Delay-Dependent Robust H_∞ Control for Uncertain 2-D Discrete State Delay Systems Described by the General Model

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Abstract

This paper considers the problem of delay-dependent robust optimal H_∞ control for a class of uncertain two-dimensional (2-D) discrete state delay systems described by the general model (GM). The parameter uncertainties are assumed to be norm-bounded. A linear matrix inequality (LMI)-based sufficient condition for the existence of delay-dependent γ-suboptimal state feedback robust H_∞ controllers which guarantees not only the asymptotic stability of the closed-loop system, but also the H_∞ noise attenuation γ over all admissible parameter uncertainties is established. Furthermore, a convex optimization problem is formulated to design a delay-dependent state feedback robust optimal H_∞ controller which minimizes the H_∞ noise attenuation γ of the closed-loop system. Finally, an illustrative example is provided to demonstrate the effectiveness of the proposed method.

Keywords

2-D Discrete System, General Model, H_∞ Control, Linear Matrix Inequality, State Delays, Uncertain System

1. Introduction

In the past decades, research on two-dimensional (2-D) discrete systems has rapidly increased due to their extensive practical applications in circuits analysis [1], digital image processing [2], signal filtering [3] and thermal power engineering [4], etc. Thus, the study of 2-D systems is an attractive problem and a number of results have been presented in the literature. Among these results the stability analysis of 2-D discrete systems has been given in [5]-[15]. In [5], the problem of robust stability analysis and stabilization for 2-D discrete uncertain systems described by the Fornasini-Marchesini

(FM) second model has been studied and the sufficient conditions for 2-D discrete uncertain systems to be robustly stable have been given in terms of linear matrix inequalities (LMIs). A necessary and sufficient condition for the stability of 2-D discrete systems described by the FM first model has been derived in [13]. The problem of stability for linear discrete 2-D singular general model (GM) has been discussed in [14] and a sufficient condition for the internal stability of the 2-D singular GM has been derived in terms of LMIs. In [15], the robust stability analysis problem for a class of uncertain 2-D discrete systems described by the FM second model has been studied and sufficient conditions in terms of LMIs have been derived to ensure the robust stability of the uncertain 2-D discrete systems. Lately, the solution to the guaranteed cost control problem for 2-D discrete uncertain systems has been presented by many authors [16]-[22].

In recent years, the H_∞ control problem for 2-D discrete systems has gained a great deal of interest and many important results have been obtained [23]-[26]. A major advantage of H_∞ control is that its performance specification takes account of the worst-case performance for system in terms of the system energy gain. This is appropriate for system robustness analysis and robust control with modeling uncertainties and disturbances than other performance specifications, such as the LQ-optimal control specification [23]. In [24], the solutions for the H_∞ control and robust stabilization problems for 2-D systems described by the Roesser model using the 2-D system bounded realness property have been presented. The problem of H_∞ static output feedback control for 2-D discrete systems described by the Roesser model and the FM second model has been addressed in [25]. In [26], the problem of robust H_∞ control for uncertain 2-D discrete systems described by the GM via output feedback controllers has been investigated.

Since, delay is encountered in many dynamic systems and is often a source of instability, much attention has been focused on the problem of stability analysis and controller design for 2-D discrete state-delayed systems in the last decades. Presently, the stability results for 2-D discrete state-delayed systems fall in two groups: delay-independent stability conditions [27]-[35] and delay-dependent ones [36]-[45]. The former refers to the stability conditions which do not depend on delay; the latter contains information on the size of delay. Generally speaking, the delay-dependent stability condition is less conservative especially when the sizes of the delays are small [45]. The problem of stability analysis for 2-D discrete state-delayed systems in the GM has been considered in [33] and sufficient condition for the stability has been derived via Lyapunov approach. In [27], a solution to the problem of delay-independent H_∞ control for 2-D state-delayed systems described by the FM second model has been presented. The problem of robust reliable control for a class of uncertain 2-D discrete switched systems with state delays and actuator faults represented by a model of Roesser type has been studied by [28]. In [35], the problem of robust guaranteed cost control via memoryless state feedback for uncertain 2-D discrete state-delayed systems described by the FM second model has been considered. Several technical errors that have occurred in the main results of [35] were corrected in [30]. A solution to the guaranteed cost control

problem via memory state feedback control laws for a class of uncertain 2-D discrete state-delayed systems described by the FM second model has been presented in [31]. The problem of delay–dependent guaranteed cost control via memoryless state feedback for uncertain 2-D discrete state-delayed systems described by the FM second model has been studied in [43]. In [41], a delay-range dependent H_∞ control and stabilization problem for a class of uncertain 2-D state-delayed systems described by the Roesser model has been proposed. The delay-dependent H_∞ control problem via a delay-dependent bounded real lemma for a class of 2-D state-delayed systems described by the FM second model has been addressed in [40]. Recently, the problems of delay-dependent H_∞ control for 2-D discrete state-delayed systems described by the FM second model and Roesser model have been considered in [42] [44], respectively and a delay-dependent optimal state feedback H_∞ controller has been obtained for both the models. It may be mentioned here that the criteria presented in [42] [44] do not consider the uncertainty in system parameters. In real-time applications, the parametric uncertainties cannot be ignored as they are the main source of instability and poor system performance. Hence, the delay-dependent robust optimal H_∞ control for uncertain 2-D discrete state delay systems is an important and challenging problem. However to the best of authors' knowledge, the delay-dependent robust optimal H_∞ control problem for uncertain 2-D discrete state delay systems represented by the GM which is structurally distinct from FM second model and Roesser model has not been addressed so far in the literature.

This paper, therefore, investigates the problem of delay-dependent robust optimal H_∞ control for a class of uncertain 2-D discrete state delay systems described by the GM. The approach adopted in this paper is as follows: We first derive an LMI-based sufficient condition for the existence of delay-dependent γ-suboptimal state feedback robust H_∞ controllers in terms of feasible solution to a certain LMI. Further, a convex optimization problem with LMI constraints is formulated to design a delay-dependent robust optimal H_∞ controller which minimizes the H_∞ noise attenuation γ of the closed-loop system. The paper is organized as follows. Section 2 formulates the problem of delay-dependent robust H_∞ control for a class of uncertain 2-D discrete state delay systems described by the GM and recalls some useful results. In Section 3, a solution to the problem of delay-dependent robust optimal H_∞ control is presented. An example illustrating the potential of the proposed technique is given in Section 4.

Notations:

Throughout the paper, the following notations are used: R^n denotes real vector space of dimension n; $R^{n \times m}$ is the set of $n \times m$ real matrices; the superscript T stands for matrix transposition; $\mathbf{0}$ denotes null matrix or null vector of appropriate dimension; I is the identity matrix of appropriate dimension; $G < 0$ stands for the matrix G which is symmetric and negative definite; diag{....} stands for a block diagonal matrix; $\|.\|$ denotes the Euclidean vector norm. The l_2 norm of a 2-D signal $\omega(i, j)$ is given by

$$\|\omega\|_2 = \sqrt{\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|\omega(i, j)\|^2}, \text{ where } \omega(i, j) \text{ is said to be in } l_2 \text{ if } \|\omega\|_2 < \infty.$$

2. Problem Formulation and Preliminaries

Consider an uncertain 2-D discrete state delay system described by the GM [46].

$$
\begin{aligned}
\boldsymbol{x}(i+1, j+1) =\; & \overline{\boldsymbol{A}}_1 \boldsymbol{x}(i, j+1) + \overline{\boldsymbol{A}}_2 \boldsymbol{x}(i+1, j) + \overline{\boldsymbol{A}}_0 \boldsymbol{x}(i, j) + \overline{\boldsymbol{A}}_{1d} \boldsymbol{x}(i-d_1, j+1) \\
& + \overline{\boldsymbol{A}}_{2d} \boldsymbol{x}(i+1, j-d_2) + \overline{\boldsymbol{A}}_{0d} \boldsymbol{x}(i-k_1, j-k_2) + \overline{\boldsymbol{B}}_1 \boldsymbol{w}(i, j+1) \\
& + \overline{\boldsymbol{B}}_2 \boldsymbol{w}(i+1, j) + \overline{\boldsymbol{B}}_0 \boldsymbol{w}(i, j) + \overline{\boldsymbol{C}}_1 \boldsymbol{u}(i, j+1) + \overline{\boldsymbol{C}}_2 \boldsymbol{u}(i+1, j) + \overline{\boldsymbol{C}}_0 \boldsymbol{u}(i, j),
\end{aligned}
\tag{1a}
$$

$$
\boldsymbol{z}(i, j) = \boldsymbol{H} \boldsymbol{x}(i, j) + \boldsymbol{L} \boldsymbol{w}(i, j),
\tag{1b}
$$

where

$$
\begin{aligned}
& \overline{\boldsymbol{A}}_1 = (\boldsymbol{A}_1 + \Delta \boldsymbol{A}_1), \overline{\boldsymbol{A}}_2 = (\boldsymbol{A}_2 + \Delta \boldsymbol{A}_2), \overline{\boldsymbol{A}}_0 = (\boldsymbol{A}_0 + \Delta \boldsymbol{A}_0), \\
& \overline{\boldsymbol{A}}_{1d} = (\boldsymbol{A}_{1d} + \Delta \boldsymbol{A}_{1d}), \overline{\boldsymbol{A}}_{2d} = (\boldsymbol{A}_{2d} + \Delta \boldsymbol{A}_{2d}), \overline{\boldsymbol{A}}_{0d} = (\boldsymbol{A}_{0d} + \Delta \boldsymbol{A}_{0d}), \\
& \overline{\boldsymbol{B}}_1 = (\boldsymbol{B}_1 + \Delta \boldsymbol{B}_1), \overline{\boldsymbol{B}}_2 = (\boldsymbol{B}_2 + \Delta \boldsymbol{B}_2), \overline{\boldsymbol{B}}_0 = (\boldsymbol{B}_0 + \Delta \boldsymbol{B}_0), \\
& \overline{\boldsymbol{C}}_1 = (\boldsymbol{C}_1 + \Delta \boldsymbol{C}_1), \overline{\boldsymbol{C}}_2 = (\boldsymbol{C}_2 + \Delta \boldsymbol{C}_2), \overline{\boldsymbol{C}}_0 = (\boldsymbol{C}_0 + \Delta \boldsymbol{C}_0),
\end{aligned}
\tag{1c}
$$

and $0 \le i, j \in \mathbf{Z}$ are horizontal and vertical coordinates, respectively, $\boldsymbol{x}(i, j) \in R^n$, $\boldsymbol{u}(i, j) \in R^m$ are state and control input, respectively, $\boldsymbol{z}(i, j) \in R^p$ is the controlled output, $\boldsymbol{w}(i, j) \in R^q$ is the noise input with bounded energy *i.e.*, it belongs to $\ell_2 \{[0, \infty), [0, \infty)\}$. The matrices $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_0, \boldsymbol{A}_{1d}, \boldsymbol{A}_{2d}, \boldsymbol{A}_{0d} \in R^{n \times n}$, $\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_0 \in R^{n \times q}$, $\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_0 \in R^{n \times m}$, $\boldsymbol{H} \in R^{p \times n}$ and $\boldsymbol{L} \in R^{p \times q}$ are known constant matrices representing the nominal plant; d_1, d_2, k_1 and k_2 are unknown constant positive integers representing delays. The matrices $\Delta \boldsymbol{A}_1$, $\Delta \boldsymbol{A}_2$, $\Delta \boldsymbol{A}_0$, $\Delta \boldsymbol{A}_{1d}$, $\Delta \boldsymbol{A}_{2d}$, $\Delta \boldsymbol{A}_{0d}$, $\Delta \boldsymbol{B}_1$, $\Delta \boldsymbol{B}_2$, $\Delta \boldsymbol{B}_0$, $\Delta \boldsymbol{C}_1$, $\Delta \boldsymbol{C}_2$ and $\Delta \boldsymbol{C}_0$ represent parameter uncertainties in the system matrices which are assumed to be of the form

$$
\left.
\begin{aligned}
& \begin{bmatrix} \Delta \boldsymbol{A}_1 & \Delta \boldsymbol{A}_2 & \Delta \boldsymbol{A}_0 & \Delta \boldsymbol{A}_{1d} & \Delta \boldsymbol{A}_{2d} & \Delta \boldsymbol{A}_{0d} \end{bmatrix} = \boldsymbol{D} \boldsymbol{F}(i, j) \begin{bmatrix} \boldsymbol{E}_1 & \boldsymbol{E}_2 & \boldsymbol{E}_0 & \boldsymbol{E}_{1d} & \boldsymbol{E}_{2d} & \boldsymbol{E}_{0d} \end{bmatrix} \\
& \begin{bmatrix} \Delta \boldsymbol{B}_1 & \Delta \boldsymbol{B}_2 & \Delta \boldsymbol{B}_0 \end{bmatrix} = \boldsymbol{D} \boldsymbol{F}(i, j) \begin{bmatrix} \boldsymbol{E}_{b1} & \boldsymbol{E}_{b2} & \boldsymbol{E}_{b0} \end{bmatrix} \\
& \begin{bmatrix} \Delta \boldsymbol{C}_1 & \Delta \boldsymbol{C}_2 & \Delta \boldsymbol{C}_0 \end{bmatrix} = \boldsymbol{D} \boldsymbol{F}(i, j) \begin{bmatrix} \boldsymbol{E}_{c1} & \boldsymbol{E}_{c2} & \boldsymbol{E}_{c0} \end{bmatrix}
\end{aligned}
\right\}, \tag{1d}
$$

where $\boldsymbol{D} \in R^{n \times k}$, $\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_0, \boldsymbol{E}_{1d}, \boldsymbol{E}_{2d}, \boldsymbol{E}_{0d} \in R^{l \times n}$, $\boldsymbol{E}_{b1}, \boldsymbol{E}_{b2}, \boldsymbol{E}_{b0} \in R^{l \times q}$ and $\boldsymbol{E}_{c1}, \boldsymbol{E}_{c2}, \boldsymbol{E}_{c0} \in R^{l \times m}$ are known structural matrices of uncertainty and $\boldsymbol{F}(i, j) \in R^{k \times l}$ is an unknown matrix representing parameter uncertainty which satisfies

$$
\boldsymbol{F}^{\mathrm{T}}(i, j) \boldsymbol{F}(i, j) \le \boldsymbol{I} \quad (\text{or equivalently, } \|\boldsymbol{F}(i, j)\| \le 1). \tag{1e}
$$

For system (1), suppose a finite set of initial conditions [33], *i.e.*, there exist positive integers r_1 and r_2, such that

$$
\left.
\begin{aligned}
& \boldsymbol{x}(i, j) = h_{ij}, \quad i \in [-d_1, 0], \quad 0 \le j \le r_1 \\
& \boldsymbol{x}(i, j) = v_{ij}, \quad j \in [-d_2, 0], \quad 0 \le i \le r_2 \\
& \boldsymbol{x}(i, j) = \omega_{ij}, \quad i \in [-k_1, 0], \; j \in [-k_2, 0] \\
& h_{00} = v_{00} = \omega_{00}.
\end{aligned}
\right\} \tag{2}
$$

Denote $X_r = \sup \{\|x(i, j)\| : i + j = r, i, j \in \mathbf{Z}\}$, we first present the definition of asymptotic stability for the system (1).

