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Fractional Order Generalized Thermoelastic Infinite Medium with Cylindrical Cavity Subjected to Harmonically Varying Heat

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Abstract

In this work, a mathematical model of an elastic material with cylindrical cavity will be constructed. The governing equations will be taken into the context of the fractional order generalized thermoelasticity theory (Youssef 2010). Laplace transform and direct approach will be used to obtain the solution when the boundary of the cavity is exposed to harmonically heat with constant angular frequency of thermal vibration. The inverse of Laplace transforms will be computed numerically using a method based on Fourier expansion techniques. Some comparisons have been shown in figures to present the effect of the fractional order parameter and the angular frequency of thermal vibration on all the studied felids.

Keywords: Thermoelasticity, Generalized Thermoelasticity, Fractional Order, Cylindrical Cavity, Harmonically Heat

1. Introduction

Recently, a considerable research effort has been expended to study anomalous diffusion, which is characterized by the time-fractional diffusion-wave equation by Kimmich [1]:

$$\rho c = \kappa I^{\alpha} c_{,ii}, \tag{1}$$

where ρ is the mass density, c is the concentration, κ is the diffusion conductivity, i is the coordinate symbol which takes the values 1, 2 and 3, the subscript "," means the derivative with respect to x_i and notion I^{α} is the Riemann-Liouville fractional integral is introduced as a natural generalization of the well-known n-fold repeated integral $I^{\alpha} f(t)$ written in a convolution-type form as in [2,3]:

$$I^{\alpha} f(t) = \begin{Bmatrix} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau & for \quad 0 < \alpha \le 2 \\ f(t) & for \quad \alpha = 0 \end{Bmatrix}, \tag{2}$$

where $\Gamma(\alpha)$ is the gamma function.

According to Kimmich [1] Equation (1) describes different cases of diffusion where $0 < \alpha < 1$ correspond to weak diffusion (sub diffusion), $\alpha = 1$ correspond to normal diffusion, $1 < \alpha < 2$ correspond to strong diffusion (super diffusion) and $\alpha = 2$ correspond to ballistic diffusion.

It should be noted that the term diffusion is often used in a more generalized sense including various transport phenomena. Equation (1) is a mathematical model of a wide range of important physical phenomena, for example, the sub-diffusive transport occur in widely different systems ranging from dielectrics and semiconductors through polymers to fractals, glasses, porous, and random media. Super diffusion is comparatively rare and has been observed in porous glasses, polymer chain, biological systems, transport of organic molecules and atomic clusters on surface [4]. One might expect the anomalous heat conduction in media where the anomalous diffusion is observed.

Fujita [5,6] considered the heat wave equation for the case of $1 \le \alpha \le 2$:

$$\rho C\, T = k I^{\alpha} T_{,ii}, \tag{3}$$

where C is the specific heat, k is the thermal conductivity and the subscript "," means the derivative with respect to the coordinates x_i.

Equation (3) can be obtained as a consequence of the non local constitutive equation for the heat flux components q_i is in the form

$$q_i = -k I^{\alpha-1} T_{,i}, \ 1 < \alpha \le 2. \quad (4)$$

Povstenko [4] used the Caputo heat wave equation defines in the form:

$$q_i = -k I^{\alpha-1} T_{,i}, \ \ 0 < \alpha \le 2, \quad (5)$$

to get the stresses corresponding to the fundamental solution of a Cauchy problem for the fractional heat conduction equation in one-dimensional and two-dimensional cases.

Some applications of fractional calculus to various problems of mechanics of solids are reviewed in the literature [7,8].

2. Theory of Fractional Order Generalized Thermoelasticity

The classical thermoelasticity is based on the principles of the theory of heat conduction which is called Fourier law, which relates the heat flux components q_i to the temperature gradient as follows:

$$q_i = -k T_{,i}. \quad (6)$$

In combination with the energy conservative law, this leads to the parabolic heat conduction equation which is considered by Povstenko [4]:

$$\rho C \dot{T} = k T_{,ii}, \quad (7)$$

where dotted above T means the derivative with respect to the time *t*.

Recently, in the non classical thermoelasticity theories, Fourier law (6) and heat conduction (7) are replaced by more general equations, have been formulated. The first well-known generalized of such a type of Lord and Shulman [9] and it takes the form:

$$q_i + \tau_o \dot{q}_i = -k \ T_{,i}, \quad (8)$$

which leads to the hyperbolic differential equation of heat conduction of Lord and Shulman [9]:

$$\rho C \left(\dot{T} + \tau_o \ddot{T}\right) = k T_{,ii}, \quad (9)$$

where τ_o is non-negative constant and is called relaxation time.

