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Distortionless Lossy Transmission Lines Terminated by in Series Connected *RCL*-Loads

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Abstract

The paper deals with a lossy transmission line terminated at both ends by non-linear *RCL* elements. The mixed problem for the hyperbolic system, describing the transmission line, to an initial value problem for a neutral equation is reduced. Sufficient conditions for the existence and uniqueness of periodic regimes are formulated. The proof is based on the finding out of suitable operator whose fixed point is a periodic solution of the neutral equation. The method has a good rate of convergence of the successive approximations even for high frequencies.

Keywords: Lossy Transmission Line, *RCL*-Nonlinear Loads, Fixed Point, Periodic Solution

1. Introduction

The principal importance of transmission lines investigations has been discussed in many papers (cf. for instance [1-8]). In a previous paper [9] we have investigated lossless transmission lines terminated by in series connected *RCL*-loads. In [10] we have considered a lossy transmission line terminated by a resistive load with exponential *V-I* characteristic. In [11] we have considered periodic regimes for lossy transmission lines terminated by parallel connected *RCL*-loads. Here we investigate lossy transmission lines terminated at both ends by in series connected *RCL*-loads but in contrast of [11] the capacitive element has a nonlinear *V-C* characteristic. Unlike of the usually accepted approach (cf. for instance [12,13]) we consider first order hyperbolic system instead of the Telegrapher's equation derived from it. First we reduce the mixed problem for the hyperbolic system to an initial value problem for neutral system of equations on the boundary [14]. Extending ideas from [15-17] we introduce operators whose fixed points are periodic solutions of the neutral system. Our treatment is based on the fixed point method (cf. [18]). All derivation are performed under assumption

$$R/L = G/C \quad (R \neq 0, G \neq 0).$$

The last condition is known as Heaviside one and it implies that the waves propagate without distortion.

We would like to mention the advantages of our method in comparison of the other used ones: lumped element method, finite element method and finite-difference time-domain method (cf. for instance [19-21]). If we use numerical methods we have to keep one and same accuracy. But here we consider nonlinearities of polynomial and transcendental type (for exponential ones cf. [10]). For such "bad" nonlinearities (cf. [2]) there are examples showing that if we want to keep the same accuracy it should be reduced step thousands of times. Here we obtain (even though approximate) an analytical solution for voltage and current beginning with simple initial approximations.

We proceed from the system:

$$C\frac{\partial u(x,t)}{\partial t} + \frac{\partial i(x,t)}{\partial x} + Gu(x,t) = 0, \qquad (1)$$

$$L\frac{\partial i(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} + Ri(x,t) = 0,$$

$$(x,t) \in \Pi = \left\{(x,t) \in R^2 : (x,t) \in [0,\Lambda] \times [0,\infty)\right\},$$

$$u(x,0) = u_0(x),\ i(x,0) = i_0(x),\ x \in [0,\Lambda] \qquad (2)$$

where L, C, R and G are prescribed specific parameters of the line and $\Lambda > 0$ is its length. Here the current $i(x,t)$ and voltage $u(x,t)$ are unknown functions. The initial conditions for the foregoing system (1) are prescribed functions $u_0(x)$, $i_0(x)$. The boundary conditions can be derived from the loads and sources at the ends of the line (cf. **Figure 1**).

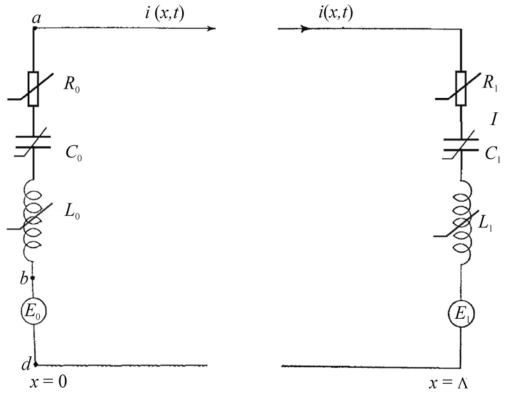

Figure 1. Lossy transmission line terminated by nonlinear *RCL*-loads.

In view of the Kirchoff's voltage law for the voltages between the nodes *a*, *b* and *d* for *x* = 0 we obtain:

$$-u(0,t) = -u_{ad} = u_{ba} - u_{bd} = u_{ba} - E_0(t), \quad (3)$$

where $E_0(t)$ is the source voltage. The voltage u_{ba} is:

$$u_{ba} = u_{R_0} + u_{C_0} + u_{\Psi_0} = R_0(i(0,t)) + u_{C_0} + \frac{\mathrm{d}\Psi_0}{\mathrm{d}t}.$$

To calculate the voltage of the condenser C_0 we proceed from the relation (assuming

$$u_{C_0}(T) \equiv u(T) = 0\ ),$$

$$i = \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\mathrm{d}(C_0(u).u)}{\mathrm{d}t} \Rightarrow \int_T^t i(x,\tau)\mathrm{d}\tau = C_0(u)\cdot u \equiv \bar{C}_0(u)$$

or

$$u_{C_0} = u(0,t) = \bar{C}_0^{-1}\left(\int_T^t i(0,\tau)\mathrm{d}\tau\right).$$

To calculate the voltage of the inductor L_0 we proceed from

$$u_{\Psi_0} = \frac{\mathrm{d}\Psi_0}{\mathrm{d}t} = (L_0(i)\cdot i) = \left(i\frac{\mathrm{d}L_0(i)}{\mathrm{d}i} + L_0(i)\right)\frac{\mathrm{d}i}{\mathrm{d}t}.$$

Therefore the first boundary condition is:

$$u_{ba} = R_0(i(0,t)) + \bar{C}_0^{-1}\left(\int_T^t i(0,\tau)\mathrm{d}\tau\right) + \left[i(0,t)\frac{\mathrm{d}L_0(i(0,t))}{\mathrm{d}i} + L_0(i(0,t))\right]\frac{\mathrm{d}i(0,t)}{\mathrm{d}t}$$

or

$$u(0,t) = E_0(t) - R_0(i(0,t)) - \bar{C}_0^{-1}\left(\int_T^t i(0,\tau)\mathrm{d}\tau\right) - \left[i(0,t)\frac{\mathrm{d}L_0(i(0,t))}{\mathrm{d}i} + L_0(i(0,t))\right]\frac{\mathrm{d}i(0,t)}{\mathrm{d}t}. \quad (4)$$

Analogously for $x = \Lambda$ (cf. **Figure 1**) we obtain

$$u(\Lambda,t) = E_1(t) + R_1(i(\Lambda,t)) + \bar{C}_1^{-1}\left(\int_T^t i(\Lambda,t)\mathrm{d}\tau\right) + \left[i(\Lambda,t)\frac{\mathrm{d}L_1(i(\Lambda,t))}{\mathrm{d}i} + L_1(i(\Lambda,t))\right]\frac{\mathrm{d}i(\Lambda,t)}{\mathrm{d}t}. \quad (5)$$

Here $R_1(.), C_1(.)$ and $L_1(.)$ are characteristics (in general, nonlinear functions) of *RCL*-loads at right end.

Now we are able to formulate the initial-boundary (mixed) value problem for the hyperbolic transmission line system of equations: to find a solution $(u(x,t), i(x,t))$ of the hyperbolic system (1) for $(x,t) \in \Pi = \{(x,t) \in R^2 : 0 \le x \le \Lambda,\ t \ge 0\}$, satisfying the initial conditions

$$u(x,0) = u_0(x),\ i(x,0) = i_0(x) \text{ for } x \in [0,\Lambda] \quad (6)$$

and the boundary conditions

$$u(0,t) = E_0(t) - R_0(i(0,t)) - \bar{C}_0^{-1}\left(\int_T^t i(0,\tau)\mathrm{d}\tau\right) - \left[i(0,t)\frac{\mathrm{d}L_0(i(0,t))}{\mathrm{d}i} + L_0(i(0,t))\right]\frac{\mathrm{d}i(0,t)}{\mathrm{d}t}, \quad (7)$$

$$u(\Lambda,t) = E_1(t) + R_1(i(\Lambda,t)) + \bar{C}_1^{-1}\left(\int_T^t i(\Lambda,t)\mathrm{d}\tau\right) + \left[i(\Lambda,t)\frac{\mathrm{d}L_1(i(\Lambda,t))}{\mathrm{d}i} + L_1(i(\Lambda,t))\right]\frac{\mathrm{d}i(\Lambda,t)}{\mathrm{d}t} \quad (8)$$

where $u_0(x), i_0(x), E_k(t), R_k(.), C_k(.)$ and $L_k(.)$ $(k = 0, 1)$ are prescribed functions.

So the system (1) and conditions (6) - (8) form a mixed problem for a lossy transmission line equations.