Definition 1. [42] The system (1) is asymptotically stable if $\lim_{r \to \infty} X_r = 0$ with

$u(i, j) = \mathbf{0}$, $w(i, j) = \mathbf{0}$ and the initial condition (2).

Introduce the following state feedback controller

$$\boldsymbol{u}(i, j) = \boldsymbol{K}\boldsymbol{x}(i, j). \tag{3}$$

Applying the controller (3) to system (1) will result in the closed-loop system:

$$\begin{aligned}
\boldsymbol{x}(i+1, j+1) = {}& \overline{\boldsymbol{A}}_{1k}\boldsymbol{x}(i, j+1) + \overline{\boldsymbol{A}}_{2k}\boldsymbol{x}(i+1, j) + \overline{\boldsymbol{A}}_{0k}\boldsymbol{x}(i, j) \\
& + \overline{\boldsymbol{A}}_{1d}\boldsymbol{x}(i-d_1, j+1) + \overline{\boldsymbol{A}}_{2d}\boldsymbol{x}(i+1, j-d_2) + \overline{\boldsymbol{A}}_{0d}\boldsymbol{x}(i-k_1, j-k_2) \\
& + \overline{\boldsymbol{B}}_1\boldsymbol{w}(i, j+1) + \overline{\boldsymbol{B}}_2\boldsymbol{w}(i+1, j) + \overline{\boldsymbol{B}}_0\boldsymbol{w}(i, j),
\end{aligned} \tag{4}$$

where

$$\overline{\boldsymbol{A}}_{1k} = \overline{\boldsymbol{A}}_1 + \overline{\boldsymbol{C}}_1\boldsymbol{K}, \quad \overline{\boldsymbol{A}}_{2k} = \overline{\boldsymbol{A}}_2 + \overline{\boldsymbol{C}}_2\boldsymbol{K}, \quad \overline{\boldsymbol{A}}_{0k} = \overline{\boldsymbol{A}}_0 + \overline{\boldsymbol{C}}_0\boldsymbol{K}.$$

To investigate the delay-dependent H_∞ control problem, we first define the vectors $\boldsymbol{s}(i, j), \boldsymbol{t}(i, j), \boldsymbol{q}(i, j) \in R^n$ such that

$$\begin{cases}
\boldsymbol{x}(i+1, j+1) = \boldsymbol{x}(i, j+1) + \boldsymbol{s}(i, j+1), \\
\boldsymbol{x}(i+1, j+1) = \boldsymbol{x}(i+1, j) + \boldsymbol{t}(i+1, j), \\
\boldsymbol{x}(i+1, j+1) = \boldsymbol{x}(i, j) + \boldsymbol{q}(i, j).
\end{cases} \tag{5}$$

The following well known lemmas are needed in the proof of our main result.

Lemma 1. [42] [44] For any matrices $\boldsymbol{M}_1, \boldsymbol{M}_2 \in R^{n \times n}$ and $0 < \boldsymbol{X} \in R^{n \times n}$, and any integer $d \geq 0$, the following summation inequality holds

$$\begin{aligned}
& -\sum_{l=-d}^{-1} \boldsymbol{s}^{\mathrm{T}}(i+l, j+1)\boldsymbol{X}\boldsymbol{s}(i+l, j+1) \\
& \leq \boldsymbol{\eta}^{\mathrm{T}}(i, j+1)\begin{bmatrix} \boldsymbol{M}_1^{\mathrm{T}} + \boldsymbol{M}_1 & -\boldsymbol{M}_1^{\mathrm{T}} + \boldsymbol{M}_2 \\ * & -\boldsymbol{M}_2^{\mathrm{T}} - \boldsymbol{M}_2 \end{bmatrix}\boldsymbol{\eta}(i, j+1) \\
& + d\boldsymbol{\eta}^{\mathrm{T}}(i, j+1)\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{X}^{-1}\boldsymbol{Y}\boldsymbol{\eta}(i, j+1),
\end{aligned} \tag{6}$$

where

$$\boldsymbol{\eta}(i, j+1) = \begin{bmatrix} \boldsymbol{x}(i, j+1) \\ \boldsymbol{x}(i-d, j+1) \end{bmatrix}, \quad \boldsymbol{Y} = \begin{bmatrix} \boldsymbol{M}_1 & \boldsymbol{M}_2 \end{bmatrix}.$$

Lemma 2. [5] Let $\boldsymbol{M}$, $\boldsymbol{E}$, $\boldsymbol{F}$ and $\boldsymbol{Y}_1$ be real matrices of appropriate dimension with $\boldsymbol{Y}_1$ satisfying $\boldsymbol{Y}_1 = \boldsymbol{Y}_1^{\mathrm{T}}$ then

$$\boldsymbol{Y}_1 + \boldsymbol{M}\boldsymbol{F}\boldsymbol{E} + \boldsymbol{E}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}} < \boldsymbol{0}, \tag{7a}$$

holds for all $\boldsymbol{F}$ satisfying $\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} \leq \boldsymbol{I}$, if and only if there exist $\varepsilon > 0$ such that

$$\boldsymbol{Y}_1 + \varepsilon\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}} + \varepsilon^{-1}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E} < \boldsymbol{0}. \tag{7b}$$

The H_∞ performance measure for system (1) with the initial condition (2) is defined as follows:

Definition 2. [42] Given a scalar $\gamma > 0$, integers $d_1^* > 0$, $d_2^* > 0$, $k_1^* > 0$, $k_2^* > 0$ and symmetric positive definite weighting matrices $\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{Q}_3, \boldsymbol{Z}_1, \boldsymbol{Z}_2, \boldsymbol{Z}_3 \in R^{n \times n}$. The closed-loop system (4), formed by system (1) with the initial condition (2) and state-feedback controller (3), is said to have an H_∞ noise attenuation γ for any delays d_1,

d_2, k_1, k_2 satisfying $0 < d_1 \le d_1^*$, $0 < d_2 \le d_2^*$, $0 < k_1 \le k_1^*$ and $0 < k_2 \le k_2^*$ if it is robustly stable and satisfies

$$J = \sup_{0 \ne w \in \ell_2} \frac{\|\overline{z}\|_2^2}{\|\overline{w}\|_2^2 + D_1(d_1^*, j) + D_2(i, d_2^*) + D_3(k_1^*, k_2^*)} < \gamma^2, \tag{8}$$

where

$$\|\overline{z}\|_2^2 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left\| \begin{bmatrix} z(i, j+1) \\ z(i+1, j) \\ z(i, j) \end{bmatrix} \right\|^2, \quad \|\overline{w}\|_2^2 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left\| \begin{bmatrix} w(i, j+1) \\ w(i+1, j) \\ w(i, j) \end{bmatrix} \right\|^2,$$

$$D_1(d_1^*, j) = \sum_{j=0}^{\infty} \left[x^{\mathrm{T}}(0, j+1) Q_1 x(0, j+1) + \sum_{\theta=-d_1^*+1}^{0} \sum_{l=\theta-1}^{-1} s^{\mathrm{T}}(l, j+1) Z_1 s(l, j+1) \right],$$

$$D_2(i, d_2^*) = \sum_{i=0}^{\infty} \left[x^{\mathrm{T}}(i+1, 0) Q_2 x(i+1, 0) + \sum_{\theta=-d_2^*+1}^{0} \sum_{l=\theta-1}^{-1} t^{\mathrm{T}}(i+1, l) Z_2 t(i+1, l) \right],$$

$$D_3(k_1^*, k_2^*) = \sum_{i=0}^{\infty} \left[x^{\mathrm{T}}(i, 0) Q_3 x(i, 0) + \sum_{\varphi_1=-k_1^*+1}^{0} \sum_{m=\varphi_1-1}^{-1} \sum_{\varphi_2=-k_2^*+1}^{0} \sum_{n=\varphi_2-1}^{-1} q^{\mathrm{T}}(i+m, n) Z_3 q(i+m, n) \right]$$

$$+ \sum_{j=0}^{\infty} \left[x^{\mathrm{T}}(0, j) Q_3 x(0, j) + \sum_{\varphi_1=-k_1^*+1}^{0} \sum_{m=\varphi_1-1}^{-1} \sum_{\varphi_2=-k_2^*+1}^{0} \sum_{n=\varphi_2-1}^{-1} q^{\mathrm{T}}(m, j+n) Z_3 q(m, j+n) \right],$$

and d_1^*, k_1^*, d_2^*, k_2^* are upper bounds of delays in states.

In the case when the initial condition is known to be zero, then the H_∞ performance measure (8) reduces to

$$J_0 = \sup_{0 \ne \omega \in l_2} \frac{\|\overline{z}\|_2}{\|\overline{w}\|_2} < \gamma. \tag{9}$$

Using the 2-D Parseval's theorem [3], (9) is equivalent to

$$\|G(z_1, z_2)\|_\infty = \sup_{\omega_1, \omega_2 \in [0, 2\pi]} \sigma_{\max} \left[G(e^{j\omega_1}, e^{j\omega_2}) \right] < \gamma, \tag{10}$$

where $\sigma_{\max}(\cdot)$ denotes the maximum singular value of the corresponding matrix and

$$G(z_1, z_2) = H \left(z_1 z_2 I_n - z_2 \overline{A}_{1k} - z_1 \overline{A}_{2k} - \overline{A}_{0k} - z_1^{-d_1} z_2 \overline{A}_{1d} - z_1 z_2^{-d_2} \overline{A}_{2d} - z_1^{-k_1} z_2^{-k_2} \overline{A}_{0d} \right)^{-1} \tag{11}$$
$$\times \left(z_2 \overline{B}_1 + z_1 \overline{B}_2 + \overline{B}_0 \right) + L$$

is the transfer function from the disturbance input $w(i, j)$ to the controlled output $z(i, j)$ for the system (4).

The objective of this paper is to design a controller of the form (3) such that the closed-loop system (4) is asymptotically stable and the H_∞ norm of the transfer function (11) from the noise input $w(i, j)$ to the controlled output $z(i, j)$ for the closed-loop system (4) is smaller than γ. Such controller is said to be a γ-suboptimal state feedback robust H_∞ controller for the uncertain 2-D discrete state delay system (1) with any state delays d_1, d_2, k_1, k_2 satisfying $0 < d_1 \le d_1^*$, $0 < d_2 \le d_2^*$,

$$0 < k_1 \le k_1^* \quad \text{and} \quad 0 < k_2 \le k_2^*.$$

3. Main Results

In this section, we first present a delay-dependent approach to solve the H_∞ control problem for the uncertain 2-D discrete state delay system (1) with any state delays d_1, d_2, k_1, k_2 satisfying $0 < d_1 \le d_1^*$, $0 < d_2 \le d_2^*$, $0 < k_1 \le k_1^*$ and $0 < k_2 \le k_2^*$. Based on Lemma 1, a sufficient condition for the existence of γ-suboptimal state feedback robust H_∞ controllers is obtained in terms of an LMI. Further, a delay-dependent state feedback robust optimal H_∞ controller is obtained by solving a convex optimization problem which minimizes the H_∞ noise attenuation γ of the closed-loop system (4).