According to equation (9), Kaliski [10] and Lord and Shulman [9] constructed the theory of generalized thermoelasticity.

In the context of the generalized thermoelasticity, the governing equations for isotropic medium are defined as follows:

The equation of motion

$$\sigma_{ij,j} + \rho F_i = \rho \ddot{u}_i. \quad (10)$$

The constitution relation

$$\sigma_{ij} = 2\mu e_{ij} + \left(\lambda e_{kk} - \gamma\theta\right)\delta_{ij}, \quad (11)$$

where λ, μ are Lamé's constant, u_i is the displacement component, F_i the body force component, $\theta = T - T_o$ is the increment of the dynamical temperature where T_o is the reference temperature, $\gamma = (3\lambda + 2\mu)\alpha_T$ where α_T is called the thermal expansion coefficient, where δ_{ij} is the Kronecker delta symbol, σ_{ij} is the stress tensor such that $\sigma_{ij} = \sigma_{ji}$ and e_{ij} is the strain tensor satisfy the relations

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \quad (12)$$

The heat flux equation

$$q_{i,i} = -\rho C \dot{\theta} - T_o \gamma \dot{e}. \quad (13)$$

The entropy increment equation per unit volume takes the form

$$\rho T_o S = \rho C \theta + T_o \gamma e, \quad (14)$$

where S is the entropy increment of the material.

The heat flux-entropy equation

$$q_{i,i} = -\rho T_o \dot{S}. \quad (15)$$

The heat equation without any heat sources

$$q_i + \tau_o \frac{\partial q_i}{\partial t} = -k I^{\alpha-1} \theta_{,i} \quad 0 < \alpha \le 2. \quad (16)$$

By using Equations (14,15,16), we have the heat equation in the form [11]:

$$k I^{\alpha-1}\theta_{,ii} = \left(\frac{\partial}{\partial t} + \tau_o \frac{\partial^2}{\partial t^2}\right)\left(\rho C \theta + T_o \gamma e\right), 0 < \alpha \le 2 \quad (17)$$

where $\begin{Bmatrix} 0 < \alpha < 1 & \text{for weak conductivity} \\ \alpha = 1 & \text{for normal conductivity} \\ 1 < \alpha \le 2 & \text{for strong conductivity} \end{Bmatrix}$

3. The Problem Formulation

Let us consider a perfectly conducting elastic infinite body with cylindrical cavity occupies the region $R \le r < \infty$ of an isotropic homogeneous medium whose state can be expressed in terms of the space variable *r* and the time variable t such that all of the state functions vanish at infinity. We will use the cylindrical system of coordinates (r, ψ, z) with the z-axis lying along the axis of the cylinder. Due to symmetry, the problem is one-dimensional with all the functions considered depending on the radial distance r and the time *t*.

The medium described above is considered to be quiescent and the surface of the cavity is subjected to harmonically varying heat and traction free described mathematically as follow:

$$\theta(R,t) = \theta_1 \cos(\omega t)\,, \tag{18}$$

and

$$\sigma_{rr}(R,t) = 0\,, \tag{19}$$

where θ_1 is constant and ω is the angular frequency of thermal vibration ($\omega = 0$ for a thermal shock).

It is assumed that there are no body forces and no heat sources in the medium. Thus, the field equations (10), (11), (12) and (17) in cylindrical case can be set as [12]:

$$(\lambda + 2\mu)\frac{\partial e}{\partial r} - \gamma \frac{\partial \theta}{\partial r} = \rho \frac{\partial^2 u}{\partial t^2}\,, \tag{20}$$

$$I^{\alpha-1}\nabla^2\theta = \left(\frac{\partial}{\partial t} + \tau_o \frac{\partial^2}{\partial t^2}\right)\left(\frac{\rho C}{k}\theta + \frac{T_o \gamma}{k} e\right), 0 < \alpha \le 2\,, \tag{21}$$

$$\sigma_{rr} = 2\mu \frac{\partial u}{\partial r} + \lambda e - \gamma\,\theta\,, \tag{22}$$

$$\sigma_{\psi\psi} = 2\mu \frac{u}{r} + \lambda e - \gamma \theta \tag{23}$$

$$\sigma_{zz} = \lambda e - \gamma\,\theta\,, \tag{24}$$

$$\sigma_{zr} = \sigma_{\psi r} = \sigma_{zz} = 0\,, \tag{25}$$

$$e = \frac{1}{r}\frac{\partial\,(r u)}{\partial\, r}\,, \tag{26}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial\, r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r}$

For convenience, we shall use the following non-dimensional variables [12]:

$$(r', u') = c_o \eta (r, u), (t', \tau_o') = c_o^2 \eta (t, \tau_o),\ \theta' = \frac{\theta}{T_o}\,\sigma', = \frac{\sigma}{\mu},$$

where $c_o^2 = \dfrac{\lambda + 2\mu}{\rho}$ and $\eta = \dfrac{\rho C}{k}$.

