}{L} = \frac{1}{1 - \frac{E}{L}\frac{L}{A}} = \frac{1}{1 - v_{10}\frac{v_5}{v_9}},$$

If reducing it, then proves the lemma.

Lemma 4. If $L_6/A \geq 0$, then

$$-\frac{v_4}{v_1} + \frac{1}{1 + v_{10}} \geq 0.$$

Proof: Substituting to v_{10},

$$E = v_{10}\left(L_1 + L_2 + L_3 + L_4 + L_5 + L_6\right) = v_{10}\left(A - E\right),$$

and we obtain

$$\frac{E}{A} = \frac{v_{10}}{1 + v_{10}}.$$

In other words,

$$\frac{L_1 + L_2 + L_3 + L_4 + L_5 + L_6}{A} = 1 - \frac{v_{10}}{1 + v_{10}},$$

or

$$\frac{L_1 + L_2 + L_3 + L_4 + L_5}{A} = \frac{v_4}{v_1},$$

Then, we get proof the lemma.

4. Formulation of a Fractional Programming

Now we consider the problem of maximizing a capital sufficiency subject to liquidity ratio, leverage, liquidity asset ratio and loan portfolio ratio, total deposits ratio, total liabilities ratio, and combine ratio.

$$f = \frac{v_1 v_2 v_9 + v_1 v_7 v_9 - v_1 v_5 v_6 v_8 - v_3 v_4 v_5 v_6}{v_3 v_4 v_5 v_6} \rightarrow \max,$$

subject to constraints:

$$g_1 = v_1 v_2 v_9 + v_1 v_7 v_9 - v_1 v_5 v_6 v_8 - v_3 v_4 v_6 v_9 = 0,$$

$$g_2 = \frac{v_5}{v_9} - \frac{v_3 v_4 v_5 v_6}{v_1 v_2 v_9 + v_1 v_7 v_9 - v_1 v_5 v_6 v_8} = 0,$$

$$g_3 = v_2 + v_4 - \frac{v_5}{v_9} \geq 0,$$

$$g_4 = \frac{v_4}{v_1} - \frac{v_3 v_4 v_5 v_6}{v_1 v_2 v_9 + v_1 v_7 v_9 - v_1 v_5 v_6 v_8} \leq 0,$$

$$g_5 = v_3 + v_8 + v_9 + \frac{v_1 v_2 v_9 + v_1 v_7 v_9 - v_1 v_5 v_6 v_8 - v_3 v_4 v_5 v_6}{v_3 v_4 v_5 v_6} - 1 \le 0,$$

$$g_6 = v_2 + v_4 + v_7 \ge 1$$

$$\begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\underline{v}_1 \le v_1 \le \overline{v}_1, \ \underline{v}_2 \le v_2 \le \overline{v}_2, \ \underline{v}_3 \le v_3 \le \overline{v}_3,$$

$$\underline{v}_4 \le v_4 \le \overline{v}_4, \ \underline{v}_5 \le v_5 \le \overline{v}_5, \ \underline{v}_6 \le v_6 \le \overline{v}_6,$$

$$\underline{v}_7 \le v_7 \le \overline{v}_7, \ \underline{v}_8 \le v_8 \le \overline{v}_8, \ \underline{v}_9 \le v_9 \le \overline{v}_9.$$

where f: capital sufficiency from lemma 1, g_1: liquidity ratio from lemma 2, g_2: leverage from lemma 3, g_3: liquidity asset ratio and loan portfolio ratio from remark 5, g_4: total deposits ratio from lemma 4, g_5: total liabilities ratio from remark 4, and g_6: combine ratio from remark 3.

5. Numerical Results

The common ratio variables used in the specification of this model were taken directly as end of 2017 balance sheet of Top 5 commercial banks of Mongolia. We have shown the formulations below in **Table 1**, how to calculate these common ratios. Indeed, in **Table 2**, results are showed by calculation variables.

Top 5 commercial banks in Mongolia shares 86 percent of a commercial bank market. We have ranked highest to lowest in **Table 2**. The ratio variables are difference due to specific of the commercial banks. The last columns of this table are mean and standard deviation calculated by the common ratio variables of Top 5 commercial banks.

We use the following formulation to calculate upper and lower bound of this fractional programming model:

$$\overline{v}_i = mean(i) + 0.5 \cdot stdev(i),$$

$$\underline{v}_i = mean(i) - 0.5 \cdot stdev(i),$$

$$i = \overline{1,9}.$$

According to the formulation, the bounds found as follows:

$$0.483 \le v_1 \le 0.544, \ 0.432 \le v_2 \le 0.507, \ 0.340 \le v_3 \le 0.429,$$

$$0.424 \le v_4 \le 0.474, \ 0.188 \le v_5 \le 0.287, \ 1.025 \le v_6 \le 1.381,$$

$$0.122 \le v_7 \le 0.211, \ 0.203 \le v_8 \le 0.281, \ 0.209 \le v_9 \le 0.312.$$

Table 2. The calculation variables.

Ratios	Khan	TDB	XAC	State	Golomt	Mean	Stdev
v_1	0.564	0.520	0.545	0.409	0.528	0.513	0.061
v_2	0.457	0.403	0.442	0.599	0.445	0.469	0.075
v_3	0.405	0.253	0.358	0.496	0.412	0.385	0.089
v_4	0.477	0.452	0.483	0.362	0.472	0.449	0.050
v_5	0.202	0.203	0.413	0.181	0.186	0.237	0.099
v_6	0.998	1.648	1.522	0.988	0.859	1.203	0.356
v_7	0.150	0.296	0.206	0.064	0.118	0.166	0.089
v_8	0.300	0.232	0.116	0.251	0.311	0.242	0.078
v_9	0.228	0.232	0.442	0.197	0.201	0.260	0.103
v_{10}	0.129	0.140	0.070	0.089	0.081	0.102	0.031

Source: Audited report of individual commercial bank.

Optimal solutions found by MATLAB were $v_{10}^* = 0.092$ $v_1^* = 0.522$, $v_2^* = 0.502$, $v_3^* = 0.340$, $v_4^* = 0.429$, $v_5^* = 0.208$, $v_6^* = 1.232$, $v_8^* = 0.281$ and $v_9^* = 0.239$.

6. Concluding Remarks

The problem of maximizing a capital sufficiency subject to the ratios of the bank indicators for the first time has been formulated a fractional programming problem. The problem is nonconvex hard optimization problem. Solution of this problem is very suitable for realistic situation and the bank asset and liability management can describe where we are going on the market by analyzing competition impacts.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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