Theorem 1. The closed-loop system (4), formed by system (1) with the initial condition (2) and state-feedback controller (3), is robustly stable and has a specified H_∞ noise attenuation γ, if there exist matrices $M_{11}, M_{12}, M_{21}, M_{22}, M_{31}, M_{32} \in R^{n \times n}$ and symmetric positive definite matrices $P, P_1, P_2, S_1, S_2, S_3 \in R^{n \times n}$ satisfying $P_1 < \gamma^2 Q_1$, $P_2 < \gamma^2 Q_2$, $0 < P - P_1 - P_2 < \gamma^2 Q_3$, $S_1 < \gamma^2 Z_1$, $S_2 < \gamma^2 Z_2$, $S_3 < \gamma^2 Z_3$ such that

$$\begin{bmatrix}
\Omega_1 & \Omega_2^{\mathrm{T}} & \Omega_3^{\mathrm{T}} & \Omega_4^{\mathrm{T}} & \hat{\Omega}_5^{\mathrm{T}} & \hat{\Omega}_6^{\mathrm{T}} & \hat{\Omega}_7^{\mathrm{T}} & \hat{\Omega}_8^{\mathrm{T}} & \Omega_9^{\mathrm{T}} & \Omega_{10}^{\mathrm{T}} & \Omega_{11}^{\mathrm{T}} \\
* & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -P^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -d_1^{*-1}S_1^{-1} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -d_2^{*-1}S_2^{-1} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -k_1^{*-1}k_2^{*-1}S_3^{-1} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -d_1^{*-1}S_1 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -d_2^{*-1}S_2 & 0 \\
* & * & * & * & * & * & * & * & * & * & -k_1^{*-1}k_2^{*-1}S_3
\end{bmatrix} < 0, \quad (12)$$

where

$$\Omega_1 = \begin{bmatrix}
\beta_1 & M_{12}-M_{11}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & -M_{12}-M_{12}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \beta_2 & M_{22}-M_{21}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -M_{22}^{\mathrm{T}}-M_{22} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \beta_3 & M_{32}-M_{31}^{\mathrm{T}} & 0 & 0 & 0 \\
* & * & * & * & * & -M_{32}^{\mathrm{T}}-M_{32} & 0 & 0 & 0 \\
* & * & * & * & * & * & -\gamma^2 I & 0 & 0 \\
* & * & * & * & * & * & * & -\gamma^2 I & 0 \\
* & * & * & * & * & * & * & * & -\gamma^2 I
\end{bmatrix},$$

$$\boldsymbol{\beta}_1 = -\boldsymbol{P}_1 + \boldsymbol{M}_{11}^{\mathrm{T}} + \boldsymbol{M}_{11},$$

$$\boldsymbol{\beta}_2 = -\boldsymbol{P}_2 + \boldsymbol{M}_{21}^{\mathrm{T}} + \boldsymbol{M}_{21},$$

$$\boldsymbol{\beta}_3 = -\boldsymbol{P} + \boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{M}_{31}^{\mathrm{T}} + \boldsymbol{M}_{31},$$

$$\boldsymbol{\Omega}_2 = \begin{bmatrix} \boldsymbol{H} & 0 & 0 & 0 & 0 & 0 & \boldsymbol{L} & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\Omega}_3 = \begin{bmatrix} 0 & 0 & \boldsymbol{H} & 0 & 0 & 0 & 0 & \boldsymbol{L} & 0 \end{bmatrix},$$

$$\boldsymbol{\Omega}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & \boldsymbol{H} & 0 & 0 & 0 & \boldsymbol{L} \end{bmatrix},$$

$$\hat{\boldsymbol{\Omega}}_5 = \begin{bmatrix} \bar{\boldsymbol{A}}_{1k} & \bar{\boldsymbol{A}}_{1d} & \bar{\boldsymbol{A}}_{2k} & \bar{\boldsymbol{A}}_{2d} & \bar{\boldsymbol{A}}_{0k} & \bar{\boldsymbol{A}}_{0d} & \bar{\boldsymbol{B}}_1 & \bar{\boldsymbol{B}}_2 & \bar{\boldsymbol{B}}_0 \end{bmatrix},$$

$$\hat{\boldsymbol{\Omega}}_6 = \begin{bmatrix} \bar{\boldsymbol{A}}_{1k} - \boldsymbol{I} & \bar{\boldsymbol{A}}_{1d} & \bar{\boldsymbol{A}}_{2k} & \bar{\boldsymbol{A}}_{2d} & \bar{\boldsymbol{A}}_{0k} & \bar{\boldsymbol{A}}_{0d} & \bar{\boldsymbol{B}}_1 & \bar{\boldsymbol{B}}_2 & \bar{\boldsymbol{B}}_0 \end{bmatrix},$$

$$\hat{\boldsymbol{\Omega}}_7 = \begin{bmatrix} \bar{\boldsymbol{A}}_{1k} & \bar{\boldsymbol{A}}_{1d} & \bar{\boldsymbol{A}}_{2k} - \boldsymbol{I} & \bar{\boldsymbol{A}}_{2d} & \bar{\boldsymbol{A}}_{0k} & \bar{\boldsymbol{A}}_{0d} & \bar{\boldsymbol{B}}_1 & \bar{\boldsymbol{B}}_2 & \bar{\boldsymbol{B}}_0 \end{bmatrix},$$

$$\hat{\boldsymbol{\Omega}}_8 = \begin{bmatrix} \bar{\boldsymbol{A}}_{1k} & \bar{\boldsymbol{A}}_{1d} & \bar{\boldsymbol{A}}_{2k} & \bar{\boldsymbol{A}}_{2d} & \bar{\boldsymbol{A}}_{0k} - \boldsymbol{I} & \bar{\boldsymbol{A}}_{0d} & \bar{\boldsymbol{B}}_1 & \bar{\boldsymbol{B}}_2 & \bar{\boldsymbol{B}}_0 \end{bmatrix},$$

$$\boldsymbol{\Omega}_9 = \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\Omega}_{10} = \begin{bmatrix} 0 & 0 & \boldsymbol{M}_{21} & \boldsymbol{M}_{22} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\Omega}_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & \boldsymbol{M}_{31} & \boldsymbol{M}_{32} & 0 & 0 & 0 \end{bmatrix}.$$

Proof. To prove that the closed-loop system (4) with $\boldsymbol{w}(i,j) = \boldsymbol{0}$ is asymptotically stable, we define a Lyapunov-Krasovskii function

$$V\big(\boldsymbol{x}(i,j)\big) = V_1\big(\boldsymbol{x}(i,j)\big) + V_2\big(\boldsymbol{x}(i,j)\big) + V_3\big(\boldsymbol{x}(i,j)\big), \tag{13}$$

where

$$V_1\big(\boldsymbol{x}(i,j)\big) = \boldsymbol{x}^{\mathrm{T}}(i,j)\boldsymbol{P}_1\boldsymbol{x}(i,j) + \sum_{\theta=-d_1+1}^{0}\sum_{l=\theta-1}^{-1} \boldsymbol{s}^{\mathrm{T}}(i+l,j)\boldsymbol{S}_1\boldsymbol{s}(i+l,j),$$

$$V_2\big(\boldsymbol{x}(i,j)\big) = \boldsymbol{x}^{\mathrm{T}}(i,j)\boldsymbol{P}_2\boldsymbol{x}(i,j) + \sum_{\theta=-d_2+1}^{0}\sum_{l=\theta-1}^{-1} \boldsymbol{t}^{\mathrm{T}}(i,j+l)\boldsymbol{S}_2\boldsymbol{t}(i,j+l),$$

$$V_3\big(\boldsymbol{x}(i,j)\big) = \boldsymbol{x}^{\mathrm{T}}(i,j)\big(\boldsymbol{P} - \boldsymbol{P}_1 - \boldsymbol{P}_2\big)\boldsymbol{x}(i,j)$$

$$+ \sum_{\varphi_1=-k_1+1}^{0}\sum_{m=\varphi_1-1}^{-1}\sum_{\varphi_2=-k_2+1}^{0}\sum_{n=\varphi_2-1}^{-1} \boldsymbol{q}^{\mathrm{T}}(i+m,j+n)\boldsymbol{S}_3\boldsymbol{q}(i+m,j+n)$$

and $\boldsymbol{P}_1 > 0$, $\boldsymbol{P}_2 > 0$, $(\boldsymbol{P} - \boldsymbol{P}_1 - \boldsymbol{P}_2) > 0$, $\boldsymbol{S}_1 > 0$, $\boldsymbol{S}_2 > 0$ and $\boldsymbol{S}_3 > 0$. Thus, it is explicit that $V\big(\boldsymbol{x}(i,j)\big) > 0$.

Along any trajectory of the system (1) with $\boldsymbol{w}(i,j) = \boldsymbol{0}$, the increment $\Delta V(i+1,j+1)$ is given by

$$\Delta V(i+1, j+1) = V_1\big(\boldsymbol{x}(i+1, j+1)\big) + V_2\big(\boldsymbol{x}(i+1, j+1)\big) + V_3\big(\boldsymbol{x}(i+1, j+1)\big)$$

$$-V_1\big(\boldsymbol{x}(i, j+1)\big) - V_2\big(\boldsymbol{x}(i+1, j)\big) - V_3\big(\boldsymbol{x}(i, j)\big)$$

$$= \begin{bmatrix} \boldsymbol{x}(i, j+1) \\ \boldsymbol{x}(i-d_1, j+1) \\ \boldsymbol{x}(i+1, j) \\ \boldsymbol{x}(i+1, j-d_2) \\ \boldsymbol{x}(i, j) \\ \boldsymbol{x}(i-k_1, j-k_2) \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} \end{bmatrix} \boldsymbol{P} \begin{bmatrix} \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} + \begin{bmatrix} -\boldsymbol{P}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\boldsymbol{P}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\boldsymbol{P}+\boldsymbol{P}_1+\boldsymbol{P}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\times \begin{bmatrix} \boldsymbol{x}(i, j+1) \\ \boldsymbol{x}(i-d_1, j+1) \\ \boldsymbol{x}(i+1, j) \\ \boldsymbol{x}(i+1, j-d_2) \\ \boldsymbol{x}(i, j) \\ \boldsymbol{x}(i-k_1, j-k_2) \end{bmatrix} + \begin{bmatrix} \boldsymbol{s}(i, j+1) \\ \boldsymbol{t}(i+1, j) \\ \boldsymbol{q}(i, j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} d_1\boldsymbol{S}_1 & 0 & 0 \\ 0 & d_2\boldsymbol{S}_2 & 0 \\ 0 & 0 & k_1k_2\boldsymbol{S}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{s}(i, j+1) \\ \boldsymbol{t}(i+1, j) \\ \boldsymbol{q}(i, j) \end{bmatrix}$$

$$- \sum_{l=-d_1}^{-1} \boldsymbol{s}^{\mathrm{T}}(i+l, j+1)\boldsymbol{S}_1\boldsymbol{s}(i+l, j+1) - \sum_{l=-d_2}^{-1} \boldsymbol{t}^{\mathrm{T}}(i+1, j+l)\boldsymbol{S}_2\boldsymbol{t}(i+1, j+l)$$

$$- \sum_{m=-k_1}^{-1} \sum_{n=-k_2}^{-1} \boldsymbol{q}^{\mathrm{T}}(i+m, j+n)\boldsymbol{S}_3\boldsymbol{q}(i+m, j+n). \tag{14}$$

Applying (5), we get

$$\begin{bmatrix} \boldsymbol{s}(i, j+1) \\ \boldsymbol{t}(i+1, j) \\ \boldsymbol{q}(i, j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} d_1\boldsymbol{S}_1 & 0 & 0 \\ 0 & d_2\boldsymbol{S}_2 & 0 \\ 0 & 0 & k_1k_2\boldsymbol{S}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{s}(i, j+1) \\ \boldsymbol{t}(i+1, j) \\ \boldsymbol{q}(i, j) \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{x}(i, j+1) \\ \boldsymbol{x}(i-d_1, j+1) \\ \boldsymbol{x}(i+1, j) \\ \boldsymbol{x}(i+1, j-d_2) \\ \boldsymbol{x}(i, j) \\ \boldsymbol{x}(i-k_1, j-k_2) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}}-\boldsymbol{I} & \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}}-\boldsymbol{I} & \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}}-\boldsymbol{I} \\ \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} d_1\boldsymbol{S}_1 & 0 & 0 \\ 0 & d_2\boldsymbol{S}_2 & 0 \\ 0 & 0 & k_1k_2\boldsymbol{S}_3 \end{bmatrix}$$

$$\times \begin{bmatrix} \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}}-\boldsymbol{I} & \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}}-\boldsymbol{I} & \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}}-\boldsymbol{I} \\ \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{x}(i, j+1) \\ \boldsymbol{x}(i-d_1, j+1) \\ \boldsymbol{x}(i+1, j) \\ \boldsymbol{x}(i+1, j-d_2) \\ \boldsymbol{x}(i, j) \\ \boldsymbol{x}(i-k_1, j-k_2) \end{bmatrix}. \tag{15}$$