2. Reducing the Mixed Problem for the Transmission Line System to an Initial Value Problem for a Nonlinear Neutral System

First we present system (1) in matrix form:

$$A_1U_t + A_2U_x + A_3U = 0 \left(U_t \equiv \frac{\partial U}{\partial t}, U_x \equiv \frac{\partial U}{\partial x}\right) \quad (9)$$

where $A_1 = \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix}$,

$$U = \begin{bmatrix} u \\ i \end{bmatrix}, U_t = \begin{bmatrix} u_t \\ i_t \end{bmatrix}, U_x = \begin{bmatrix} u_x \\ i_x \end{bmatrix}.$$

Since $|A_1| = LC \ne 0$, then multiplying Equation (9) by A_1^{-1} we obtain

$$U_t + AU_x + BU = 0 \tag{10}$$

where $A = \begin{bmatrix} 0 & 1/C \\ 1/L & 0 \end{bmatrix}$, $B = \begin{bmatrix} G/C & 0 \\ 0 & R/L \end{bmatrix}$. In order to transform the matrix $A = A_1^{-1} A_2 = \begin{bmatrix} 0 & 1/C \\ 1/L & 0 \end{bmatrix}$ in a diagonal form we solve the characteristic equation:

$$\begin{vmatrix} -\lambda & 1/C \\ 1/L & -\lambda \end{vmatrix} = 0$$

whose roots are $\lambda_1 = 1/\sqrt{LC}$ and $\lambda_2 = -1/\sqrt{LC}$. Denote the matrix formed by eigenvectors by

$$H = \begin{bmatrix} \sqrt{C} & \sqrt{L} \\ -\sqrt{C} & \sqrt{L} \end{bmatrix}$$

and its inverse one—by

$$H^{-1} = \frac{1}{2}\begin{bmatrix} 1/\sqrt{C} & -1/\sqrt{C} \\ 1/\sqrt{L} & 1/\sqrt{L} \end{bmatrix}.$$

If we denote by

$$A^{can} = \begin{bmatrix} 1/\sqrt{LC} & 0 \\ 0 & -1/\sqrt{LC} \end{bmatrix},$$

then $A^{can} = HAH^{-1}$.

Introduce new variables $Z = HU$ (or $U = H^{-1}Z$):

$$Z = \begin{bmatrix} V(x,t) \\ I(x,t) \end{bmatrix}, U = \begin{bmatrix} u(x,t) \\ i(x,t) \end{bmatrix}.$$

Then

$$\left| \begin{aligned} &V(x,t) = \sqrt{C}\, u(x,t) + \sqrt{L}\, i(x,t) \\ &I(x,t) = -\sqrt{C}\, u(x,t) + \sqrt{L}\, i(x,t), \end{aligned} \right. \tag{11}$$

$$\left| \begin{aligned} &u(x,t) = V(x,t)/2\sqrt{C} - I(x,t)/2\sqrt{C} \\ &i(x,t) = V(x,t)/2\sqrt{L} + I(x,t)/2\sqrt{L}. \end{aligned} \right. \tag{12}$$

Replacing $U = H^{-1}Z$ in Equation (10) we obtain

$$\frac{\partial(H^{-1}Z)}{\partial t} + A\frac{\partial(H^{-1}Z)}{\partial x} + B(H^{-1}Z) = 0 .$$

Since H^{-1} is a constant matrix we have:

$$H^{-1}Z_t + (AH^{-1})Z_x + (BH^{-1})Z = 0.$$

After multiplication from the left by H we obtain

$$Z_t + H(AH^{-1})Z_x + H(BH^{-1})Z = 0 \text{, i.e.}$$

$$Z_t + A^{can}Z_x + (HBH^{-1})Z = 0 . \tag{13}$$

But

$$HBH^{-1} = \frac{1}{2}\begin{bmatrix} (G/C)+(R/L) & (-G/C)+(R/L) \\ (-G/C)+(R/L) & (G/C)+(R/L) \end{bmatrix}$$

and then

$$\begin{bmatrix} V_t \\ I_t \end{bmatrix} + \begin{bmatrix} 1/\sqrt{LC} & 0 \\ 0 & -1/\sqrt{LC} \end{bmatrix}\begin{bmatrix} V_x \\ I_x \end{bmatrix} + \frac{1}{2}\begin{bmatrix} (G/C)+(R/L) & (-G/C)+(R/L) \\ (-G/C)+(R/L) & (G/C)+(R/L) \end{bmatrix}\begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We consider distortionless lossy transmission lines that means the following Heaviside condition is fulfilled:

$$R/L = G/C .$$

Then HBH^{-1} can be simplified and the last system becomes:

$$\begin{bmatrix} V_t \\ I_t \end{bmatrix} + \begin{bmatrix} 1/\sqrt{LC} & 0 \\ 0 & -1/\sqrt{LC} \end{bmatrix}\begin{bmatrix} V_x \\ I_x \end{bmatrix} + \begin{bmatrix} R/L & 0 \\ 0 & R/L \end{bmatrix}\begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{aligned} &V_t + \left(1/\sqrt{LC}\right)V_x + (R/L)V = 0 \\ &I_t - \left(1/\sqrt{LC}\right)I_x + (R/L)I = 0. \end{aligned} \tag{14}$$

The new initial conditions we obtain from conditions (6) and system (12):

$$\begin{aligned} V(x,0) &= \sqrt{C}\, u(x,0) + \sqrt{L}\, i(x,0) \\ &= \sqrt{C}\, u_0(x) + \sqrt{L}\, i_0(x) \equiv V_0(x),\ x \in [0,\Lambda], \end{aligned} \tag{15}$$

$$\begin{aligned} I(x,0) &= -\sqrt{C}\, u(x,0) + \sqrt{L}\, i(x,0) \\ &= -\sqrt{C}\, u_0(x) + \sqrt{L}\, i_0(x) \equiv I_0(x),\ x \in [0,\Lambda]. \end{aligned} \tag{16}$$

Further on we set

$$\begin{aligned} &W(x,t) = e^{(R/L)t}V(x,t),\ J(x,t) = e^{(R/L)t}I(x,t) \\ &\Leftrightarrow V(x,t) = e^{-(R/L)t}W(x,t),\ I(x,t) = e^{-(R/L)t}J(x,t). \end{aligned} \tag{17}$$

Then system (14) can be written in the form:

$$W_t + vW_x = 0,\quad J_t - vJ_x = 0\ \left(v = 1/\sqrt{LC}\right). \tag{18}$$

The initial conditions (15), (16) remain the same ones:

$$W(0,t) = V(0,t) = V_0(x),\ J(0,t) = I(0,t) = I_0(x).$$

From system (12) and denotation (17) we obtain

$$\begin{aligned} u(x,t) &= \left[W(x,t) - J(x,t)\right]e^{-(R/L)t}/\left(2\sqrt{C}\right), \\ i(x,t) &= \left[W(x,t) + J(x,t)\right]e^{-(R/L)t}/\left(2\sqrt{L}\right) \end{aligned} \tag{19}$$

and then

$$W(x,t) = e^{(R/L)t}\sqrt{C}\, u(x,t) + e^{(R/L)t}\sqrt{L}\, i(x,t) ,$$

$$J(x,t) = -e^{(R/L)t}\sqrt{C}\, u(x,t) + e^{(R/L)t}\sqrt{L}\, i(x,t) .$$

Replacing in system (19) $x=0$ we have

$$u(0,t)=\left[W(0,t)-J(0,t)\right]e^{-(R/L)t}\Big/\left(2\sqrt{C}\right),$$
$$i(0,t)=\left[W(0,t)+J(0,t)\right]e^{-(R/L)t}\Big/\left(2\sqrt{L}\right) \quad (20)$$

and replacing $x=\Lambda$ —

$$u(\Lambda,t)=\left[W(\Lambda,t)-J(\Lambda,t)\right]e^{-(R/L)t}\Big/\left(2\sqrt{C}\right),$$
$$i(\Lambda,t)=\left[W(\Lambda,t)+J(\Lambda,t)\right]e^{-(R/L)t}\Big/\left(2\sqrt{L}\right). \quad (21)$$

Now replace expressions (20) into the first boundary condition and obtain:

$$\begin{aligned}&\left[W(0,t)-J(0,t)\right]e^{-(R/L)t}\Big/\left(2\sqrt{C}\right)\\ &=E_0(t)-R_0\left(\left(W(0,t)+J(0,t)\right)e^{-(R/L)t}\Big/\left(2\sqrt{L}\right)\right)\\ &\quad-\bar{C}_0^{-1}\left(\int_T^t\left[W(0,\tau)+J(0,\tau)\right]e^{-(R/L)t}\Big/\left(2\sqrt{L}\right)\mathrm{d}\tau\right)\\ &\quad-\left[\left(i(0,t)\right)\frac{\mathrm{d}L_0\left(i(0,t)\right)}{\mathrm{d}i}+L_0\left(i(0,t)\right)\right]\\ &\quad\times\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(W(0,t)+J(0,t)\right)e^{-(R/L)t}\Big/\left(2\sqrt{L}\right)\right).\end{aligned} \quad (22)$$