Equations (20-26) assume the form (where the primes are suppressed for simplicity)

$$\frac{\partial e}{\partial r} - a \frac{\partial \theta}{\partial r} = \frac{\partial^2 u}{\partial t^2}\,, \tag{27}$$

$$I^{\alpha-1}\nabla^2\theta = \left(\frac{\partial}{\partial t} + \tau_o \frac{\partial^2}{\partial t^2}\right)(\theta + \varepsilon e), 0 < \alpha \le 2\,, \tag{28}$$

$$\sigma_{rr} = \beta^2 \frac{\partial u}{\partial r} + (\beta^2 - 2)\frac{u}{r} - a\beta^2\,\theta\,, \tag{29}$$

$$\sigma_{\psi\psi} = (\beta^2 - 2)\frac{\partial u}{\partial r} + \beta^2 \frac{u}{r} - a\beta^2\,\theta\,, \tag{30}$$

$$\sigma_{zz} = (\beta^2 - 2)e - a\beta^2\theta\,, \tag{31}$$

where $a = \dfrac{\gamma T_o}{\lambda + 2\mu}, \varepsilon = \dfrac{\gamma}{\rho C},\ \beta = \left(\dfrac{\lambda + 2\mu}{\mu}\right)^{1/2}$ and $\gamma = (3\lambda + 2\mu)\alpha_T$.

4. Formulation in the Laplace Transform Domain

Taking the Laplace transform for the both sides of the Equations (27-31), this is defined as follows:

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt\,, \tag{32}$$

where the rule for the Laplace transform of the Riemann-Liouville fractional integral for zero initial function reads from Povstenko [4]:

$$L\{I^\alpha f(t)\} = \frac{1}{s^\alpha} L\{f(t)\} = \frac{\bar{f}(s)}{s^\alpha},\ \alpha > 0\,. \tag{33}$$

Then, we have

$$\frac{d\bar{e}}{dr} - \alpha_2 \frac{d\bar{\theta}}{dr} = \alpha_1 \bar{u}\,, \tag{34}$$

$$\nabla^2\bar{\theta} = \alpha_3\,\bar{\theta} + \alpha_4\,\bar{e}\,, \tag{35}$$

$$\bar{\sigma}_{rr} = \beta^2 \frac{d\bar{u}}{dr} + (\beta^2 - 2)\frac{\bar{u}}{r} - \beta^2\alpha_2\,\bar{\theta} \tag{36}$$

$$\bar{\sigma}_{\psi\psi} = (\beta^2 - 2)\frac{d\bar{u}}{dr} + \beta^2 \frac{\bar{u}}{r} - \beta^2 \alpha_2\,\bar{\theta}\,, \tag{37}$$

$$\bar{\sigma}_{zz} = (\beta^2 - 2)\bar{e} - \beta^2\alpha_2\bar{\theta}\,, \tag{38}$$

$$\bar{\theta}(R,s) = \alpha_5(s)\,, \tag{39}$$

$$\bar{\sigma}_{rr}(R,s) = 0\,, \tag{40}$$

where $\nabla^2 = \dfrac{d^2}{d\,r^2} + \dfrac{1}{r}\dfrac{d}{d\,r}, \alpha_1 = s^2, \alpha_2 = a$,

$\alpha_3 = (s^\alpha + \tau_0 s^{\alpha+1})$, $\alpha_4 = \varepsilon\,\alpha_3$, $\alpha_5 = \dfrac{s}{s^2 + \omega^2}$,

and an over bar symbol denotes its Laplace transform and s denotes the Laplace transform parameter.