Now applying Lemma 1, we get the following summation inequalities

$$-\sum_{l=-d_1}^{-1}\left(\boldsymbol{s}^{\mathrm{T}}\left(i+l,j\right)\boldsymbol{S}_1\boldsymbol{s}\left(i+l,j\right)\right)\leq\begin{bmatrix}\boldsymbol{x}\left(i,j+1\right)\\\boldsymbol{x}\left(i-d_1,j+1\right)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{M}_{11}^{\mathrm{T}}+\boldsymbol{M}_{11}&-\boldsymbol{M}_{11}^{\mathrm{T}}+\boldsymbol{M}_{12}\\ *&-\boldsymbol{M}_{12}^{\mathrm{T}}-\boldsymbol{M}_{12}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\left(i,j+1\right)\\\boldsymbol{x}\left(i-d_1,j+1\right)\end{bmatrix}$$

$$+d_1\begin{bmatrix}\boldsymbol{x}\left(i,j+1\right)\\\boldsymbol{x}\left(i-d_1,j+1\right)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{M}_{11}^{\mathrm{T}}\\\boldsymbol{M}_{12}^{\mathrm{T}}\end{bmatrix}\boldsymbol{S}_1^{-1}\begin{bmatrix}\boldsymbol{M}_{11}&\boldsymbol{M}_{12}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\left(i,j+1\right)\\\boldsymbol{x}\left(i-d_1,j+1\right)\end{bmatrix}, \tag{16}$$

$$-\sum_{l=-d_2}^{-1}\left(\boldsymbol{t}^{\mathrm{T}}\left(i+1,j+l\right)\boldsymbol{S}_2\boldsymbol{t}\left(i+1,j+l\right)\right)\leq\begin{bmatrix}\boldsymbol{x}\left(i+1,j\right)\\\boldsymbol{x}\left(i+1,j-d_2\right)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{M}_{21}^{\mathrm{T}}+\boldsymbol{M}_{21}&-\boldsymbol{M}_{21}^{\mathrm{T}}+\boldsymbol{M}_{22}\\ *&-\boldsymbol{M}_{22}^{\mathrm{T}}-\boldsymbol{M}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\left(i+1,j\right)\\\boldsymbol{x}\left(i+1,j-d_2\right)\end{bmatrix}$$

$$+d_2\begin{bmatrix}\boldsymbol{x}\left(i+1,j\right)\\\boldsymbol{x}\left(i+1,j-d_2\right)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{M}_{21}^{\mathrm{T}}\\\boldsymbol{M}_{22}^{\mathrm{T}}\end{bmatrix}\boldsymbol{S}_2^{-1}\begin{bmatrix}\boldsymbol{M}_{21}&\boldsymbol{M}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\left(i+1,j\right)\\\boldsymbol{x}\left(i+1,j-d_2\right)\end{bmatrix}, \tag{17}$$

and

$$-\sum_{m=-k_1}^{-1}\sum_{n=-k_2}^{-1}\left(\boldsymbol{q}^{\mathrm{T}}\left(i+m,j+n\right)\boldsymbol{S}_3\boldsymbol{q}\left(i+m,j+n\right)\right)\leq\begin{bmatrix}\boldsymbol{x}\left(i,j\right)\\\boldsymbol{x}\left(i-k_1,j-k_2\right)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{M}_{31}^{\mathrm{T}}+\boldsymbol{M}_{31}&-\boldsymbol{M}_{31}^{\mathrm{T}}+\boldsymbol{M}_{32}\\ *&-\boldsymbol{M}_{32}^{\mathrm{T}}-\boldsymbol{M}_{32}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\left(i,j\right)\\\boldsymbol{x}\left(i-k_1,j-k_2\right)\end{bmatrix}$$

$$+k_1k_2\begin{bmatrix}\boldsymbol{x}\left(i,j\right)\\\boldsymbol{x}\left(i-k_1,j-k_2\right)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{M}_{31}^{\mathrm{T}}\\\boldsymbol{M}_{32}^{\mathrm{T}}\end{bmatrix}\boldsymbol{S}_3^{-1}\begin{bmatrix}\boldsymbol{M}_{31}&\boldsymbol{M}_{32}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\left(i,j\right)\\\boldsymbol{x}\left(i-k_1,j-k_2\right)\end{bmatrix}, \tag{18}$$

where $\boldsymbol{M}_{11},\boldsymbol{M}_{12},\boldsymbol{M}_{21},\boldsymbol{M}_{22},\boldsymbol{M}_{31},\boldsymbol{M}_{32}\in R^{n\times n}$.

Now, substituting (15)-(18) in (14) yields

$$\Delta V\left(i+1,j+1\right)\leq\begin{bmatrix}\boldsymbol{x}\left(i,j+1\right)\\\boldsymbol{x}\left(i-d_1,j+1\right)\\\boldsymbol{x}\left(i+1,j\right)\\\boldsymbol{x}\left(i+1,j-d_2\right)\\\boldsymbol{x}\left(i,j\right)\\\boldsymbol{x}\left(i-k_1,j-k_2\right)\end{bmatrix}^{\mathrm{T}}\left(\begin{bmatrix}\boldsymbol{\beta}_1&\boldsymbol{M}_{12}-\boldsymbol{M}_{11}^{\mathrm{T}}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ *&-\boldsymbol{M}_{12}-\boldsymbol{M}_{12}^{\mathrm{T}}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ *&*&\boldsymbol{\beta}_2&\boldsymbol{M}_{22}-\boldsymbol{M}_{21}^{\mathrm{T}}&\boldsymbol{0}&\boldsymbol{0}\\ *&*&*&-\boldsymbol{M}_{22}^{\mathrm{T}}-\boldsymbol{M}_{22}&\boldsymbol{0}&\boldsymbol{0}\\ *&*&*&*&\boldsymbol{\beta}_3&\boldsymbol{M}_{32}-\boldsymbol{M}_{31}^{\mathrm{T}}\\ *&*&*&*&*&-\boldsymbol{M}_{32}^{T}-\boldsymbol{M}_{32}\end{bmatrix}\right.$$

$$+\begin{bmatrix}\overline{\boldsymbol{A}}_{1k}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{2k}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{0k}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}}\end{bmatrix}\boldsymbol{P}\begin{bmatrix}\overline{\boldsymbol{A}}_{1k}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{2k}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{0k}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}+\begin{bmatrix}\overline{\boldsymbol{A}}_{1k}^{\mathrm{T}}-\boldsymbol{I}&\overline{\boldsymbol{A}}_{1k}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{1k}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{2k}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{2k}^{\mathrm{T}}-\boldsymbol{I}&\overline{\boldsymbol{A}}_{2k}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{0k}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{0k}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{0k}^{\mathrm{T}}-\boldsymbol{I}\\\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}d_1\boldsymbol{S}_1&\boldsymbol{0}&\boldsymbol{0}\\\boldsymbol{0}&d_2\boldsymbol{S}_2&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{0}&k_1k_2\boldsymbol{S}_3\end{bmatrix}\tag{19a}$$

$$\times\begin{bmatrix}\overline{\boldsymbol{A}}_{1k}^{\mathrm{T}}-\boldsymbol{I}&\overline{\boldsymbol{A}}_{1k}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{1k}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{2k}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{2k}^{\mathrm{T}}-\boldsymbol{I}&\overline{\boldsymbol{A}}_{2k}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}}\\\overline{\boldsymbol{A}}_{0k}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{0k}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{0k}^{\mathrm{T}}-\boldsymbol{I}\\\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}}&\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}+\begin{bmatrix}\boldsymbol{M}_{11}^{\mathrm{T}}&\boldsymbol{0}&\boldsymbol{0}\\\boldsymbol{M}_{12}^{\mathrm{T}}&\boldsymbol{0}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{M}_{21}^{\mathrm{T}}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{M}_{22}^{\mathrm{T}}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{M}_{31}^{\mathrm{T}}\\\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{M}_{32}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}d_1\boldsymbol{S}_1^{-1}&\boldsymbol{0}&\boldsymbol{0}\\\boldsymbol{0}&d_2\boldsymbol{S}_2^{-1}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{0}&k_1k_2\boldsymbol{S}_3^{-1}\end{bmatrix}$$

$$\times \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{21} & \boldsymbol{M}_{22} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{31} & \boldsymbol{M}_{32} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(i, j+1) \\ \boldsymbol{x}(i-d_1, j+1) \\ \boldsymbol{x}(i+1, j) \\ \boldsymbol{x}(i+1, j-d_2) \\ \boldsymbol{x}(i, j) \\ \boldsymbol{x}(i-k_1, j-k_2) \end{bmatrix}. \tag{19a}$$

Applying Schur complement, it follows from matrix inequality (12) that

$$\begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{M}_{12} - \boldsymbol{M}_{11}^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & -\boldsymbol{M}_{12} - \boldsymbol{M}_{12}^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & \boldsymbol{\beta}_2 & \boldsymbol{M}_{22} - \boldsymbol{M}_{21}^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & -\boldsymbol{M}_{22}^{\mathrm{T}} - \boldsymbol{M}_{22} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & * & \boldsymbol{\beta}_3 & \boldsymbol{M}_{32} - \boldsymbol{M}_{31}^{\mathrm{T}} \\ * & * & * & * & * & -\boldsymbol{M}_{32}^{T} - \boldsymbol{M}_{32} \end{bmatrix} + \begin{bmatrix} \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} \end{bmatrix} \boldsymbol{P} \begin{bmatrix} \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

$$+ \begin{bmatrix} \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}} - \boldsymbol{I} & \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}} - \boldsymbol{I} & \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}} - \boldsymbol{I} \\ \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} d_1 \boldsymbol{S}_1 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & d_2 \boldsymbol{S}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & k_1 k_2 \boldsymbol{S}_3 \end{bmatrix} \begin{bmatrix} \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}} - \boldsymbol{I} & \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{1k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}} - \boldsymbol{I} & \overline{\boldsymbol{A}}_{2k}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} \\ \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0k}^{\mathrm{T}} - \boldsymbol{I} \\ \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} & \overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tag{19b}$$

$$+ \begin{bmatrix} \boldsymbol{M}_{11}^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{M}_{12}^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_{21}^{\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_{22}^{\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{31}^{\mathrm{T}} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{32}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} d_1 \boldsymbol{S}_1^{-1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & d_2 \boldsymbol{S}_2^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & k_1 k_2 \boldsymbol{S}_3^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{21} & \boldsymbol{M}_{22} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{31} & \boldsymbol{M}_{32} \end{bmatrix} < \boldsymbol{0}.$$

Thus, from (19b), it implies that

$$\Delta V(i+1, j+1) < 0, \tag{20}$$

holds for any delays d_1, d_2, k_1 and k_2 satisfying $0 < d_1 \leq d_1^*$, $0 < d_2 \leq d_2^*$, $0 < k_1 \leq k_1^*$ and $0 < k_2 \leq k_2^*$.

Hence, the closed-loop system (4) with $\boldsymbol{w}(i, j) = \boldsymbol{0}$ is asymptotically stable.

Next, we establish the H_∞ performance measure of the closed-loop system (4) for $\boldsymbol{w}(i, j) \in \ell_2 \{[0, \infty), [0, \infty)\}$.

We consider

$$\Delta V(i+1,j+1)+\begin{bmatrix} z(i,j+1) \\ z(i+1,j) \\ z(i,j) \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} z(i,j+1) \\ z(i+1,j) \\ z(i,j) \end{bmatrix}\gamma^2\begin{bmatrix} w(i,j+1) \\ w(i+1,j) \\ w(i,j) \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} w(i,j+1) \\ w(i+1,j) \\ w(i,j) \end{bmatrix}=\begin{bmatrix} x(i,j+1) \\ x(i-d_1,j+1) \\ x(i+1,j) \\ x(i+1,j-d_2) \\ x(i,j) \\ x(i-k_1,j-k_2) \\ w(i,j+1) \\ w(i+1,j) \\ w(i,j) \end{bmatrix}\left(\begin{bmatrix} \overline{A}_{1k}^{\mathrm{T}} \\ \overline{A}_{1d}^{\mathrm{T}} \\ \overline{A}_{2k}^{\mathrm{T}} \\ \overline{A}_{2d}^{\mathrm{T}} \\ \overline{A}_{0k}^{\mathrm{T}} \\ \overline{A}_{0d}^{\mathrm{T}} \\ \overline{B}_{1}^{\mathrm{T}} \\ \overline{B}_{2}^{\mathrm{T}} \\ \overline{B}_{0}^{\mathrm{T}} \end{bmatrix}P\begin{bmatrix} \overline{A}_{1k}^{\mathrm{T}} \\ \overline{A}_{1d}^{\mathrm{T}} \\ \overline{A}_{2k}^{\mathrm{T}} \\ \overline{A}_{2d}^{\mathrm{T}} \\ \overline{A}_{0k}^{\mathrm{T}} \\ \overline{A}_{0d}^{\mathrm{T}} \\ \overline{B}_{1}^{\mathrm{T}} \\ \overline{B}_{2}^{\mathrm{T}} \\ \overline{B}_{0}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}\right.$$

$$+\begin{bmatrix} \pi_1 & M_{12}-M_{11}^{\mathrm{T}} & 0 & 0 & 0 & 0 & H^{\mathrm{T}}L & 0 & 0 \\ 0 & -M_{12}-M_{12}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi_2 & M_{22}-M_{21}^{\mathrm{T}} & 0 & 0 & 0 & H^{\mathrm{T}}L & 0 \\ 0 & 0 & 0 & -M_{22}^{\mathrm{T}}-M_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi_3 & M_{32}-M_{31}^{\mathrm{T}} & 0 & 0 & H^{\mathrm{T}}L \\ 0 & 0 & 0 & 0 & 0 & -M_{32}^{\mathrm{T}}-M_{32} & 0 & 0 & 0 \\ L^{\mathrm{T}}H & 0 & 0 & 0 & 0 & 0 & L^{\mathrm{T}}L-\gamma^2 I & 0 & 0 \\ 0 & 0 & L^{\mathrm{T}}H & 0 & 0 & 0 & 0 & L^{\mathrm{T}}L-\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & L^{\mathrm{T}}H & 0 & 0 & 0 & L^{\mathrm{T}}L-\gamma^2 I \end{bmatrix}$$