Replacing expressions (21) into the second boundary condition we obtain the following equation

$$\begin{aligned}&\left[W(\Lambda,t)-J(\Lambda,t)\right]e^{-(R/L)t}\Big/\left(2\sqrt{C}\right)\\ &=E_1(t)+R_1\left((W(\Lambda,t)+J(\Lambda,t))e^{-(R/L)t}\Big/\left(2\sqrt{L}\right)\right)\\ &\quad+\bar{C}_1^{-1}\left(\int_T^t\left[W(\Lambda,s)+J(\Lambda,s)\right]e^{-(R/L)s}\Big/\left(2\sqrt{L}\right)\mathrm{d}s\right)\\ &\quad+\left[i(\Lambda,t)\frac{\mathrm{d}L_1\left(i(\Lambda,t)\right)}{\mathrm{d}i}+L_1\left(i(\Lambda,t)\right)\right]\\ &\quad\times\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(W(\Lambda,t)+J(\Lambda,t)\right)e^{-(R/L)t}\Big/\left(2\sqrt{L}\right)\right).\end{aligned} \quad (23)$$

Introduce denotations

$$\begin{aligned}\tilde{L}_0(t)&=i(0,t)\frac{\mathrm{d}L_0\left(i(0,t)\right)}{\mathrm{d}i}+L_0\left(i(0,t)\right)\\ &=\left(\left(W(0,t)+J(0,t)\right)e^{-(R/L)t}\Big/\left(2\sqrt{L}\right)\right)\frac{\mathrm{d}L_0\left(i(0,t)\right)}{\mathrm{d}i}\\ &\quad+L_0\left((W(0,t)+J(0,t))e^{-(R/L)t}\Big/\left(2\sqrt{L}\right)\right),\end{aligned}$$

$$\begin{aligned}\tilde{L}_1(t)&=i(\Lambda,t)\frac{\mathrm{d}L_1\left(i(\Lambda,t)\right)}{\mathrm{d}i}+L_1\left(i(\Lambda,t)\right)\\ &=\left(\left(W(\Lambda,t)+J(\Lambda,t)\right)e^{-(R/L)t}\Big/\left(2\sqrt{L}\right)\right)\frac{\mathrm{d}L_1\left(i(\Lambda,t)\right)}{\mathrm{d}i}\\ &\quad+L_1\left(\left(W(\Lambda,t)+J(\Lambda,t)\right)e^{-(R/L)t}\Big/\left(2\sqrt{L}\right)\right).\end{aligned}$$

We assume that the unknown functions are $W(\Lambda,t)=W(t)$, $J(0,t)=J(t)$. An integration along the charateristics yields $W(\Lambda,t+T)=W(0,t)$, $J(\Lambda,t)=J(0,t+T)$ and then Equations (22) and (23) become

$$\begin{aligned}&\left(W(0,t+T)-J(0,t)\right)e^{-(R/L)t}\Big/\left(2\sqrt{C}\right)\\ &=E_0(t)-R_0\left(\left(W(0,t+T)-J(0,t)\right)e^{-(R/L)t}\Big/\left(2\sqrt{C}\right)\right)\\ &\quad-\bar{C}_0^{-1}\left(\int_T^t\left(\left(W(0,t+T)+J(0,\tau)\right)e^{-(R/L)\tau}\Big/\left(2\sqrt{L}\right)\right)\mathrm{d}\tau\right)\\ &\quad-\tilde{L}_0(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(W(0,t+T)+J(0,\tau)\right)e^{-(R/L)\tau}\Big/\left(2\sqrt{L}\right)\right)\end{aligned}$$

and

$$\begin{aligned}&\left(W(\Lambda,t)-J(0,t+T)\right)e^{-(R/L)t}\Big/\left(2\sqrt{C}\right)\\ &=E_1(t)+R_1\left(\left(W(\Lambda,t)+J(0,t+T)\right)e^{-(R/L)t}\Big/\left(2\sqrt{L}\right)\right)\\ &\quad+\bar{C}_1^{-1}\left(\int_T^t\left(\left[W(\Lambda,\tau)+J(0,\tau+T)\right]e^{-(R/L)\tau}\Big/\left(2\sqrt{L}\right)\right)\mathrm{d}\tau\right.\\ &\quad\left.+\tilde{L}_1(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(W(\Lambda,t)+J(0,t+T)\right)e^{-(R/L)t}\Big/\left(2\sqrt{L}\right)\right)\right).\end{aligned}$$

Then we put $t+T\equiv t$ and change the variables in the integrals:

$$\begin{aligned}&\left(W(t)-J(t-T)\right)e^{-(R/L)(t-T)}\Big/\left(2\sqrt{C}\right)\\ &=E_0(t-T)-R_0\left(\left(W(t)+J(t-T)\right)e^{-(R/L)(t-T)}\Big/\left(2\sqrt{L}\right)\right)\\ &\quad-\bar{C}_0^{-1}\int_T^t\left(\left(W(\theta)+J(\theta-T)\right)e^{-(R/L)(\theta-T)}\Big/\left(2\sqrt{L}\right)\right)\mathrm{d}\theta\\ &\quad-\tilde{L}_0(t-T)\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(W(t)+J(t-T)\right)e^{-(R/L)(t-T)}\Big/\left(2\sqrt{L}\right)\right),\end{aligned}$$

$$\begin{aligned}&\left(W(t-T)-J(t)\right)e^{-(R/L)(t-T)}\Big/\left(2\sqrt{C}\right)\\ &=E_1(t-T)+R_1\left(\left(W(t-T)+J(t)\right)e^{-(R/L)(t-T)}\Big/\left(2\sqrt{C}\right)\right)\\ &\quad+\bar{C}_1^{-1}\int_T^t\left(\left(W(\theta-T)+J(\theta)\right)e^{-(R/L)(\theta-T)}\Big/\left(2\sqrt{L}\right)\right)\mathrm{d}\theta\\ &\quad+\tilde{L}_1(t-T)\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(W(t-T)+J(t)\right)e^{-(R/L)(t-T)}\Big/\left(2\sqrt{L}\right)\right).\end{aligned}$$

To solve the above equations with respect to the derivatives $\mathrm{d}W(t)/\mathrm{d}t$ and $\mathrm{d}J(t)/\mathrm{d}t$ we have to divide the above equations into $\tilde{L}_0(t-T)$ and $\tilde{L}_1(t-T)$, respectively.

But

$$u_{\Psi_k}=\frac{\mathrm{d}\Psi_k}{\mathrm{d}t}=\frac{\mathrm{d}\tilde{L}_k(i)}{\mathrm{d}t}\equiv\frac{\mathrm{d}L_k(i).i}{\mathrm{d}t}=\left(i\frac{\mathrm{d}L_k(i)}{\mathrm{d}i}+L_k(i)\right)\frac{\mathrm{d}i}{\mathrm{d}t}$$

$(k=0,1)$ where $L_k(i)=\sum_{n=0}^{m}a_n^{(k)}i^n$. Then since

$$\Psi_k(i)=i.L_k(i)=i.\sum_{n=0}^{m}a_n^{(k)}i^n=\sum_{n=0}^{m}a_n^{(k)}i^{n+1}.$$

we get

$$\begin{aligned}\frac{\mathrm{d}\tilde{L}_k(i)}{\mathrm{d}i}&=i\frac{\mathrm{d}L_k(i)}{\mathrm{d}i}+L_k(i)\\&=i\sum_{n=0}^{m}na_n^{(k)}i^{n-1}+\sum_{n=0}^{m}a_n^{(k)}i^n\\&=\sum_{n=0}^{m}(n+1)a_n^{(k)}i^n,\ (k=0,1).\end{aligned}$$

Introduce denotations

$$\Pi_0(W,J)(t)=e^{-(R/L)(t-T)}\left(W(t)+J(t-T)\right)\Big/\left(2\sqrt{L}\right),$$

$$\Pi_1(W,J)(t)=e^{-(R/L)(t-T)}\left(W(t-T)+J(t)\right)\Big/\left(2\sqrt{L}\right).$$

Therefore

$$\begin{aligned}\tilde{L}_0(t-T)&=\Pi_0(W,J)(t)\frac{\mathrm{d}L_0\left(\Pi_0(W,J)(t)\right)}{\mathrm{d}i}\\&\quad+L_0\left(\Pi_0(W,J)(t)\right)\\&=\sum_{n=0}^{m}(n+1)a_n^{(0)}\left(\Pi_0(W,J)(t)\right)^n,\end{aligned}$$

$$\begin{aligned}\tilde{L}_1(t-T)&=\Pi_1(W,J)(t)\frac{\mathrm{d}L_1\left(\Pi_1(W,J)(t)\right)}{\mathrm{d}i}\\&\quad+L_1\left(\Pi_1(W,J)(t)\right)\\&=\sum_{n=0}^{m}(n+1)a_n^{(1)}\left(\Pi_1(W,J)(t)\right)^n.\end{aligned}$$