Eliminating u from the Equations (26,34,35), we get

$$(\nabla^2 - \alpha_1)\bar{e} = \alpha_2 \nabla^2 \bar{\theta}\,, \tag{41}$$

$$\left(\nabla^2-\alpha_3\right)\bar{\theta}=\alpha_4\,\bar{e}\,. \quad (42)$$

Eliminating $\bar{e}$ from Equations (41,42), we obtain

$$\left[\nabla^4-\left(\alpha_1+\alpha_2\alpha_4+\alpha_3\right)\nabla^2+\alpha_1\alpha_3\right]\bar{\theta}=0\,. \quad (43)$$

In a similar manner, we can show that e satisfies the equation

$$\left[\nabla^4-\left(\alpha_1+\alpha_2\alpha_4+\alpha_3\right)\nabla^2+\alpha_1\alpha_3\right]\bar{e}=0\,. \quad (44)$$

The bounded solutions of Equations (41,42) at infinity can be written in the form

$$\bar{\theta}=\sum_{i=1}^{2}A_i\left(p_i^2-\alpha_1\right)K_0\left(p_i r\right), \quad (45)$$

$$\bar{e}=\sum_{i=1}^{2}B_i\,K_0\left(p_i r\right), \quad (46)$$

where $K_o(.)$ is the modified Bessel function of the second kind of order zero. A_1, A_2, B_1 and B_2 are all parameters depending on the parameter s of the Laplace transform, p_1^2 and p_2^2 are the roots of the characteristic equation

$$p^4-\left[\alpha_1+\alpha_2\alpha_4+\alpha_3\right]p^2+\alpha_1\alpha_3=0\,, \quad (47)$$

and satisfy the relations

$$p_1^2+p_2^2=\alpha_1+\alpha_2\alpha_4+\alpha_3\ ,$$

$$p_1^2p_2^2=\alpha_1\alpha_3\,.$$

Using Equation (41), we obtain

$$B_i=\alpha_2\ p_i^2\ A_i,\text{i=1,2.} \quad (48)$$

Thus, we have

$$\bar{e}=\alpha_2\sum_{i=1}^{2}A_i\,p_i^2\,K_0\left(p_i r\right). \quad (49)$$

Substituting from Equation (49) into the Laplace transform of Equation (26), we obtain

$$\bar{u}=-\alpha_2\sum_{i=1}^{2}A_i\,p_iK_1\left(p_i r\right)\ , \quad (50)$$

where $K_1(.)$ is the modified Bessel function of the second kind of order one.

In deriving Equation (50), we have used the following well-known relation of the Bessel function:

$$\int z\,K_0\left(z\right)dz=-z\,K_1\left(z\right),$$

Finally, substituting from Equations (45,49,50) into Equations (36-38), we obtain the stress components in the form:

$$\bar{\sigma}_{rr}=\alpha_2\sum_{i=1}^{2}A_i\left[\beta^2\,\alpha_1K_0\left(p_i r\right)+\frac{2}{r}p_i\,K_1\left(p_i\,r\right)\right], \quad (51)$$

$$\bar{\sigma}_{\psi\psi}=\alpha_2\sum_{i=1}^{2}A_i\left[\left(\beta^2\,\alpha_1-2p_i^2\right)K_0\left(p_i r\right)-\frac{2}{r}p_i\,K_1\left(p_i\,r\right)\right], \quad (52)$$

$$\bar{\sigma}_{zz}=\alpha_2\sum_{i=1}^{2}\left(\beta^2\alpha_1-2p_i^2\right)A_iK_0\left(p_i r\right). \quad (53)$$

Using the boundary conditions (39,40), we get

$$\sum_{i=1}^{2}A_i\left(p_i^2-\alpha_1\right)K_0\left(p_i R\right)=\alpha_5\,, \quad (54)$$

$$\sum_{i=1}^{2}A_i\left[\beta^2\alpha_1K_0\left(p_i R\right)+\frac{2}{r}p_i\,K_1\left(p_i R\right)\right]=0\,. \quad (55)$$

Then, we have

$$\begin{bmatrix}A_1\\A_2\end{bmatrix}=\begin{bmatrix}l_{11}&l_{12}\\l_{21}&l_{22}\end{bmatrix}^{-1}\begin{bmatrix}\alpha_5\\0\end{bmatrix}, \quad (56)$$

where

$$l_{11}=\left(p_1^2-\alpha_1\right)K_0\left(p_1R\right),\ \ l_{12}=\left(p_2^2-\alpha_1\right)K_0\left(p_2R\right),$$

$$l_{21}=\left[\beta^2\,\alpha_2\alpha_1K_0\left(p_1r\right)+\frac{2}{R}\alpha_2p_1\,K_1\left(p_1R\right)\right],$$

$$l_{22}=\left[\beta^2\,\alpha_2\alpha_1K_0\left(p_2R\right)+\frac{2}{R}\alpha_2p_2\,K_1\left(p_2R\right)\right].$$