$$+\begin{bmatrix} \overline{A}_{1k}^{\mathrm{T}}-I & \overline{A}_{1k}^{\mathrm{T}} & \overline{A}_{1k}^{\mathrm{T}} \\ \overline{A}_{1d}^{\mathrm{T}} & \overline{A}_{1d}^{\mathrm{T}} & \overline{A}_{1d}^{\mathrm{T}} \\ \overline{A}_{2k}^{\mathrm{T}} & \overline{A}_{2k}^{\mathrm{T}}-I & \overline{A}_{2k}^{\mathrm{T}} \\ \overline{A}_{2d}^{\mathrm{T}} & \overline{A}_{2d}^{\mathrm{T}} & \overline{A}_{2d}^{\mathrm{T}} \\ \overline{A}_{0k}^{\mathrm{T}} & \overline{A}_{0k}^{\mathrm{T}} & \overline{A}_{0k}^{\mathrm{T}}-I \\ \overline{A}_{0d}^{\mathrm{T}} & \overline{A}_{0d}^{\mathrm{T}} & \overline{A}_{0d}^{\mathrm{T}} \\ \overline{B}_{1}^{\mathrm{T}} & \overline{B}_{1}^{\mathrm{T}} & \overline{B}_{1}^{\mathrm{T}} \\ \overline{B}_{2}^{\mathrm{T}} & \overline{B}_{2}^{\mathrm{T}} & \overline{B}_{2}^{\mathrm{T}} \\ \overline{B}_{0}^{\mathrm{T}} & \overline{B}_{0}^{\mathrm{T}} & \overline{B}_{0}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} d_1 S_1 & 0 & 0 \\ 0 & d_2 S_2 & 0 \\ 0 & 0 & k_1 k_2 S_3 \end{bmatrix}\begin{bmatrix} \overline{A}_{1k}^{\mathrm{T}}-I & \overline{A}_{1k}^{\mathrm{T}} & \overline{A}_{1k}^{\mathrm{T}} \\ \overline{A}_{1d}^{\mathrm{T}} & \overline{A}_{1d}^{\mathrm{T}} & \overline{A}_{1d}^{\mathrm{T}} \\ \overline{A}_{2k}^{\mathrm{T}} & \overline{A}_{2k}^{\mathrm{T}}-I & \overline{A}_{2k}^{\mathrm{T}} \\ \overline{A}_{2d}^{\mathrm{T}} & \overline{A}_{2d}^{\mathrm{T}} & \overline{A}_{2d}^{\mathrm{T}} \\ \overline{A}_{0k}^{\mathrm{T}} & \overline{A}_{0k}^{\mathrm{T}} & \overline{A}_{0k}^{\mathrm{T}}-I \\ \overline{A}_{0d}^{\mathrm{T}} & \overline{A}_{0d}^{\mathrm{T}} & \overline{A}_{0d}^{\mathrm{T}} \\ \overline{B}_{1}^{\mathrm{T}} & \overline{B}_{1}^{\mathrm{T}} & \overline{B}_{1}^{\mathrm{T}} \\ \overline{B}_{2}^{\mathrm{T}} & \overline{B}_{2}^{\mathrm{T}} & \overline{B}_{2}^{\mathrm{T}} \\ \overline{B}_{0}^{\mathrm{T}} & \overline{B}_{0}^{\mathrm{T}} & \overline{B}_{0}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

$$+\begin{bmatrix} M_{11}^{\mathrm{T}} & 0 & 0 \\ M_{12}^{\mathrm{T}} & 0 & 0 \\ 0 & M_{21}^{\mathrm{T}} & 0 \\ 0 & M_{22}^{\mathrm{T}} & 0 \\ 0 & 0 & M_{31}^{\mathrm{T}} \\ 0 & 0 & M_{32}^{\mathrm{T}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} d_1 S_1^{-1} & 0 & 0 \\ 0 & d_2 S_2^{-1} & 0 \\ 0 & 0 & k_1 k_2 S_3^{-1} \end{bmatrix}\begin{bmatrix} M_{11}^{\mathrm{T}} & 0 & 0 \\ M_{12}^{\mathrm{T}} & 0 & 0 \\ 0 & M_{21}^{\mathrm{T}} & 0 \\ 0 & M_{22}^{\mathrm{T}} & 0 \\ 0 & 0 & M_{31}^{\mathrm{T}} \\ 0 & 0 & M_{32}^{\mathrm{T}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}\right)\begin{bmatrix} x(i,j+1) \\ x(i-d_1,j+1) \\ x(i+1,j) \\ x(i+1,j-d_2) \\ x(i,j) \\ x(i-k_1,j-k_2) \\ w(i,j+1) \\ w(i+1,j) \\ w(i,j) \end{bmatrix},$$

(21)

where

$$\pi_1 = \left(-P_1 + H^{\mathrm{T}}H + M_{11}^{\mathrm{T}} + M_{11}\right),$$

$$\pi_2 = \left(-P_2 + H^{\mathrm{T}}H + M_{21}^{\mathrm{T}} + M_{21}\right),$$

$$\boldsymbol{\pi}_3 = \left(-\boldsymbol{P} + \boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{H}^{\mathrm{T}}\boldsymbol{H} + \boldsymbol{M}_{31}^{\mathrm{T}} + \boldsymbol{M}_{31}\right).$$

It follows from inequality (12) that

$$\Delta V(i+1,j+1) + \begin{bmatrix} z(i,j+1) \\ z(i+1,j) \\ z(i,j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} z(i,j+1) \\ z(i+1,j) \\ z(i,j) \end{bmatrix} - \gamma^2 \begin{bmatrix} w(i,j+1) \\ w(i+1,j) \\ w(i,j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} w(i,j+1) \\ w(i+1,j) \\ w(i,j) \end{bmatrix} < 0. \tag{22}$$

Summing the inequality (22) over $i, j = 0 \to \infty$ gives

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Delta V(i+1,j+1) + \|\bar{z}\|_2^2 - \gamma^2 \|\bar{w}\|_2^2 < 0. \tag{23}$$

Now,

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Delta V(i+1,j+1)$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Big[V_1\big(x(i+1,j+1)\big) + V_2\big(x(i+1,j+1)\big) + V_3\big(x(i+1,j+1)\big) \tag{24}$$

$$- V_1\big(x(i,j+1)\big) - V_2\big(x(i+1,j)\big) - V_3\big(x(i,j)\big)\Big]$$

$$= \sum_{j=0}^{\infty}\left[-\left(x^{\mathrm{T}}(0,j+1)P_1 x(0,j+1) + \sum_{\theta=-d_1^*+1}^{0}\sum_{l=\theta-1}^{-1} s^{\mathrm{T}}(l,j+1)S_1 s(l,j+1) \right) \right]$$

$$+ \sum_{i=0}^{\infty}\left[-\left(x^{\mathrm{T}}(i+1,0)P_2 x(i+1,0) + \sum_{\theta=-d_2^*+1}^{0}\sum_{l=\theta-1}^{-1} t^{\mathrm{T}}(i+1,l)S_2 t(i+1,l) \right) \right] \tag{25}$$

$$+ \sum_{i=0}^{\infty}\left[-\left(x^{\mathrm{T}}(i,0)(P-P_1-P_2)x(i,0) + \sum_{\varphi_1=-k_1^*+1}^{0}\sum_{m=\varphi_1-1}^{-1}\sum_{\varphi_2=-k_2^*+1}^{0}\sum_{n=\varphi_2-1}^{-1} q^{\mathrm{T}}(i+m,n)S_3 q(i+m,n) \right) \right]$$

$$+ \sum_{j=0}^{\infty}\left[-\left(x^{\mathrm{T}}(0,j)(P-P_1-P_2)x(0,j) + \sum_{\varphi_1=-k_1^*+1}^{0}\sum_{m=\varphi_1-1}^{-1}\sum_{\varphi_2=-k_2^*+1}^{0}\sum_{n=\varphi_2-1}^{-1} q^{\mathrm{T}}(m,j+n)S_3 q(m,j+n) \right) \right].$$

Thus, by using (25) in (23), we get

$$\|\bar{z}\|_2^2 - \gamma^2 \|\bar{w}\|_2^2 < -\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Delta V(i+1,j+1)$$

$$= \sum_{j=0}^{\infty}\left[\left(x^{\mathrm{T}}(0,j+1)P_1 x(0,j+1) + \sum_{\theta=d_1^*+1}^{0}\sum_{l=\theta-1}^{-1} s^{\mathrm{T}}(l,j+1)S_1 s(l,j+1) \right) \right]$$

$$+ \sum_{i=0}^{\infty}\left[\left(x^{\mathrm{T}}(i+1,0)P_2 x(i+1,0) + \sum_{\theta=d_2^*+1}^{0}\sum_{l=\theta-1}^{-1} t^{\mathrm{T}}(i+1,l)S_2 t(i+1,l) \right) \right] \tag{26}$$

$$+ \sum_{i=0}^{\infty}\left[\left(x^{\mathrm{T}}(i,0)(P-P_1-P_2)x(i,0) + \sum_{\varphi_1=-k_1^*+1}^{0}\sum_{m=\varphi_1-1}^{-1}\sum_{\varphi_2=-k_2^*+1}^{0}\sum_{n=\varphi_2-1}^{-1} q^{\mathrm{T}}(i+m,n)S_3 q(i+m,n) \right) \right]$$

$$+ \sum_{j=0}^{\infty}\left[\left(x^{\mathrm{T}}(0,j)(P-P_1-P_2)x(0,j) + \sum_{\varphi_1=-k_1^*+1}^{0}\sum_{m=\varphi_1-1}^{-1}\sum_{\varphi_2=-k_2^*+1}^{0}\sum_{n=\varphi_2-1}^{-1} q^{\mathrm{T}}(m,j+n)S_3 q(m,j+n) \right) \right].$$

Since, $P_1 < \gamma^2 Q_1$, $P_2 < \gamma^2 Q_2$, $0 < P - P_1 - P_2 < \gamma^2 Q_3$, $S_1 < \gamma^2 Z_1$, $S_2 < \gamma^2 Z_2$ and $S_3 < \gamma^2 Z_3$, therefore, it follows from Definition 2 that the closed-loop system (4) guarantees the H_∞ noise attenuation γ. This completes the proof of Theorem 1.

In the following, we will show that the above derived sufficient condition for existence of delay-dependent robust H_∞ controllers can be transformed to the solvability of an LMI.

Theorem 2. Consider the closed-loop system (4) with the initial condition (2). Given scalars $\gamma > 0$, $\varepsilon > 0$ and integers $d_1^* > 0$, $d_2^* > 0$, $k_1^* > 0$ and $k_2^* > 0$, if there exist matrices $N_{11}, N_{12}, N_{21}, N_{22}, N_{31}, N_{32} \in R^{n \times n}$, $N \in R^{m \times n}$ and symmetric positive definite matrices $\bar{P}, \bar{P}_1, \bar{P}_2, \bar{S}_1, \bar{S}_2, \bar{S}_3 \in R^{n \times n}$ such that

$$
\begin{bmatrix}
\psi_1 & \psi_2^{\mathrm{T}} & \psi_3^{\mathrm{T}} & \psi_4^{\mathrm{T}} & \psi_5^{\mathrm{T}} & d_1^* \psi_6^{\mathrm{T}} & d_2^* \psi_7^{\mathrm{T}} & k_1^* k_2^* \psi_8^{\mathrm{T}} & d_1^* \psi_9^{\mathrm{T}} & d_2^* \psi_{10}^{\mathrm{T}} & k_1^* k_2^* \psi_{11}^{\mathrm{T}} & \psi_{12}^{\mathrm{T}} & 0 \\
* & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\bar{P} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\varepsilon}D^{\mathrm{T}} \\
* & * & * & * & * & -d_1^* \bar{S}_1 & 0 & 0 & 0 & 0 & 0 & 0 & d_1^* \sqrt{\varepsilon}D^{\mathrm{T}} \\
* & * & * & * & * & * & -d_2^* \bar{S}_2 & 0 & 0 & 0 & 0 & 0 & d_2^* \sqrt{\varepsilon}D^{\mathrm{T}} \\
* & * & * & * & * & * & * & -k_1^* k_2^* \bar{S}_3 & 0 & 0 & 0 & 0 & k_1^* k_2^* \sqrt{\varepsilon}D^{\mathrm{T}} \\
* & * & * & * & * & * & * & * & -d_1^* \bar{S}_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -d_2^* \bar{S}_2 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -k_1^* k_2^* \bar{S}_3 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -\varepsilon I & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -I
\end{bmatrix} < 0,
\qquad (27)
$$

where

$$
\psi_1 = \begin{bmatrix}
-\bar{P}_1 & \bar{P} + N_{11}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & N_{12} - N_{12}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -\bar{P}_2 & \bar{P} + N_{21}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & N_{22} - N_{22}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \bar{P}_1 + \bar{P}_2 - \bar{P} & \bar{P} + N_{31}^{\mathrm{T}} & 0 & 0 & 0 \\
* & * & * & * & * & N_{32} - N_{32}^{T} & 0 & 0 & 0 \\
* & * & * & * & * & * & -\gamma^2 I & 0 & 0 \\
* & * & * & * & * & * & * & -\gamma^2 I & 0 \\
* & * & * & * & * & * & * & * & -\gamma^2 I
\end{bmatrix},
$$

$$
\psi_2 = \begin{bmatrix} H\bar{P} & 0 & 0 & 0 & 0 & 0 & L & 0 & 0 \end{bmatrix},
$$

$$
\psi_3 = \begin{bmatrix} 0 & 0 & H\bar{P} & 0 & 0 & 0 & 0 & L & 0 \end{bmatrix},
$$

$$\psi_4 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{H}\overline{\boldsymbol{P}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{L} \end{bmatrix},$$