We have to ensure a strict positive lower bound for $\tilde{L}_k(t-T),\ (k=0,1)$. We can find an interval $|i|\le i_0$ such that the inequalities

$$\begin{aligned}&\left|\Pi_0(W,J)(t)\right|\le e^{(\mu-(R/L))T_0}\left(W_0+J_0e^{-\mu T}\right)\Big/\left(2\sqrt{L}\right)\le i_0,\\&\left|\Pi_1(W,J)(t)\right|\le e^{(\mu-(R/L))T_0}\left(W_0e^{-\mu T}+J_0\right)\Big/\left(2\sqrt{L}\right)\le i_0\end{aligned}\qquad(24)$$

imply

$$\tilde{L}_k(t-T)=\sum_{n=0}^{m}(n+1)a_n^{(k)}\left(\Pi_k(W,J)(t)\right)^n\ge\bar{L}_k>0$$

$$\Rightarrow 1/\tilde{L}_k(t-T)\le 1/\bar{L}_k\le 1/\bar{L}\ \ (k=0,1)$$

where $1/\bar{L}=\max\left\{1/\bar{L}_k:k=0,1\right\}$.

This can be done if the polynomials have suitable properties (cf. 4. Numerical example).

We also briefly recall:

$$i=\frac{\mathrm{d}q}{\mathrm{d}t}=\frac{\mathrm{d}\left(C_0(u).u\right)}{\mathrm{d}t}\Rightarrow\int_T^t i(x,\tau)\mathrm{d}\tau=C_0(u).u\equiv\bar{C}_0(u)$$

and

$$u_{C_0}=u(0,t)=\bar{C}_0^{-1}\left(\int_T^t i(0,\tau)\mathrm{d}\tau\right)$$

where $c_k,\Phi_k,\ h\in[2,3]$,

$$C_k(u)=\frac{c_k}{\sqrt[h]{1-(u/\Phi_k)}}=\frac{c_k\sqrt[h]{\Phi_k}}{\sqrt[h]{\Phi_k-u}}\ \ (k=0,1)$$

are constants and

$$|u|\le W_0<\Phi=\min\{\Phi_0,\Phi_1\}\le\left(h/(h-1)\right)\Phi.$$

The derivatives of the functions $\bar{C}_k(u)=uC_k(u)$ are

$$\begin{aligned}\frac{\mathrm{d}\bar{C}_k(u)}{\mathrm{d}u}&=C_k(u)+u\frac{\mathrm{d}C_k(u)}{\mathrm{d}u}\\&=\frac{c_k\sqrt[h]{\Phi_k}\left(h\Phi_k-hu+u\right)}{h\left(\Phi_k-u\right)^{\frac{1}{h}+1}},u\in[-W_0,W_0].\end{aligned}$$

Since $\frac{\mathrm{d}\bar{C}_k(u)}{\mathrm{d}u}>0$ the inverse function exists for $|u|<\frac{h\Phi_k}{h-1}$.

Since

$$\bar{C}_k(u)=uC_k(u)=\frac{c_ku\sqrt[h]{\Phi_k}}{\sqrt[h]{\Phi_k-u}}:[-W_0,W_0]$$

$$\rightarrow\left[\frac{-c_k\sqrt[h]{\Phi_k}\ W_0}{\sqrt[h]{\Phi_k+W_0}},\frac{c_k\sqrt[h]{\Phi_k}\ W_0}{\sqrt[h]{\Phi_k-W_0}}\right]$$

then

$$\bar{C}_k^{-1}(I):\left[\frac{-c_k\sqrt[h]{\Phi_k}\ W_0}{\sqrt[h]{\Phi_k+W_0}},\frac{c_k\sqrt[h]{\Phi_k}\ W_0}{\sqrt[h]{\Phi_k-W_0}}\right]\rightarrow[-W_0,W_0],$$

or $\left|\bar{C}_k^{-1}(I)\right|\le W_0$.

The explicit form of the inverse function for $h=2$ is:

$$\begin{aligned}&c_k\sqrt{\Phi_k}\,u\Big/\sqrt{\Phi_k-u}=I\\&\Rightarrow c_k^2\Phi_ku^2=\Phi_kI^2-I^2u\\&\Rightarrow c_k^2\Phi_ku^2+I^2u-\Phi_kI^2=0,\\&\bar{C}_k^{-1}(I)\\&\quad=\frac{-I^2+\sqrt{I^4+4c_k^2\Phi_k^2I^2}}{2c_k^2\Phi_k}\ (k=0,1).\end{aligned}$$

We need the derivative given below (see Equation (26)) and the following estimates:

$$|I| \le \frac{c_k\sqrt{\Phi_k}\,W_0}{\sqrt{\Phi_k + W_0}} = \min\left\{\left|\frac{-c_k\sqrt[h]{\Phi_k}\,W_0}{\sqrt[h]{\Phi_k + W_0}}\right|,\ \left|\frac{c_k\sqrt[h]{\Phi_k}\,W_0}{\sqrt[h]{\Phi_k - W_0}}\right|\right\}$$

and

$$\begin{aligned}\left|\frac{\mathrm{d}\bar{C}_k^{-1}(I)}{\mathrm{d}I}\right| &\le \frac{1}{c_k^2\Phi_k}\left(|I| + \frac{I^2 + 4c_k^2\Phi_k^2}{\sqrt{I^2 + 4c_k^2\Phi_k^2}}\right)\\ &\le \frac{1}{c_k^2\Phi_k}\left(|I| + \sqrt{I^2 + 4c_k^2\Phi_k^2}\right)\\ &\le \frac{1}{c_k^2\Phi_k}\left(\frac{c_k\sqrt{\Phi_k}\,W_0}{\sqrt{\Phi_k + W_0}} + \sqrt{\left(\frac{c_k\sqrt{\Phi_k}\,W_0}{\sqrt{\Phi_k + W_0}}\right)^2 + 4c_k^2\Phi_k^2}\right)\\ &\le \frac{2}{c_k}\sqrt{1 + \frac{W_0}{\Phi_k}}\,(k = 0,1).\end{aligned} \tag{25}$$

Therefore the arguments

$$\int_T^t \left(\left[W(\theta) + J(\theta - T)\right]e^{-(R/L)(\theta - T)}\Big/\left(2\sqrt{L}\right)\right)\mathrm{d}\theta,.$$

$$\int_T^t \left(\left[W(\theta - T) + J(\theta)\right]e^{-(R/L)(\theta - T)}\Big/\left(2\sqrt{L}\right)\right)\mathrm{d}\theta$$

of $\bar{C}_0^{-1}(.), \bar{C}_1^{-1}(.)$ should satisfy the inequalities

$$\left|\int_T^t \left(\left[W(\theta) + J(\theta - T)\right]e^{-(R/L)(\theta - T)}\Big/\left(2\sqrt{L}\right)\right)\mathrm{d}\theta\right| \le \frac{c_k\sqrt{\Phi_k}\,W_0}{\sqrt{\Phi_k + W_0}},$$

$$\left|\int_T^t \left(\left[W(\theta - T) + J(\theta)\right]e^{-(R/L)(\theta - T)}\Big/\left(2\sqrt{L}\right)\right)\mathrm{d}\theta\right| \le \frac{c_k\sqrt{\Phi_k}\,W_0}{\sqrt{\Phi_k + W_0}}$$

or in view of the inequality

$$\left|\int_T^t \left(\left[W(\theta - T) + J(\theta)\right]e^{-(R/L)(\theta - T)}\Big/\left(2\sqrt{L}\right)\right)\mathrm{d}\theta\right|$$

$$\le \left(\left(W_0 + J_0e^{-\mu T}\right)\Big/\left(2\sqrt{L}\right)\right)\left(\left(e^{(\mu - (R/L))T_0} - 1\right)\Big/\left(\mu - (R/L)\right)\right)$$

we obtain

$$\frac{W_0 + J_0e^{-\mu T}}{2\sqrt{L}}\,\frac{e^{(\mu - (R/L))T_0} - 1}{\mu - (R/L)} \le \frac{c_k\sqrt{\Phi_k}\,W_0}{\sqrt{\Phi_k + W_0}}$$

and

$$\frac{W_0e^{-\mu T} + J_0}{2\sqrt{L}}\,\frac{e^{(\mu - (R/L))T_0} - 1}{\mu - (R/L)} \le \frac{c_k\sqrt{\Phi_k}\,W_0}{\sqrt{\Phi_k + W_0}}\,.$$