Then, we have

$$A_1=\frac{\alpha_5}{G}\left[\beta^2\,\alpha_2\alpha_1K_0\left(p_2R\right)+\frac{2}{R}\alpha_2p_2\,K_1\left(p_2R\right)\right],$$

and

$$A_2=-\frac{\alpha_5}{G}\left[\beta^2\,\alpha_2\alpha_1K_0\left(p_1r\right)+\frac{2}{R}\alpha_2p_1\,K_1\left(p_1R\right)\right],$$

where

$$G=\begin{bmatrix}\beta^2\,\alpha_2\alpha_1K_0\left(p_2R\right)+\\\frac{2}{R}\alpha_2p_2\,K_1\left(p_2R\right)\end{bmatrix}\left[\left(p_1^2-\alpha_1\right)K_0\left(p_1R\right)\right]-\begin{bmatrix}\beta^2\,\alpha_2\alpha_1K_0\left(p_1r\right)+\\\frac{2}{R}\alpha_2p_1\,K_1\left(p_1R\right)\end{bmatrix}\left[\left(p_2^2-\alpha_1\right)K_0\left(p_2R\right)\right].$$

Those complete the solution in the Laplace transform space.

5. Numerical Inversion of the Laplace Transform

In order to invert the Laplace transform, we adopt a numerical inversion method based on a Fourier series expansion [13].

By this method the inverse $f(t)$ of the Laplace transform $\bar{f}(s)$ is approximated by

$$f(t) = \frac{e^{ct}}{t_1}\left[\frac{1}{2}\bar{f}(c) + R1\sum_{k=1}^{N}\bar{f}\left(c+\frac{ik\pi}{t_1}\right)\exp\left(\frac{ik\pi t}{t_1}\right)\right] \quad (57)$$

$$,0 < t_1 < 2t,$$

where N is a sufficiently large integer representing the number of terms in the truncated Fourier series, chosen such that

$$\exp(ct)\ R1\left[\bar{f}\left(c+\frac{iN\pi}{t_1}\right)\exp\left(\frac{iN\pi t}{t_1}\right)\right] \le \varepsilon_1, \quad (58)$$

where ε_1 is a prescribed small positive number that corresponds to the degree of accuracy required. The parameter c is a positive free parameter that must be greater than the real part of all the singularities of $\bar{f}(s)$. The optimal choice of c was obtained according to the criteria described in [13].

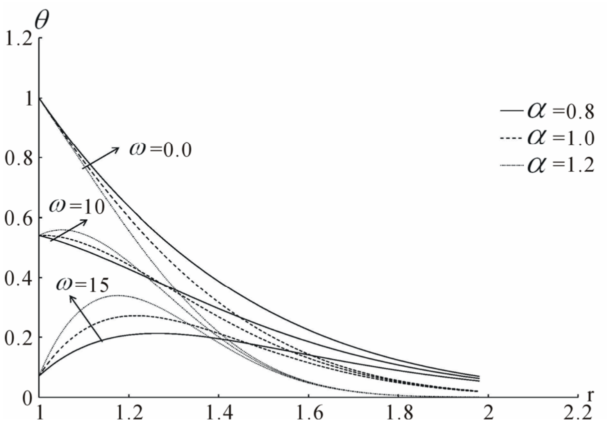

Figure 1. The temperature distribution.

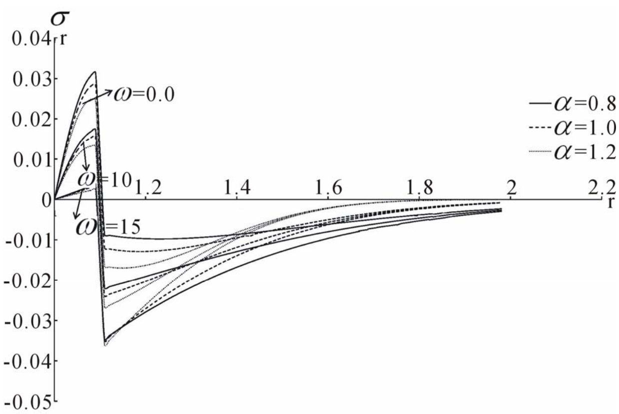

Figure 2. The radial stress distribution.