$$\psi_5 = \begin{bmatrix} \boldsymbol{A}_1\overline{\boldsymbol{P}} + \boldsymbol{C}_1\boldsymbol{N} - \boldsymbol{A}_{1d}\boldsymbol{N}_{11} & \boldsymbol{A}_{1d}\boldsymbol{N}_{12} & \boldsymbol{A}_2\overline{\boldsymbol{P}} + \boldsymbol{C}_2\boldsymbol{N} - \boldsymbol{A}_{2d}\boldsymbol{N}_{21} & \boldsymbol{A}_{2d}\boldsymbol{N}_{22} \\ \boldsymbol{A}_0\overline{\boldsymbol{P}} + \boldsymbol{C}_0\boldsymbol{N} - \boldsymbol{A}_{0d}\boldsymbol{N}_{31} & \boldsymbol{A}_{0d}\boldsymbol{N}_{32} & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_0 \end{bmatrix},$$

$$\psi_6 = \begin{bmatrix} \boldsymbol{A}_1\overline{\boldsymbol{P}} + \boldsymbol{C}_1\boldsymbol{N} - \boldsymbol{A}_{1d}\boldsymbol{N}_{11} - \overline{\boldsymbol{P}} & \boldsymbol{A}_{1d}\boldsymbol{N}_{12} & \boldsymbol{A}_2\overline{\boldsymbol{P}} + \boldsymbol{C}_2\boldsymbol{N} - \boldsymbol{A}_{2d}\boldsymbol{N}_{21} & \boldsymbol{A}_{2d}\boldsymbol{N}_{22} \\ \boldsymbol{A}_0\overline{\boldsymbol{P}} + \boldsymbol{C}_0\boldsymbol{N} - \boldsymbol{A}_{0d}\boldsymbol{N}_{31} & \boldsymbol{A}_{0d}\boldsymbol{N}_{32} & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_0 \end{bmatrix},$$

$$\psi_7 = \begin{bmatrix} \boldsymbol{A}_1\overline{\boldsymbol{P}} + \boldsymbol{C}_1\boldsymbol{N} - \boldsymbol{A}_{1d}\boldsymbol{N}_{11} & \boldsymbol{A}_{1d}\boldsymbol{N}_{12} & \boldsymbol{A}_2\overline{\boldsymbol{P}} + \boldsymbol{C}_2\boldsymbol{N} - \boldsymbol{A}_{2d}\boldsymbol{N}_{21} - \overline{\boldsymbol{P}} & \boldsymbol{A}_{2d}\boldsymbol{N}_{22} \\ \boldsymbol{A}_0\overline{\boldsymbol{P}} + \boldsymbol{C}_0\boldsymbol{N} - \boldsymbol{A}_{0d}\boldsymbol{N}_{31} & \boldsymbol{A}_{0d}\boldsymbol{N}_{32} & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_0 \end{bmatrix},$$

$$\psi_8 = \begin{bmatrix} \boldsymbol{A}_1\overline{\boldsymbol{P}} + \boldsymbol{C}_1\boldsymbol{N} - \boldsymbol{A}_{1d}\boldsymbol{N}_{11} & \boldsymbol{A}_{1d}\boldsymbol{N}_{12} & \boldsymbol{A}_2\overline{\boldsymbol{P}} + \boldsymbol{C}_2\boldsymbol{N} - \boldsymbol{A}_{2d}\boldsymbol{N}_{21} & \boldsymbol{A}_{2d}\boldsymbol{N}_{22} \\ \boldsymbol{A}_0\overline{\boldsymbol{P}} + \boldsymbol{C}_0\boldsymbol{N} - \boldsymbol{A}_{0d}\boldsymbol{N}_{31} - \overline{\boldsymbol{P}} & \boldsymbol{A}_{0d}\boldsymbol{N}_{32} & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_0 \end{bmatrix},$$

$$\psi_9 = \begin{bmatrix} \mathbf{0} & \overline{\boldsymbol{S}}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\psi_{10} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{S}}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\psi_{11} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{S}}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\psi_{12} = \begin{bmatrix} \left(\boldsymbol{E}_1\overline{\boldsymbol{P}} + \boldsymbol{E}_{c1}\boldsymbol{N} - \boldsymbol{E}_{1d}\boldsymbol{N}_{11}\right) & \boldsymbol{E}_{1d}\boldsymbol{N}_{12} & \left(\boldsymbol{E}_2\overline{\boldsymbol{P}} + \boldsymbol{E}_{c2}\boldsymbol{N} - \boldsymbol{E}_{2d}\boldsymbol{N}_{21}\right) & \boldsymbol{E}_{2d}\boldsymbol{N}_{22} \\ \left(\boldsymbol{E}_0\overline{\boldsymbol{P}} + \boldsymbol{E}_{c0}\boldsymbol{N} - \boldsymbol{E}_{0d}\boldsymbol{N}_{31}\right) & \boldsymbol{E}_{0d}\boldsymbol{N}_{32} & \boldsymbol{E}_{b1} & \boldsymbol{E}_{b2} & \boldsymbol{E}_{b0} \end{bmatrix},$$

then the closed-loop system (4) with $0 < d_1 \le d_1^*$, $0 < d_2 \le d_2^*$, $0 < k_1 \le k_1^*$ and $0 < k_2 \le k_2^*$ has a specified H_∞ noise attenuation γ and the controller (3) with $\boldsymbol{K} = \boldsymbol{N}\overline{\boldsymbol{P}}^{-1}$ is a γ-suboptimal state feedback robust H_∞ controller for the uncertain 2-D discrete state delay system (1).

Proof. It follows from matrix inequality (12) that $\boldsymbol{M}_{12}$, $\boldsymbol{M}_{22}$ and $\boldsymbol{M}_{32}$ are reversible. So, let us assume that

$$\mu_1 = \begin{bmatrix} \boldsymbol{P} & \mathbf{0} \\ \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{P}^{-1} & \mathbf{0} \\ -\boldsymbol{M}_{12}^{-1}\boldsymbol{M}_{11}\boldsymbol{P}^{-1} & \boldsymbol{M}_{12}^{-1} \end{bmatrix}, \tag{28a}$$

$$\mu_2 = \begin{bmatrix} \boldsymbol{P} & \mathbf{0} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{P}^{-1} & \mathbf{0} \\ -\boldsymbol{M}_{22}^{-1}\boldsymbol{M}_{21}\boldsymbol{P}^{-1} & \boldsymbol{M}_{22}^{-1} \end{bmatrix}, \tag{28b}$$

$$\mu_3 = \begin{bmatrix} \boldsymbol{P} & \mathbf{0} \\ \boldsymbol{M}_{31} & \boldsymbol{M}_{32} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{P}^{-1} & \mathbf{0} \\ -\boldsymbol{M}_{32}^{-1}\boldsymbol{M}_{31}\boldsymbol{P}^{-1} & \boldsymbol{M}_{32}^{-1} \end{bmatrix}. \tag{28c}$$

Pre-multiplying and post-multiplying both sides of matrix inequality (12) by

$$\text{diag}\left\{\mu_1^{\mathrm{T}}, \mu_2^{\mathrm{T}}, \mu_3^{\mathrm{T}}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \text{diag}\left\{d_1^*\boldsymbol{I}, d_2^*\boldsymbol{I}, k_1^*k_2^*\boldsymbol{I}\right\}, \text{diag}\left\{d_1^*\boldsymbol{S}_1^{-1}, d_2^*\boldsymbol{S}_2^{-1}, k_1^*k_2^*\boldsymbol{S}_3^{-1}\right\}\right\}$$

and its transpose, respectively, we get

$$
\begin{bmatrix}
\left(-P^{-1}P_1P^{-1}\right) & \left(P^{-1}+\left(M_{12}^{-1}M_{11}P^{-1}\right)^{\mathrm{T}}\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P^{-1}H^{\mathrm{T}} & 0 & 0 \\
* & -\left(\left(M_{12}^{-1}\right)^{\mathrm{T}}-M_{12}^{-1}\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \left(-P^{-1}P_2P^{-1}\right) & \left(P^{-1}+\left(M_{22}^{-1}M_{21}P^{-1}\right)^{\mathrm{T}}\right) & 0 & 0 & 0 & 0 & 0 & 0 & P^{-1}H^{\mathrm{T}} & 0 \\
* & * & * & -\left(\left(M_{22}^{-1}\right)^{\mathrm{T}}-M_{22}^{-1}\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \left(-P^{-1}\left(P-P_1-P_2\right)P^{-1}\right) & \left(P^{-1}+\left(M_{32}^{-1}M_{31}P^{-1}\right)^{\mathrm{T}}\right) & 0 & 0 & 0 & 0 & 0 & P^{-1}H^{\mathrm{T}} \\
* & * & * & * & * & -\left(\left(M_{32}^{-1}\right)^{\mathrm{T}}-M_{32}^{-1}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\gamma^2 I & 0 & 0 & L^{\mathrm{T}} & 0 & 0 \\
* & * & * & * & * & * & * & -\gamma^2 I & 0 & 0 & L^{\mathrm{T}} & 0 \\
* & * & * & * & * & * & * & * & -\gamma^2 I & 0 & 0 & L^{\mathrm{T}} \\
* & * & * & * & * & * & * & * & * & -I & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -I & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -I \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & *
\end{bmatrix}
$$

$$
\begin{aligned}
&\left(\boldsymbol{P}^{-1}\overline{\boldsymbol{A}}_{1k}^{\mathrm{T}}-\left(\boldsymbol{M}_{12}^{-1}\boldsymbol{M}_{11}\boldsymbol{P}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}}\right) \quad d_1^*\left(\boldsymbol{P}^{-1}\overline{\boldsymbol{A}}_{1k}^{\mathrm{T}}-\boldsymbol{P}^{-1}-\left(\boldsymbol{M}_{12}^{-1}\boldsymbol{M}_{11}\boldsymbol{P}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}}\right) \quad d_2^*\left(\boldsymbol{P}^{-1}\overline{\boldsymbol{A}}_{1k}^{\mathrm{T}}-\left(\boldsymbol{M}_{12}^{-1}\boldsymbol{M}_{11}\boldsymbol{P}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}}\right)\\[4pt]
&\qquad\left(\boldsymbol{M}_{12}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} \qquad\qquad\qquad d_1^*\left(\boldsymbol{M}_{12}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} \qquad\qquad\qquad d_2^*\left(\boldsymbol{M}_{12}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}}\\[4pt]
&\left(\boldsymbol{P}^{-1}\overline{\boldsymbol{A}}_{2k}^{\mathrm{T}}-\left(\boldsymbol{M}_{22}^{-1}\boldsymbol{M}_{21}\boldsymbol{P}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}}\right) \quad d_1^*\left(\boldsymbol{P}^{-1}\overline{\boldsymbol{A}}_{2k}^{\mathrm{T}}-\left(\boldsymbol{M}_{22}^{-1}\boldsymbol{M}_{21}\boldsymbol{P}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}}\right) \quad d_2^*\left(\boldsymbol{P}^{-1}\overline{\boldsymbol{A}}_{2k}^{\mathrm{T}}-\boldsymbol{P}^{-1}-\left(\boldsymbol{M}_{22}^{-1}\boldsymbol{M}_{21}\boldsymbol{P}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}}\right)\\[4pt]
&\qquad\left(\boldsymbol{M}_{22}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} \qquad\qquad\qquad d_1^*\left(\boldsymbol{M}_{22}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} \qquad\qquad\qquad d_2^*\left(\boldsymbol{M}_{22}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}}\\[4pt]
&\left(\boldsymbol{P}^{-1}\overline{\boldsymbol{A}}_{0k}^{\mathrm{T}}-\left(\boldsymbol{M}_{32}^{-1}\boldsymbol{M}_{31}\boldsymbol{P}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}}\right) \quad d_1^*\left(\boldsymbol{P}^{-1}\overline{\boldsymbol{A}}_{0k}^{\mathrm{T}}-\left(\boldsymbol{M}_{32}^{-1}\boldsymbol{M}_{31}\boldsymbol{P}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}}\right) \quad d_2^*\left(\boldsymbol{P}^{-1}\overline{\boldsymbol{A}}_{0k}^{\mathrm{T}}-\left(\boldsymbol{M}_{32}^{-1}\boldsymbol{M}_{31}\boldsymbol{P}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}}\right)\\[4pt]
&\qquad\left(\boldsymbol{M}_{32}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} \qquad\qquad\qquad d_1^*\left(\boldsymbol{M}_{32}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} \qquad\qquad\qquad d_2^*\left(\boldsymbol{M}_{32}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}}\\[4pt]
&\qquad\overline{\boldsymbol{B}}_1^{\mathrm{T}} \qquad\qquad\qquad\qquad d_1^*\overline{\boldsymbol{B}}_1^{\mathrm{T}} \qquad\qquad\qquad\qquad d_2^*\overline{\boldsymbol{B}}_1^{\mathrm{T}}\\
&\qquad\overline{\boldsymbol{B}}_2^{\mathrm{T}} \qquad\qquad\qquad\qquad d_1^*\overline{\boldsymbol{B}}_2^{T} \qquad\qquad\qquad\qquad d_2^*\overline{\boldsymbol{B}}_2^{T}\\
&\qquad\overline{\boldsymbol{B}}_0^{\mathrm{T}} \qquad\qquad\qquad\qquad d_1^*\overline{\boldsymbol{B}}_0^{T} \qquad\qquad\qquad\qquad d_2^*\overline{\boldsymbol{B}}_0^{T}\\
&\qquad\mathbf{0} \qquad\qquad\qquad\qquad\quad \mathbf{0} \qquad\qquad\qquad\qquad\quad \mathbf{0}\\
&\qquad\mathbf{0} \qquad\qquad\qquad\qquad\quad \mathbf{0} \qquad\qquad\qquad\qquad\quad \mathbf{0}\\
&\qquad\mathbf{0} \qquad\qquad\qquad\qquad\quad \mathbf{0} \qquad\qquad\qquad\qquad\quad \mathbf{0}\\
&\quad-\boldsymbol{P}^{-1} \qquad\qquad\qquad\quad \mathbf{0} \qquad\qquad\qquad\qquad\quad \mathbf{0}\\
&\qquad* \qquad\qquad\qquad\quad -d_1^{*-1}\boldsymbol{S}_1^{-1} \qquad\qquad\qquad\quad \mathbf{0}\\
&\qquad* \qquad\qquad\qquad\qquad * \qquad\qquad\qquad\quad -d_2^{*-1}\boldsymbol{S}_2^{-1}\\
&\qquad* \qquad\qquad\qquad\qquad * \qquad\qquad\qquad\qquad\quad *\\
&\qquad* \qquad\qquad\qquad\qquad * \qquad\qquad\qquad\qquad\quad *\\
&\qquad* \qquad\qquad\qquad\qquad * \qquad\qquad\qquad\qquad\quad *\\
&\qquad* \qquad\qquad\qquad\qquad * \qquad\qquad\qquad\qquad\quad *
\end{aligned}
$$