For the *I-V* characteristics we assume that they are polynomial functions

$$R_k(i) = \sum_{n=1}^{m} r_n^{(k)} i^n, (k = 0,1)\,.$$

Now we are able to formulate a periodic problem for the obtained neutral system:

$$\frac{\mathrm{d}\bar{C}_k^{-1}(I)}{\mathrm{d}I} = \begin{cases}\dfrac{1}{c_k^2\Phi_k}\left(-I - \dfrac{I^2 + 2c_k^2\Phi_k^2}{\sqrt{I^2 + 4c_k^2\Phi_k^2}}\right), I \in \left[\dfrac{-c_k\sqrt{\Phi_k}\,W_0}{\sqrt{\Phi_k + W_0}}, 0\right)\\[2ex] \dfrac{1}{c_k^2\Phi_k}\left(-I + \dfrac{I^2 + 2c_k^2\Phi_k^2}{\sqrt{I^2 + 4c_k^2\Phi_k^2}}\right), I \in \left(0,\ \dfrac{c_k\sqrt{\Phi_k}\,W_0}{\sqrt{\Phi_k - W_0}}\right]\end{cases}, \tag{26}$$

$$\begin{aligned}\frac{\mathrm{d}W(t)}{\mathrm{d}t} =& -\frac{\mathrm{d}J(t - T)}{\mathrm{d}t} + \frac{R}{L}W(t) + \frac{R}{L}J(t - T) + \left(2e^{(R/L)(t - T)}\sqrt{L}E_0(t - T) - \left(W(t) - J(t - T)\right)Z_0\right)\Big/\tilde{L}_0\\ &- \frac{2e^{(R/L)(t - T)}\sqrt{L}}{\tilde{L}_0}R_0\left(\frac{e^{-(R/L)(t - T)}}{2\sqrt{L}}\left(W(t) + J(t - T)\right)\right)\\ &- \frac{2e^{(R/L)(t - T)}\sqrt{L}}{\tilde{L}_0}\bar{C}_0^{-1}\left(\int_T^t \frac{e^{-(R/L)(t - T)}}{2\sqrt{L}}\left(W(\theta) + J(\theta - T)\right)\mathrm{d}\theta\right) \equiv U(W, J)(t).\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{\mathrm{d}J(t)}{\mathrm{d}t} =& -\frac{\mathrm{d}W(t - T)}{\mathrm{d}t} + \frac{R}{L}W(t - T) + \frac{R}{L}J(t) - 2e^{(R/L)(t - T)}\sqrt{L}E_1(t - T)\Big/\tilde{L}_1\\ &+ \left(\left(W(t - T) - J(t)\right)Z_0\right)\Big/\tilde{L}_1 - \frac{2e^{(R/L)(t - T)}\sqrt{L}}{\tilde{L}_1}R_1\frac{e^{-(R/L)(t - T)}}{2\sqrt{L}}\left(W(t - T) + J(t)\right)\\ &- \frac{2e^{-(R/L)(t - T)}\sqrt{L}}{\tilde{L}_1}\bar{C}_1^{-1}\left(\int_T^t \frac{e^{-(R/L)(t - T)}}{2\sqrt{L}}\left(W(\theta - T) + J(\theta)\right)\mathrm{d}\theta\right) \equiv I(W, J)(s),\ t \in \left[T, T + T_0\right],\end{aligned} \tag{28}$$

$$W(t)=\upsilon_0(t),\ \dot{W}(t)=\dot{\upsilon}_0(t),$$
$$J(t)=\iota_0(t),\ \dot{J}(t)=\dot{\iota}_0(t),\quad t\in[0,T].$$

The initial functions $\upsilon_0(t)$, $\iota_0(t)$ can be obtained from the initial conditions (7) by shifting along the characteristics of the initial functions $u_0(x)$, $i_0(x)$ (cf. [9-11]).

3. Main Results

Here we formulate conditions for the existence-uniqueness of a periodic solution of neutral functional differential system (27), (28) (for definition of neutral equation see [15]). First we define a suitable operator generating by the right-hand sides of the Equations (27), (28). We find a T_0 – periodic solution of Equations (27), (28) on the interval $[T,T+T_0)$ coinciding with prescribed T_0 – periodic function on $[0,T]$ and then one can continue it periodically because our system is autonomous one.

By $L^{1,\infty}_{T_0}[T,\infty)$ we mean the space consisting of all measurable essentially bounded T_0 – periodic functions whose derivatives are also essentially bounded and T_0 – periodic.

Introduce the sets:

$$M_u=\left\{f\in L^{1,\infty}_{T_0}[T,\infty):\int_{T_0}^{T+T_0} f(t)\mathrm{d}t=0\wedge f(t)=\upsilon_0(t),\ t\in[-T,T]\right\},$$

$$M_i=\left\{f\in L^{1,\infty}_{T_0}[T,\infty):\int_{T}^{T+T_0} f(t)\mathrm{d}t=0\wedge f(t)=\iota_0(t),\ t\in[-T,T]\right\}.$$

Lemma 1. If $f(.)\in M_u$ (respectively $f(.)\in M_i$) then $F(t)=\int_T^t f(\tau)\mathrm{d}\tau$ is T_0 – periodic function.

Assumption (E) $E_k(.)\in L^{1,\infty}_{T_0}[-T,\infty)$, $\int_T^{T+T_0} E_k(t)\mathrm{d}t=0$, $|E_k(t)|\le W_0 e^{\mu(t-T)}, t\in[T,T+T_0]$, $E_k(-T)=0\ (k=0,1)$.

Assumption (IN) $\upsilon_0(t),\iota_0(t)$ are essentially bounded T_0 – periodic functions where $T=mT_0, m\in\{2,3,4,5,\cdots\}$, $\upsilon_0(0)=\iota_0(0)=0$, $|\upsilon_0 t|\le W_0$, $|\iota_0(t)|\le J_0$. Here W_0, J_0 are prescribed positive constants.

Lemma 2. If the assumptions **(IN)** and **(E)** are fulfilled and $(W,J)\in M_u\times M_i$ then $R_0(\Pi_0(W,J)(t))$, $R_1(\Pi_1(W,J)(t))$, $\Psi_0(t)=\int_T^t(\Pi_0(W,J)(\tau))\mathrm{d}\tau$ and $\Psi_1(t)=\int_T^t(\Pi_1(W,J)(\tau))\mathrm{d}\tau$ are T_0 – periodic functions.

Define the operator $B=(B_u,B_i)$ on the set $M_u\times M_i$ by the formulas:

$$B_u(W,J)(t):=\upsilon_0(t), t\in[0,T],$$
$$B_i(W,J)(t):=\iota_0(t), t\in[0,T],$$

$$B_u(W,J)(t):=\int_T^t U(W,J)(s)\mathrm{d}s-\left(\frac{t-T}{T_0}-\frac{1}{2}\right)\int_T^{T+T_0}U(W,J)(s)\mathrm{d}s-\frac{1}{T_0}\int_T^{T+T_0}\int_T^t U(W,J)(s)\mathrm{d}s\mathrm{d}t,\ t\in[T,T+T_0]$$

$$B_i(W,J)(t):=\int_T^t I(W,J)(s)\mathrm{d}s-\left(\frac{t-T}{T_0}-\frac{1}{2}\right)\int_T^{T+T_0}I(W,J)(s)\mathrm{d}s-\frac{1}{T_0}\int_T^{T+T_0}\int_T^t I(W,J)(s)\mathrm{d}s\mathrm{d}t,\ t\in[T,T+T_0].$$

Remark 1. It is easy to see that changing the integration order one obtains

$$\int_T^{T+T_0}\int_T^t U(W,J)(s)\mathrm{d}s\mathrm{d}t=(T+T_0)\int_T^{T+T_0}U(W,J)(s)\mathrm{d}s-\int_T^{T+T_0}sU(W,J)(s)\mathrm{d}s$$

and

$$\int_T^{T+T_0}\int_T^t I(W,J)(s)\mathrm{d}s\mathrm{d}t=(T+T_0)\int_T^{T+T_0}I(W,J)(s)\mathrm{d}s-\int_T^{T+T_0}sI(W,J)(s)\mathrm{d}s.$$

Lemma 3. The periodic problem (27), (28) has a solution $(W(.),J(.))\in M_u\times M_i$ iff the operator B has a fixed point $(W,J)\in M_u\times M_i$, that is, $W=B_u(W,J)$, $J=B_i(W,J)$.

Introduce the sets

$$X_u=\left\{f\in M_u:|f(t)|\le W_0 e^{\mu(t-T)}, t\in[T,T+T_0]\right\},$$

$$X_i=\left\{f\in M_i:|f(t)|\le J_0 e^{\mu(t-T)}, t\in[T,T+T_0]\right\}.$$

Here the constant $\mu>0$ will be prescribed below.