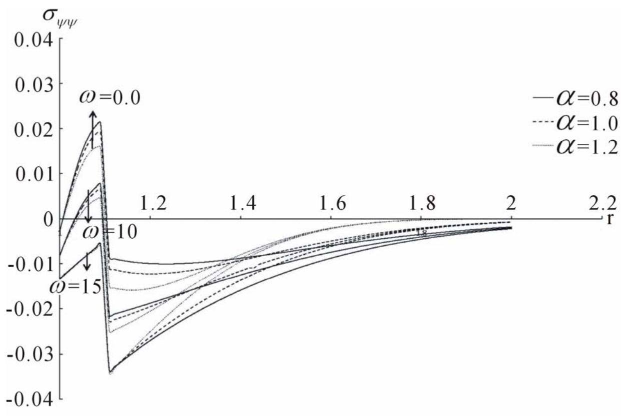

Figure 3. The stress $\sigma_{\psi\psi}$ distribution.

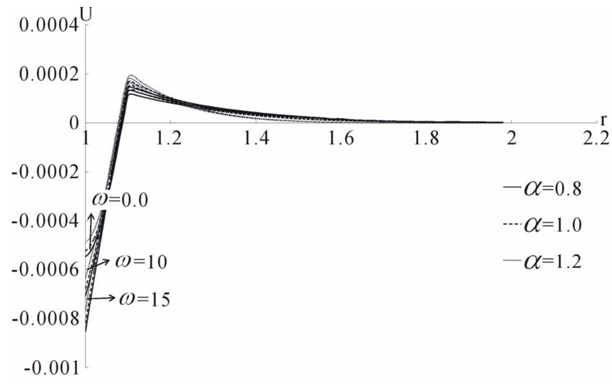

Figure 4. The displacement distribution.

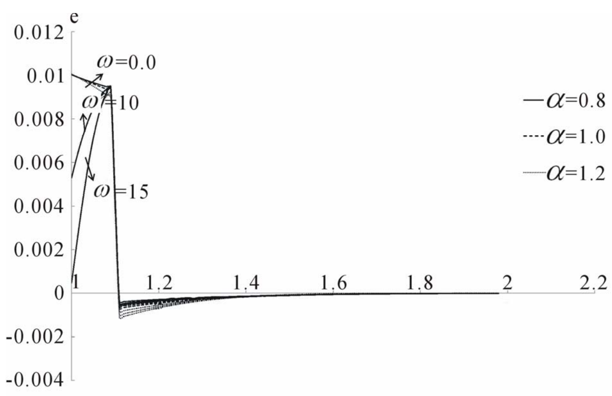

Figure 5. The strain distribution.

6. Numerical Results and Discussion

With a view to illustrating the analytical procedure presented earlier, we now consider a numerical example for which computational results are given. The results depict the variation of temperature, stress, displacement and strain fields in the context of Youssef model [11].

For this purpose, copper is taken as the thermoelastic material for which we take the following values of the different physical constants [12]:

$K = 386\ kg \cdot m \cdot k^{-1} \cdot s^{-3}$, $\alpha_T = 1.78\,(10)^{-5}\, k^{-1}$,

$\rho = 8954\ kg \cdot m^{-3}$, $T_o = 293k$, $C = 383.1\ m^2 \cdot k^{-1} \cdot s^{-2}$,

$\mu = 3.86\,(10)^{10}\ kg \cdot m^{-1} \cdot s^{-2}$,

$\lambda = 7.76\,(10)^{10}\ kg \cdot m^{-1} \cdot s^{-2}$.

From the above values we get the nondimensional values for our problem as:

$a = 0.0104441, \varepsilon = 1.618, \beta^2 = 4$.

The results of the temperature, the stresses, the displacement and the strain are shown in **Figures 1-5** respectively with wide range of non dimensional distance r from $r = R = 1.0$ up to $r = 2.0$, non dimensional time t = 0.08 and non dimensional relaxation time $\tau_o = 0.001$ with different values of the parameter $\alpha = (0.8, 1.0, 1.2)$ which describe the three types of conductivity (weak conductivity, normal conductivity, strong conductivity), respectively and with different values of the parameter $\omega = (0, 10, 15)$. We can see the significant effect of the parameter α and the angular frequency of thermal vibration ω on all the studied fields.

7. Conclusion

We considered a perfectly conducting elastic isotropic homogeneous infinite body with cylindrical cavity in the context of the fractional order generalized thermoelasticity theory (Youssef model). The effect of the fractional parameter and the angular frequency of thermal vibration on all the studied fields are very significant. New classification of the materials must be constructed according to the fractional parameter which describes the ability of the material to conduct the heat.

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