$$
\begin{aligned}
&k_1^*k_2^*\left(\boldsymbol{P}^{-1}\overline{\boldsymbol{A}}_{1k}^{\mathrm{T}}-\left(\boldsymbol{M}_{12}^{-1}\boldsymbol{M}_{11}\boldsymbol{P}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}}\right) \qquad\mathbf{0}\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&\qquad k_1^*k_2^*\left(\boldsymbol{M}_{12}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{1d}^{\mathrm{T}} \qquad\qquad\qquad d_1^*\left(\boldsymbol{S}_1^{-1}\right)^{\mathrm{T}}\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&k_1^*k_2^*\left(\boldsymbol{P}^{-1}\overline{\boldsymbol{A}}_{2k}^{\mathrm{T}}-\left(\boldsymbol{M}_{22}^{-1}\boldsymbol{M}_{21}\boldsymbol{P}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}}\right) \qquad\mathbf{0}\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&\qquad k_1^*k_2^*\left(\boldsymbol{M}_{22}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{2d}^{\mathrm{T}} \qquad\qquad\qquad\mathbf{0}\qquad\quad d_2^*\left(\boldsymbol{S}_2^{-1}\right)^{\mathrm{T}}\qquad\mathbf{0}\\
&k_1^*k_2^*\left(\boldsymbol{P}^{-1}\overline{\boldsymbol{A}}_{0k}^{\mathrm{T}}-\boldsymbol{P}^{-1}-\left(\boldsymbol{M}_{32}^{-1}\boldsymbol{M}_{31}\boldsymbol{P}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}}\right) \qquad\mathbf{0}\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&\qquad k_1^*k_2^*\left(\boldsymbol{M}_{32}^{-1}\right)^{\mathrm{T}}\overline{\boldsymbol{A}}_{0d}^{\mathrm{T}} \qquad\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\qquad k_1^*k_2^*\left(\boldsymbol{S}_3^{-1}\right)^{\mathrm{T}}\\
&\qquad k_1^*k_2^*\overline{\boldsymbol{B}}_1^{\mathrm{T}} \qquad\qquad\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&\qquad k_1^*k_2^*\overline{\boldsymbol{B}}_2^{\mathrm{T}} \qquad\qquad\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&\qquad k_1^*k_2^*\overline{\boldsymbol{B}}_0^{\mathrm{T}} \qquad\qquad\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&\qquad\mathbf{0}\qquad\qquad\qquad\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&\qquad\mathbf{0}\qquad\qquad\qquad\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&\qquad\mathbf{0}\qquad\qquad\qquad\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&\qquad\mathbf{0}\qquad\qquad\qquad\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&\qquad\mathbf{0}\qquad\qquad\qquad\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&\qquad\mathbf{0}\qquad\qquad\qquad\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&-k_1^{*-1}k_2^{*-1}\boldsymbol{S}_3^{-1} \qquad\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&\qquad* \qquad\qquad\qquad\qquad -d_1^{*-1}\boldsymbol{S}_1^{-1}\qquad\mathbf{0}\qquad\qquad\mathbf{0}\\
&\qquad* \qquad\qquad\qquad\qquad\quad * \qquad\quad -d_2^{*-1}\boldsymbol{S}_2^{-1}\qquad\mathbf{0}\\
&\qquad* \qquad\qquad\qquad\qquad\quad * \qquad\qquad\quad * \qquad -k_1^{*-1}k_2^{*-1}\boldsymbol{S}_3^{-1}
\end{aligned} \;<\mathbf{0}. \tag{29}
$$

Denoting $\bar{P} = P^{-1}$, $\bar{P}_1 = P^{-1}P_1P^{-1}$, $\bar{P}_2 = P^{-1}P_2P^{-1}$, $S_1^{-1} = \bar{S}_1$, $S_2^{-1} = \bar{S}_2$, $S_3^{-1} = \bar{S}_3$, $N_{11} = M_{12}^{-1}M_{11}P^{-1}$, $N_{12} = M_{12}^{-1}$, $N_{21} = M_{22}^{-1}M_{21}P^{-1}$, $N_{22} = M_{22}^{-1}$, $N_{31} = M_{32}^{-1}M_{31}P^{-1}$, $N_{32} = M_{32}^{-1}$ and $N = K\bar{P}$, we can re-write (29) as

$$\begin{bmatrix}
\boldsymbol{\psi}_1 & \boldsymbol{\psi}_2^{\mathrm{T}} & \boldsymbol{\psi}_3^{\mathrm{T}} & \boldsymbol{\psi}_4^{\mathrm{T}} & \hat{\boldsymbol{\psi}}_5^{\mathrm{T}} & d_1^*\hat{\boldsymbol{\psi}}_6^{\mathrm{T}} & d_2^*\hat{\boldsymbol{\psi}}_7^{\mathrm{T}} & k_1^*k_2^*\hat{\boldsymbol{\psi}}_8^{\mathrm{T}} & d_1^*\boldsymbol{\psi}_9^{\mathrm{T}} & d_2^*\boldsymbol{\psi}_{10}^{\mathrm{T}} & k_1^*k_2^*\boldsymbol{\psi}_{11}^{\mathrm{T}} \\
* & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\bar{P} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -d_1^*\bar{S}_1 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -d_2^*\bar{S}_2 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -k_1^*k_2^*\bar{S}_3 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -d_1^*\bar{S}_1 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -d_2^*\bar{S}_2 & 0 \\
* & * & * & * & * & * & * & * & * & * & -k_1^*k_2^*\bar{S}_3
\end{bmatrix} < \mathbf{0}, \quad (30)$$

where

$$\hat{\boldsymbol{\psi}}_5 = \left[\bar{A}_1\bar{P} + \bar{C}_1N - \bar{A}_{1d}N_{11} \quad \bar{A}_{1d}N_{12} \quad \bar{A}_2\bar{P} + \bar{C}_2N - \bar{A}_{2d}N_{21} \quad \bar{A}_{2d}N_{22} \right.$$
$$\left. \bar{A}_0\bar{P} + \bar{C}_0N - \bar{A}_{0d}N_{31} \quad \bar{A}_{0d}N_{32} \quad \bar{B}_1 \quad \bar{B}_2 \quad \bar{B}_0 \right],$$

$$\hat{\boldsymbol{\psi}}_6 = \left[\bar{A}_1\bar{P} + \bar{C}_1N - \bar{A}_{1d}N_{11} - \bar{P} \quad \bar{A}_{1d}N_{12} \quad \bar{A}_2\bar{P} + \bar{C}_2N - \bar{A}_{2d}N_{21} \quad \bar{A}_{2d}N_{22} \right.$$
$$\left. \bar{A}_0\bar{P} + \bar{C}_0N - \bar{A}_{0d}N_{31} \quad \bar{A}_{0d}N_{32} \quad \bar{B}_1 \quad \bar{B}_2 \quad \bar{B}_0 \right],$$

$$\hat{\boldsymbol{\psi}}_7 = \left[\bar{A}_1\bar{P} + \bar{C}_1N - \bar{A}_{1d}N_{11} \quad \bar{A}_{1d}N_{12} \quad \bar{A}_2\bar{P} + \bar{C}_2N - \bar{A}_{2d}N_{21} - \bar{P} \quad \bar{A}_{2d}N_{22} \right.$$
$$\left. \bar{A}_0\bar{P} + \bar{C}_0N - \bar{A}_{0d}N_{31} \quad \bar{A}_{0d}N_{32} \quad \bar{B}_1 \quad \bar{B}_2 \quad \bar{B}_0 \right],$$

$$\hat{\boldsymbol{\psi}}_8 = \left[\bar{A}_1\bar{P} + \bar{C}_1N - \bar{A}_{1d}N_{11} \quad \bar{A}_{1d}N_{12} \quad \bar{A}_2\bar{P} + \bar{C}_2N - \bar{A}_{2d}N_{21} \quad \bar{A}_{2d}N_{22} \right.$$
$$\left. \bar{A}_0\bar{P} + \bar{C}_0N - \bar{A}_{0d}N_{31} - \bar{P} \quad \bar{A}_{0d}N_{32} \quad \bar{B}_1 \quad \bar{B}_2 \quad \bar{B}_0 \right].$$

Using (1d) and (1e), (30) can be expressed as

$$Y_2 + MFE_c + E_c^{\mathrm{T}}F^{\mathrm{T}}M^{\mathrm{T}} < \mathbf{0}, \tag{31}$$

where

$$Y_2 = \begin{bmatrix}
\boldsymbol{\psi}_1 & \boldsymbol{\psi}_2^{\mathrm{T}} & \boldsymbol{\psi}_3^{\mathrm{T}} & \boldsymbol{\psi}_4^{\mathrm{T}} & \boldsymbol{\psi}_5^{\mathrm{T}} & d_1^*\boldsymbol{\psi}_6^{\mathrm{T}} & d_2^*\boldsymbol{\psi}_7^{\mathrm{T}} & k_1^*k_2^*\boldsymbol{\psi}_8^{\mathrm{T}} & d_1^*\boldsymbol{\psi}_9^{\mathrm{T}} & d_2^*\boldsymbol{\psi}_{10}^{\mathrm{T}} & k_1^*k_2^*\boldsymbol{\psi}_{11}^{\mathrm{T}} \\
* & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\bar{P} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -d_1^*\bar{S}_1 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -d_2^*\bar{S}_2 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -k_1^*k_2^*\bar{S}_3 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -d_1^*\bar{S}_1 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -d_2^*\bar{S}_2 & 0 \\
* & * & * & * & * & * & * & * & * & * & -k_1^*k_2^*\bar{S}_3
\end{bmatrix},$$

$$M = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{D}^{\mathrm{T}} & d_1^*\boldsymbol{D}^{\mathrm{T}} & d_2^*\boldsymbol{D}^{\mathrm{T}} & k_1^*k_2^*\boldsymbol{D}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathrm{T}},$$

$$E_c = \Big[\big(\boldsymbol{E}_1\bar{\boldsymbol{P}} + \boldsymbol{E}_{c1}\boldsymbol{N} - \boldsymbol{E}_{1d}\boldsymbol{N}_{11} \big) \quad \boldsymbol{E}_{1d}\boldsymbol{N}_{12} \quad \big(\boldsymbol{E}_2\bar{\boldsymbol{P}} + \boldsymbol{E}_{c2}\boldsymbol{N} - \boldsymbol{E}_{2d}\boldsymbol{N}_{21} \big) \quad \boldsymbol{E}_{2d}\boldsymbol{N}_{22}$$

$$\big(\boldsymbol{E}_0\bar{\boldsymbol{P}} + \boldsymbol{E}_{c0}\boldsymbol{N} - \boldsymbol{E}_{0d}\boldsymbol{N}_{31} \big) \quad \boldsymbol{E}_{0d}\boldsymbol{N}_{32} \quad \boldsymbol{E}_{b1} \quad \boldsymbol{E}_{b2} \quad \boldsymbol{E}_{b0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \Big].$$

Therefore, using Lemma 2, (31) can be rearranged as

$$\left[\begin{array}{cccccc}
\boldsymbol{\psi}_1 + \varepsilon^{-1}\boldsymbol{\psi}_{12}^{\mathrm{T}}\boldsymbol{\psi}_{12} & \boldsymbol{\psi}_2^{\mathrm{T}} & \boldsymbol{\psi}_3^{\mathrm{T}} & \boldsymbol{\psi}_4^{\mathrm{T}} & \boldsymbol{\psi}_5^{\mathrm{T}} & d_1^*\boldsymbol{\psi}_6^{\mathrm{T}} \\
* & -\boldsymbol{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & -\boldsymbol{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & -\boldsymbol{I} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & \varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} - \bar{\boldsymbol{P}} & d_1^*\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} \\
* & * & * & * & d_1^*\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} & d_1^{*2}\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} - d_1^*\bar{\boldsymbol{S}}_1 \\
* & * & * & * & d_2^*\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} & d_1^*d_2^*\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} \\
* & * & * & * & \big(k_1^*k_2^*\big)\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} & \big(d_1^*k_1^*k_2^*\big)\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & *
\end{array} \right. \tag{32}$$

$$\left. \begin{array}{ccccc}
d_2^*\boldsymbol{\psi}_7^{\mathrm{T}} & k_1^*k_2^*\boldsymbol{\psi}_8^{\mathrm{T}} & \boldsymbol{\psi}_9^{\mathrm{T}} & \boldsymbol{\psi}_{10}^{\mathrm{T}} & \boldsymbol{\psi}_{11}^{\mathrm{T}} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
d_2^*\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} & \big(k_1^*k_2^*\big)\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
d_1^*d_2^*\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} & \big(d_1^*k_1^*k_2^*\big)\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
d_2^{*2}\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} - d_2^*\bar{\boldsymbol{S}}_2 & \big(d_2^*k_1^*k_2^*\big)\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\big(d_2^*k_1^*k_2^*\big)^2\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} & \big(k_1^*k_2^*\big)^2\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} - k_1^*k_2^*\bar{\boldsymbol{S}}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & -d_1^*\bar{\boldsymbol{S}}_1 & \mathbf{0} & \mathbf{0} \\
* & * & * & -d_2^*\bar{\boldsymbol{S}}_2 & \mathbf{0} \\
* & * & * & * & -k_1^*k_2^*\bar{\boldsymbol{S}}_3
\end{array} \right] < 0.$$

The equivalence of (32) and (27) follows trivially from Schur complements. This completes the proof of Theorem 2.