The sets X_u and X_i turn out into metric spaces with metrics

$$\rho(W,\bar{W})=\operatorname{ess\,sup}\left\{e^{-\mu(t-T)}|W(t)-\bar{W}(t)|:t\in[T,T+T_0]\right\},$$

$$\rho(\dot{W},\dot{\bar{W}})=\operatorname{ess\,sup}\left\{e^{-\mu(t-T)}|\dot{W}(t)-\dot{\bar{W}}(t)|:t\in[T,T+T_0]\right\},$$

$$\rho(J,\bar{J})=\operatorname{ess\,sup}\left\{e^{-\mu(t-T)}|J(t)-\bar{J}(t)|:t\in[T,T+T_0]\right\},$$

$$\rho(\dot J,\dot{\bar J}) = \text{ess }\sup\left\{e^{-\mu(t-T)}\left|\dot J(t)-\dot{\bar J}(t)\right| : t\in[T,T+T_0]\right\}.$$

Define a metric on $X_u \times X_i$ in the following way:

$$\rho\big((W,J),(\bar W,\bar J)\big) = \max\left\{\rho\left(W,\bar W\right),\rho\left(J,\bar J\right),\rho\left(\dot W,\dot{\bar W}\right),\rho\left(\dot J,\dot{\bar J}\right)\right\}.$$

Assumption (LC): $\dfrac{e^{(\mu-(R/L))T_0}\left(W_0+J_0e^{-\mu T}\right)}{2\sqrt{L}} \le i_0,$

$$\frac{e^{(\mu-(R/L))T_0}\left(W_0e^{-\mu T}+J_0\right)}{2\sqrt{L}} \le i_0,$$

$$\frac{W_0+J_0e^{-\mu T}}{2\sqrt{L}}\,\frac{e^{(\mu-R/L)T_0}-1}{\mu-(R/L)} \le \frac{c_k\sqrt{\Phi_k}\,W_0}{\sqrt{\Phi_k+W_0}},$$

$$\frac{W_0e^{-\mu T}+J_0}{2\sqrt{L}}\,\frac{e^{(\mu-(R/L))T_0}-1}{\mu-(R/L)} \le \frac{c_k\sqrt{\Phi_k}\,W_0}{\sqrt{\Phi_k+W_0}},\quad \mu>\frac{R}{L}.$$

Theorem 1. Let the assumptions **(LC)**, **(E)** and **(IN)** be fulfilled. Then there exists a unique T_0 – periodic solution $(W,J)\in X_u\times X_i$ of the problem (27), (28).

***Proof*:** Define the operator $B\equiv(B_u,B_i): X_u\times X_i \to X_u\times X_i$ by the above formulas. In what follows we show

$$\left|W(t)\right|\le W_0e^{\mu(t-T)} \wedge \left|J(t)\right|\le J_0e^{\mu(t-T)}$$

$$\Rightarrow \left|B_u\right|\le W_0e^{\mu(t-T)} \wedge \left|B_i\right|\le J_0e^{\mu(t-T)}.$$

Using inequality (24) and proceeding as in [11] we obtain (for sufficiently large $\mu>(R/L)$):

$$\begin{aligned}
\left|B_u(W,J)(t)\right| &\le \left|\int_T^t U(W,J)(s)\mathrm{d}s\right| + \frac{1}{2}\left|\int_T^{T+T_0} U(W,J)(s)\mathrm{d}s\right| \\
&+\frac{1}{T_0}\int_T^{T+T_0}\left|\int_T^t U(W,J)(s)\mathrm{d}s\right|\mathrm{d}t \le e^{\mu(t-T)}\left[\frac{e^{\mu_0}+\mu_0-1}{\mu_0}e^{-\mu T}J_0\right.\\
&+\frac{(\mu_0+2)e^{\mu_0}+\mu_0-2}{2\mu_0}\left(\frac{R}{L}+\frac{Z_0}{\bar L_0}\right)\frac{\left(W_0+J_0e^{-\mu T}\right)}{\mu}+\frac{2\sqrt{L}\,W_0(e^{-\mu T}+1)}{\bar L_0\left(\mu+R/L\right)}\mathrm{e}^{(R/L)T_0}\frac{(\mu_0+2)e^{\mu_0}+2\mu_0-2}{\mu_0}\\
&\left.+\frac{1}{\bar L_0}\sum_{n=1}^m\frac{\left|r_n^{(0)}\right|\left(W_0+J_0e^{-\mu T}\right)^n}{n\left(2\sqrt{L}\right)^{n-1}\left(\mu-\left(((n-1)R)/(nR)\right)\right)}\times\left(e^{(n-1)(\mu-(R/L))T_0}\frac{2e^{\mu_0}+3\mu_0-2}{2\mu_0}-\frac{1}{2}\right)\right]\le e^{\mu(t-T)}W_0.
\end{aligned}$$

For the second component of the operator we obtain

$$\begin{aligned}
\left|B_i(W,J)(t)\right| &\le \left|\int_T^t I(W,J)(s)\mathrm{d}s\right| + \frac{1}{2}\left|\int_T^{T+T_0} I(W,J)(s)\mathrm{d}s\right| + \frac{1}{T_0}\int_T^{T+T_0}\left|\int_T^t I(W,J)(s)\mathrm{d}s\right|\mathrm{d}t\\
&\le e^{\mu(t-T)}\left[W_0e^{-\mu T}\left(e^{\mu_0}+\mu_0-1\right)\big/\mu_0 + \left(W_0\left(e^{-\mu T}+1\right)\big/\mu\right)\right.\\
&\times\left((R/L)+\left(Z_0/\bar L_1\right)\right)\left((\mu_0+2)e^{\mu_0}+\mu_0-2\right)\big/(2\mu_0)+\left(4e^{(R/L)T_0}+e^{(\mu+(R/L))T_0}-1\right)\frac{\sqrt{L}\,W_0\left(e^{-\mu T}+1\right)}{\bar L_1\left((R/L)+\mu\right)}\\
&\left.+\left(2e^{\mu_0}+3\mu_0-2\right)/\left(2\mu_0\bar L_1\right)\times\sum_{n=1}^m\frac{\left|r_n^{(1)}\right|\left(W_0e^{-\mu T}+J_0\right)^n}{\left(2\sqrt{L}\right)^{n-1}n}\,\frac{e^{(n\mu-(n-1)(R/L))T_0}-1}{\mu-\left(((n-1)R)/(nR)\right)}\right]\le J_0e^{\mu(t-T)}.
\end{aligned}$$

Therefore the operator *B* maps the set $X_u\times X_i$ into itself. In what follows we show that *B* is contractive operator. We need the following preliminary inequalities

$$\left|\Pi_0(W,J)(t)-\Pi_0\left(\bar W,\bar J\right)(t)\right|\le\frac{e^{(\mu-(R/L))(t-T)}}{2\mu\sqrt{L}}\left(\rho\left(\dot W,\dot{\bar W}\right)+\rho\left(\dot J,\dot{\bar J}\right)e^{-\mu T}\right),$$

$$\left|\Pi_1(W,J)(t)-\Pi_1\left(\bar W,\bar J\right)(t)\right|\le\frac{e^{(\mu-(R/L))(t-T)}}{2\mu\sqrt{L}}\left(\rho\left(\dot W,\dot{\bar W}\right)e^{-\mu T}+\rho\left(\dot J,\dot{\bar J}\right)\right),$$

$$\left|\left[\Pi_0(W,J)(t)\right]^n - \left[\Pi_0\left(\bar{W},\bar{J}\right)(t)\right]^n\right|$$
$$\le n\left[e^{(\mu-(R/L))T_0}\left(W_0+J_0e^{-\mu T}\right)\Big/\left(2\sqrt{L}\right)\right]^{n-1}\times e^{(\mu-(R/L))(t-T)}\left(\rho\left(\dot{W},\dot{\bar{W}}\right)+\rho\left(\dot{J},\dot{\bar{J}}\right)e^{-\mu T}\right)\Big/\left(2\mu\sqrt{L}\right).$$
$$\left|\left[\Pi_1(W,J)(t)\right]^n - \left[\Pi_1\left(\bar{W},\bar{J}\right)(t)\right]^n\right|$$
$$\le n\left[e^{(\mu-(R/L))T_0}\left(W_0e^{-\mu T}+J_0\right)\Big/\left(2\sqrt{L}\right)\right]^{n-1}\times e^{(\mu-(R/L))(t-T)}\left(\rho\left(\dot{W},\dot{\bar{W}}\right)e^{-\mu T}+\rho\left(\dot{J},\dot{\bar{J}}\right)\right)\Big/\left(2\mu\sqrt{L}\right).$$

Then for the first operator component we obtain:

$$\left|B_u\left(W,J\right)(t)-B_u\left(\bar{W},\bar{J}\right)(t)\right|$$
$$\le\int_T^t\left|U\left(W,J\right)(s)-U\left(\bar{W},\bar{J}\right)(s)\right|\mathrm{d}s+\left(\frac{t-T}{T_0}-\frac{1}{2}\right)\left|\int_T^{T+T_0}U\left(W,J\right)(s)\mathrm{d}s-\int_T^{T+T_0}U\left(\bar{W},\bar{J}\right)(s)\mathrm{d}s\right|$$
$$+\frac{1}{T_0}\int_T^{T+T_0}\left|\int_T^tU\left(W,J\right)(s)\mathrm{d}s-\int_T^tU\left(\bar{W},\bar{J}\right)(s)\mathrm{d}s\right|\mathrm{d}t$$
$$\le e^{\mu(t-T)}\left[\rho\left(W,J\right),\left(\bar{W},\bar{J}\right))\frac{e^{\mu_0}+\mu_0-1}{\mu_0}\frac{e^{-\mu T}}{\mu}+\frac{(\mu_0+2)e^{\mu_0}+\mu_0-2}{2\mu_0}\frac{1+e^{-\mu T}}{\mu^2}\right.$$
$$\left.\cdot\left(\left(R/L\right)+\left(Z_0/\bar{L}_0\right)+\frac{1}{\bar{L}_0}\sum_{n=1}^m\left|r_n^{(0)}\right|\frac{ne^{(n-1)(\mu-(R/L))T_0}(W_0+J_0e^{-\mu T})^{n-1}}{\left(2\sqrt{L}\right)^{n-1}}+2\sqrt{1+\left(W_0/\Phi_0\right)}\Big/\left(c_0\bar{L}_0\left(\mu-(R/L)\right)\right)\right)\right]$$
$$\equiv e^{\mu(t-T)}K_u\,\rho\left(\left(W,J\right),\left(\bar{W},\bar{J}\right)\right).$$

Further on we have

$$\left|B_i(W,J)(t)-B_i(\bar{W},\bar{J})(t)\right|$$
$$\le\int_T^t\left|I(W,J)(s)-I(\bar{W},\bar{J})(s)\right|\mathrm{d}s+\left(\frac{t-T}{T_0}-\frac{1}{2}\right)\left|\int_T^{T+T_0}I(W,J)(s)\mathrm{d}s-\int_T^{T+T_0}I(\bar{W},\bar{J})(s)\mathrm{d}s\right|$$
$$+\frac{1}{T_0}\int_T^{T+T_0}\left|\int_T^tI(W,J)(s)\mathrm{d}s-\int_T^tI(W,\bar{J})(s)\mathrm{d}s\right|\mathrm{d}t$$
$$\le e^{\mu(t-T)}\rho((W,J),(\bar{J},\bar{W}))\left[\frac{e^{\mu_0}+\mu_0-1}{\mu_0}\frac{e^{-\mu T}}{\mu}+\frac{(\mu_0+2)e^{\mu_0}+\mu_0-2}{2\mu_0}\frac{(e^{-\mu T}+1)}{\mu^2}\right.$$
$$\left.\cdot\left(\left(\left(R/L\right)+\left(Z_0/\bar{L}_0\right)+\frac{2}{\bar{L}_1}\sum_{n=1}^m\left|r_n^{(1)}\right|\frac{ne^{(n-1)(\mu-(R/L))T_0}(W_0e^{-\mu T}+J_0)^{n-1}}{\left(2\sqrt{L}\right)^{n-1}}+\frac{2}{c_1\bar{L}_1\left(\mu-(R/L)\right)}\sqrt{1+\left(W_0/\Phi_1\right)}\right)\right]$$
$$\equiv e^{\mu(t-T)}K_i\,\rho\left(\left(W,J\right),\left(\bar{J},\bar{W}\right)\right).$$

Finally we have to obtain an estimate for $t\in[T,T+T_0]$. It is easy to prove the following inequalities:

$$\left|\dot{B}_u(W,J)(t)-\dot{B}_u(\bar{W},\bar{J})(t)\right|$$
$$\le\left|U(W,J)(t)-U(\bar{W},\bar{J})(t)\right|+\frac{1}{T_0}\left|\int_T^{T+T_0}U(W,J)(s)\mathrm{d}s-\int_T^{T+T_0}U(\bar{W},\bar{J})(s)\mathrm{d}s\right|$$
$$\le e^{\mu(t-T)}\rho((W,J),(\bar{W},\bar{J}))\left[e^{-\mu T}+\frac{e^{\mu_0}+\mu_0-1}{\mu_0}\frac{\left(1+e^{-\mu T}\right)}{\mu}\times\left(\frac{R}{L}+\frac{Z_0}{\bar{L}_0}+\frac{1}{\bar{L}_0}\sum_{n=1}^{m}\left|r_n^{(0)}\right|\frac{ne^{(n-1)(\mu-(R/L))T_0}\left(W_0+J_0e^{-\mu T}\right)^{n-1}}{\left(2\sqrt{L}\right)^{n-1}}+\frac{2\sqrt{1+W_0/\Phi_0}}{\bar{L}_0c_0\left(\mu-(R/L)\right)}\right)\right]$$
$$\equiv e^{\mu(t-T)}\dot{K}_u\rho\left((W,J),(\bar{J},\bar{W})\right).$$

For the derivative of the second component of B we obtain

$$\left|\dot{B}_i(W,J)(t)-\dot{B}_i(\bar{W},\bar{J})(t)\right|$$
$$\le\left|I(W,J)(t)-I(\bar{W},\bar{J})(t)\right|+\frac{1}{T_0}\left|\int_T^{T+T_0}I(W,J)(s)\mathrm{d}s-\int_T^{T+T_0}I(\bar{W},\bar{J})(s)\mathrm{d}s\right|$$
$$\le e^{\mu(t-T)}\rho((W,J),(\bar{W},\bar{J}))\left[e^{-\mu T}+\frac{e^{\mu_0}+\mu_0-1}{\mu_0}\frac{(e^{-\mu T}+1)}{\mu}\times\left(\frac{R}{L}+\frac{Z_0}{\bar{L}_1}+\frac{1}{\bar{L}_1}\sum_{n=1}^{m}n\left|r_n^{(1)}\right|\frac{e^{(n-1)\left(\mu-\frac{R}{L}\right)T_0}(W_0e^{-\mu T}+J_0)^{n-1}}{(2\sqrt{L})^{n-1}}+\frac{2\sqrt{1+W_0/\Phi_1}}{c_1\bar{L}_1\left(\mu-(R/L)\right)}\right)\right]$$
$$\equiv e^{\mu(t-T)}\dot{K}_i\,\rho\left((W,J),(\bar{J},\bar{W})\right).$$

The above inequalities imply

$$\rho(B(W,J),B(\bar{W},\bar{J}))\le K\,\rho((W,J),(\bar{W},\bar{J}))$$

where $K=\max\left\{K_u,K_i,\dot{K}_u,\dot{K}_i\right\}$ should be chosen smaller than 1.

Therefore B is contractive operator and has a unique fixed point in M (cf. [18]). It is a unique periodic solution of system (26), (27).

Theorem 1 is thus proved.

4. Numerical Example

For a transmission line with length $\Lambda=1\,\mathrm{m}$,

$L=0.45\,\mu H/\mathrm{m},\ C=80\,pF/\mathrm{m}$, cross-section area

$S=6\ \mathrm{mm}^2$, specific resistance for the cuprum

$\rho_c=0.0175$, that is, $R=\left(\rho_c\Lambda\right)/S\approx 3.10^{-3}\,\Omega$,

$v=1/\sqrt{LC}=1/(6\cdot10^{-9})=1.66\cdot10^8$,

$Z_0=\sqrt{L/C}=75\,\Omega$. Then $T=\Lambda\sqrt{LC}=6\cdot10^{-9}$ sec.

Let us check the propagation of millimeter waves $\lambda_0=(1/6)\cdot10^{-3}\,\mathrm{m}$. We have

$$f_0=1/\left(\lambda_0\sqrt{LC}\right)=10^{12}\,Hz\Rightarrow T_0=1/f_0=10^{-12}.$$

We choose $\mu=2\cdot10^{12}$. Then $\mu T_0=\mu_0=2$ and $T=6\cdot10^{-9}\cdot2\cdot10^{12}T_0=12000T_0$. Consequently

$\mu T=2\cdot10^{12}\cdot6\cdot10^{-9}\approx12\cdot10^3\Rightarrow e^{-\mu T}\approx0,\ RT/L\approx0,$

$R/L=0.0029/(0.45\cdot10^{-6})\approx0.6\cdot10^4,\ e^{\pm(RT/L)}\approx e^0\approx1$

$\mu\pm(R/L)=2\cdot10^{12}\pm10^4=10^{12}(2\pm10^{-8})\approx2\cdot10^{12},$

$(R/L)T_0\approx0\Rightarrow e^{(R/L)T_0}-1\approx6.6\cdot10^{-6},$

$\mu-(R/L)\approx10^{12},\ e^{(\mu-(R/L))T_0}=e^{\mu_0}e^{-(R/L)T_0}\approx e^2.$

We choose resistive elements with the following *V-I* characteristics $R_0(u)=R_1(u)=0.028u-0.125u^3$ *i.e.* $r_1=0.028,\ r_2=0,\ r_3=0.125$, and inductive element with $L_0(i)=L_1(i)=3i-(1/12)i^3$. Then

$$\tilde{L}_0(i)=i\left(\mathrm{d}L_0(i)/\mathrm{d}i\right)+L_0(i)=6i-(1/3)i^3.$$

If we choose $i_0=1$ one obtains

$$6i-(1/3)i^3>6-(1/3)=17/3$$

and consequently $1/\bar{L}=3/17$.