Theorem 2 provides a parameterized representation of a set of γ-suboptimal robust H_∞ state feedback controllers (if they exist) in terms of feasible solutions to the LMI (27). Based on Theorem 2, the design problem of a delay-dependent robust optimal H_∞ controller can be formulated as

$$\begin{aligned} & \text{minimize } \gamma^2 \\ & \text{s. t. } (27), \end{aligned} \tag{33}$$

which ensures the minimization of H_∞ noise attenuation γ of the closed-loop system (4).

4. An Illustrative Example

In this section, we present an application example to demonstrate the effectiveness of our proposed result.

Example 1. In this example, we shall illustrate the applicability of Theorem 2 to the control of thermal process in heat exchanger [4] [47] which can be expressed by the partial differential equation with time and space delays:

$$
\frac{\partial T(x,t)}{\partial x} = -\frac{\partial T(x,t)}{\partial t} - a_0 T(x,t) - a_1 T(x,t-\tau_1) - a_2 T(x-x_{d_1},t)
$$
$$
-a_3 T(x-x_{d_2},t-\tau_2) + bu(x,t), \tag{34}
$$

where $T(x,t)$ is the temperature at space $x \in [0, x_f]$ and time $t \in [0, \infty)$, $u(x,t)$ is input function, τ_1 and τ_2 are the time delays, x_{d_1} and x_{d_2} are the space delays, and a_0, a_1, a_2, a_3, b are the real coefficients. Taking

$$
T(i,j) = T(i\Delta x, j\Delta t), \quad u(i,j) = u(i\Delta x, j\Delta t), \tag{35a}
$$

$$
\frac{\partial T(x,t)}{\partial x} \approx \frac{T(i,j) - T(i-1,j)}{\Delta x}, \quad \frac{\partial T(x,t)}{\partial t} \approx \frac{T(i,j+1) - T(i,j)}{\Delta t}, \tag{35b}
$$

(34) can be expressed in the following form:

$$
T(i,j+1) = \left(1 - \frac{\Delta t}{\Delta x} - a_0 \Delta t\right) T(i,j) + \frac{\Delta t}{\Delta x} T(i-1,j) - a_1 \Delta t T(i,j-d_1)
$$
$$
-a_2 \Delta t T(i-d_2,j) - a_3 \Delta t T(i-k_1,j-k_2) + b\Delta t u(i,j), \tag{36}
$$

where $d_1 = \mathrm{int}(x_{d_1}/\Delta x)$, $d_2 = \mathrm{int}(\tau_1/\Delta t + 1)$, $k_1 = \mathrm{int}(x_{d_2}/\Delta x)$ and, $k_2 = \mathrm{int}(\tau_2/\Delta t + 1)$, $\mathrm{int}(.)$ is the integer function.

It is assumed that the surface of the heat exchanger is insulated and the heat flow through it is in steady state condition, then we could take the boundary conditions as $\frac{\partial T(x,t)}{\partial x} = 0$ and $\frac{\partial T(x,t)}{\partial t} = 0$, respectively.

Denote $x^{\mathrm{T}}(i,j) = \begin{bmatrix} T^{\mathrm{T}}(i-1,j) & T^{\mathrm{T}}(i,j) \end{bmatrix}$, it is easy to verify that (36) can be converted into the following discrete form:

$$
x(i+1,j+1) = \begin{bmatrix} 0 & 0 \\ \frac{\Delta t}{\Delta x} & 1 - \frac{\Delta t}{\Delta x} - a_0\Delta t \end{bmatrix} x(i+1,j) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(i,j+1) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x(i,j)
$$
$$
+ \begin{bmatrix} 0 & 0 \\ 0 & -a_1\Delta t \end{bmatrix} x(i+1,j-d_2) + \begin{bmatrix} 0 & 0 \\ 0 & -a_2\Delta t \end{bmatrix} x(i-d_1,j+1) \tag{37}
$$
$$
+ \begin{bmatrix} 0 & 0 \\ 0 & -a_3\Delta t \end{bmatrix} x(i-k_1,j-k_2) + \begin{bmatrix} 0 \\ b\Delta t \end{bmatrix} u(i,j).
$$

Next, consider the problem of delay-dependent H_∞ control for uncertain two-dimensional (2-D) discrete state-delayed systems characterized by (37). Let $a_0 = 1$, $a_1 = 1.2$, $a_2 = 0.1$, $a_3 = -0.9$, $b = 1.2$, $\Delta t = 0.1$, $\Delta x = 0.4$, $d_1 = 3$, $d_2 = 1$, $k_1 = 1$, $k_2 = 1$ and the initial state satisfies the condition (2) with $r_1 = 3$, $r_2 = 2$.

To consider the problem of H_∞ disturbance attenuation, the thermal process is modeled in the form (1) with

$$\boldsymbol{B}_1 = \begin{bmatrix} 0 \\ 0.004 \end{bmatrix}, \ \boldsymbol{B}_2 = \begin{bmatrix} 0 \\ 0.004 \end{bmatrix}, \ \boldsymbol{B}_0 = \begin{bmatrix} 0 \\ 0.003 \end{bmatrix}, \ \boldsymbol{H} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \ \boldsymbol{L} = 0.5. \quad (38)$$

It is also assumed that the above system is subjected to the parameter uncertainties of the form (1c) and (1d) with

$$\boldsymbol{D} = \begin{bmatrix} 0.002 & 0.001 \\ 0.01 & 0.02 \end{bmatrix}, \ \boldsymbol{E}_1 = \begin{bmatrix} 0.004 & 0 \\ 0 & 0.004 \end{bmatrix}, \ \boldsymbol{E}_2 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.003 \end{bmatrix}, \ \boldsymbol{E}_0 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.002 \end{bmatrix},$$

$$\boldsymbol{E}_{1d} = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.002 \end{bmatrix}, \ \boldsymbol{E}_{2d} = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.003 \end{bmatrix}, \ \boldsymbol{E}_{0d} = \begin{bmatrix} -0.006 & 0 \\ 0 & -0.006 \end{bmatrix}, \ \boldsymbol{E}_{b1} = \begin{bmatrix} -0.5 \\ 0 \end{bmatrix}, \quad (39)$$

$$\boldsymbol{E}_{b2} =, \begin{bmatrix} 0 \\ -0.4 \end{bmatrix}, \ \boldsymbol{E}_{b0} = \begin{bmatrix} 0 \\ 0.007 \end{bmatrix}, \ \boldsymbol{E}_{c1} = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix}, \ \boldsymbol{E}_{c2} = \begin{bmatrix} 0.7 \\ 0 \end{bmatrix}, \ \boldsymbol{E}_{c0} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}.$$

Using the MATLAB LMI toolbox [48] [49], it is found that the optimization problem (33) is feasible for the present example and the optimal solution is given by

$$\bar{\boldsymbol{P}} = \begin{bmatrix} 0.0646 & -0.0083 \\ -0.0083 & 0.0160 \end{bmatrix}, \ \bar{\boldsymbol{P}}_1 = \begin{bmatrix} 0.0197 & -0.0029 \\ -0.0029 & 0.0083 \end{bmatrix}, \ \bar{\boldsymbol{P}}_2 = \begin{bmatrix} 0.0169 & -0.0009 \\ -0.0009 & 0.0028 \end{bmatrix},$$

$$\bar{\boldsymbol{S}}_1 = 10^6 \times \begin{bmatrix} 3.7202 & -0.0003 \\ -0.0003 & 3.7219 \end{bmatrix}, \ \bar{\boldsymbol{S}}_2 = 10^7 \times \begin{bmatrix} 5.1206 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix}, \ \bar{\boldsymbol{S}}_3 = 10^7 \times \begin{bmatrix} 5.1236 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix},$$

$$\boldsymbol{N}_{11} = \begin{bmatrix} -0.1294 & 0.0334 \\ 0.0334 & -0.0091 \end{bmatrix}, \ \boldsymbol{N}_{12} = 10^6 \times \begin{bmatrix} 8.0094 & -0.0007 \\ -0.0007 & 8.0134 \end{bmatrix}, \ \boldsymbol{N}_{21} = \begin{bmatrix} -0.0649 & 0.0084 \\ 0.0084 & -0.0160 \end{bmatrix}, \quad (40)$$

$$\boldsymbol{N}_{22} = 10^6 \times \begin{bmatrix} 5.9719 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix}, \ \boldsymbol{N}_{31} = \begin{bmatrix} -0.0938 & 0.0177 \\ 0.0177 & -0.0189 \end{bmatrix}, \ \boldsymbol{N}_{32} = 10^6 \times \begin{bmatrix} 5.9734 & -0.0000 \\ -0.0000 & 0.0000 \end{bmatrix},$$

$$\boldsymbol{N} = \begin{bmatrix} -0.1141 & -0.0445 \end{bmatrix}, \ \varepsilon = 6.8287 \times 10^6, \ \gamma = 0.5793,$$

and a delay-dependent optimal H_∞ state feedback controller

$$\boldsymbol{u}(i,j) = \begin{bmatrix} -2.2779 & -3.9781 \end{bmatrix} \boldsymbol{x}(i,j). \quad (41)$$

When $d_1 = 3$, $d_2 = 1$, $k_1 = 1$, $k_2 = 1$, the frequency response $\boldsymbol{G}\left(e^{j\omega_1}, e^{j\omega_2}\right)$ from the disturbance input $w(i,j)$ to the controlled output $z(i,j)$ for the system (4) is shown in **Figure 1**. It is apparent from **Figure 1** that the maximum value of $\left| \boldsymbol{G}\left(e^{j\omega_1}, e^{j\omega_2}\right) \right|$ is 0.5081, which is below the specified level of attenuation $\gamma = 0.5793$.

5. Conclusion

A solution to delay-dependent robust optimal H_∞ control problem for a class of uncertain 2-D discrete state delay systems described by the GM with norm-bounded uncertainties has been presented. A sufficient condition for the existence of delay-dependent γ-suboptimal state feedback robust H_∞ controllers has been established under the LMI framework. A delay-dependent state feedback robust optimal H_∞ controller is obtained through a convex optimization problem which can be solved by using MATLAB LMI toolbox [48] [49]. Finally, an application example has been provided to illustrate the effectiveness of the proposed technique.
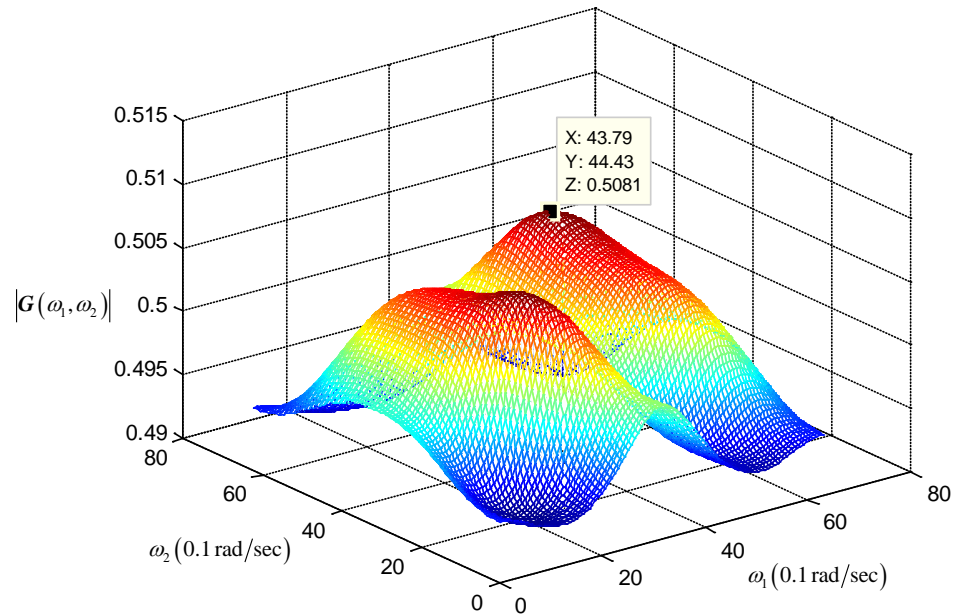
Figure 1. The frequency response $G\left(e^{j\omega_1}, e^{j\omega_2}\right)$.

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