Let us take

$$C_0(u)=C_1(u)=c/\sqrt{1-(u/\Phi)}=c\sqrt{\Phi}/\sqrt{\Phi-u},$$

where $h=2,\ c=50\,pF=5\cdot10^{-11}F$ and $\Phi=0.4\ V\Rightarrow$

$W_0 < 0.4$. Then the inequalities of Theorem 1 become

$$e^{(\mu-(R/L))T_0}\left(W_0+J_0e^{-\mu T}\right)\Big/\left(2\sqrt{L}\right)\le 1$$

$$\Rightarrow e^2W_0\Big/\left(2\sqrt{L}\right)\le 1 \Rightarrow W_0\le 2\sqrt{L}\big/e^2\approx 0.18\cdot 10^{-3},$$

$$e^{(\mu-R/L)T_0}\left(W_0e^{-\mu T}+J_0\right)\Big/\left(2\sqrt{L}\right)\le 1$$

$$\Rightarrow e^2J_0\Big/\left(2\sqrt{L}\right)\le 1 \Rightarrow J_0\le 2\sqrt{L}\big/e^2\approx 0.18\cdot 10^{-3}$$

$$\frac{W_0}{2\sqrt{L}}\frac{e^2-1}{\mu}\le\frac{cW_0\sqrt{\Phi}}{\sqrt{\Phi+W_0}}\Leftrightarrow\sqrt{\Phi+W_0}\le\frac{2c\mu\sqrt{L\Phi}}{e^2-1}$$

$$\Rightarrow W_0\le\min\left\{13.25^2-0.4;\ 0.18\cdot 10^{-3}\right\}=0.18\cdot 10^{-3},$$

$$\frac{J_0}{2\sqrt{L}}\frac{e^2-1}{\mu}\le\frac{cW_0\sqrt{\Phi}}{\sqrt{\Phi+W_0}}$$

$$\Rightarrow J_0\le\frac{2\cdot c\sqrt{L}\sqrt{\Phi}\cdot W_0\mu}{(e^2-1)\sqrt{\Phi+W_0}}\approx 6\cdot 10^{-6};$$

and

$$\frac{1}{\mu}\left[e^2\left(\frac{R}{L}+\frac{Z_0}{\bar{L}}\right)+\frac{(4e^2+2)\sqrt{L}}{\bar{L}}+\frac{1}{\bar{L}}\left(0,028\frac{e^2+1}{2}+\frac{0,125W_0^2}{L}\frac{e^6+2e^4-1}{24}\right)\right]\le 1,$$

$$\frac{1}{\mu}\left[\left(\frac{R}{L}+\frac{Z_0}{\bar{L}}\right)e^2W_0+\frac{(e^2+3)\sqrt{L}\,W_0}{\bar{L}}+\frac{e^2+2}{2\bar{L}}\left(0.028\,J_0(e^2-1)+\frac{0,125\,J_0^3}{12L}(e^6-1)\right)\right]\le J_0,$$

$$\dot{K}_u=\frac{e^2+1}{2\mu}\left[\frac{R}{L}+\frac{Z_0}{\bar{L}}+\frac{1}{\bar{L}}\left(0.028+0.125\frac{3e^4W_0^2}{4L}\right)+2\sqrt{1+(W_0/\Phi)}/(\mu\bar{L}c)\right]<1,$$

$$\dot{K}_i=\frac{e^2+1}{2\mu}\left[\frac{R}{L}+\frac{Z_0}{\bar{L}}+\frac{1}{\bar{L}}\left(0.028+0.125\frac{3e^4J_0^2}{4L}\right)+2\sqrt{1+(W_0/\Phi)}/(\mu\bar{L}c)\right]<1.$$

Taking into account $R/L+Z_0/\bar{L}=10^4(0.6+13.24\cdot 10^{-4})\approx 0.6\cdot 10^4$, $W_0<0.18\cdot 10^{-3}$ and $J_0\le 3\cdot 10^{-6}$ we see that

$$1/(2\cdot 10^{12})\left[4.44\cdot 10^4+5.58\sqrt{0.45\cdot 10^{-6}}+(13/17)(0.118+3.4)\right]\le 1,$$

$$10^{-6}\left[8+0.83\left(0.54\cdot 10^{-6}+(3.375\cdot 10^{-12})404/5.4\right)\right]\le 6,$$

$$K_u,K_i<\dot{K}_i\le\dot{K}_u\approx\approx(2,1/10^{12})(0.6\cdot 10^4+0.072+0.0036)13\cdot 10^{-7}<1.$$

Most often the initial approximation is chosen to be simple functions, namely:

$$W^{(0)}(t)=\begin{cases}W_0\sin\omega_0t,\ t\in[0,T]\\ 0,\ t\in[T,T+T_0]\end{cases};$$

$$J^{(0)}(t)=\begin{cases}J_0\sin\omega_0t, t\in[0,T]\\ 0,\ t\in[T,T+T_0]\end{cases}$$

($\omega_0=2\pi/T_0$) and $E_0(t)=E_1(t)=E_0\sin\omega_0t$.

Then we have $W^{(n+1)}(t)=B_u(W^{(n)},J^{(n)})$, $J^{(n+1)}(t)=B_i(W^{(n)},J^{(n)})$ $(n=0,1,2,\cdots)$ and

$$\rho((W^{(n+1)},J^{(n+1)}),(W^{(n)},J^{(n)}))$$

$$\le\left(1.13\cdot 10^{-7}\right)^n\Big/\left(1-1.13\cdot 10^{-7}\right)\rho\left(\left(W^{(1)},J^{(1)}\right),\left(W^{(0)},J^{(0)}\right)\right),$$

$(n=1,2\cdots)$

We notice that $\tilde{L}_0(i)=\tilde{L}_1(i)=6i-(1/3)i^3$ imply

$$\tilde{L}_0(t-T)=6\left(e^{-(R/L)(t-T)}\left(W(t)+J(t-T)\right)\Big/\left(2\sqrt{L}\right)\right)-(1/3)\left(e^{-(R/L)(t-T)}\left(W(t)+J(t-T)\right)\Big/\left(2\sqrt{L}\right)\right)^3,$$

$$\tilde{L}_1(t-T)=6\left(e^{-(R/L)(t-T)}\left(W(t-T)+J(t)\right)\Big/\left(2\sqrt{L}\right)\right)-(1/3)\left(e^{-(R/L)(t-T)}\left(W(t-T)+J(t)\right)\Big/\left(2\sqrt{L}\right)\right)^3$$

and therefore (recall $|\Pi_0|\le i_0=1, |\Pi_1|\le i_0=1$):

$$\left|2\sqrt{L}\int_T^t e^{(R/L)(s-T)}R_0\left(\Pi_0(W^{(0)},J^{(0)})(s)\right)\Big/\left(\tilde{L}_0(s-T)\right)\mathrm{d}s\right|$$

$$\le 2\sqrt{L}\int_T^t e^{(R/L)(s-T)}\left|R_0(i_0)\right|\Big/\left(\tilde{L}_0(s-T)\right)\mathrm{d}s$$

$$\le 2\sqrt{L}\int_T^t e^{(R/L)(s-T)}\left(0,028+0,125\right)\Big/(\bar{L})\mathrm{d}s$$

$$\le 0.31L\sqrt{L}\left(e^{(R/L)T_0}-1\right)\Big/\left(\bar{L}R\right)\le 0.074\cdot 10^{-12}\approx 0.$$

Analogously

$$\left|2\sqrt{L}\int_T^t e^{(R/L)(s-T)}R_1\left(\Pi_1(W^{(0)},J^{(0)})(s)\right)\Big/\tilde{L}_1(s-T)\mathrm{d}s\right|\le\left(6\sqrt{L}/17\right)(0.028+0.125)\left(e^{(R/L)T_0}-1\right)L\Big/R\approx 0.$$

We find the first approximation

$$W^{(1)}(t)=B_u(W^{(0)},J^{(0)})(t)=\int_T^t U(W^{(0)},J^{(0)})(s)\mathrm{d}s-\left(\frac{t-T}{T_0}-\frac{1}{2}\right)\int_T^{T+T_0}U(W^{(0)},J^{(0)})(s)\mathrm{d}s$$
$$-\frac{1}{T_0}\int_T^{T+T_0}\int_T^t U(W^{(0)},J^{(0)})(s)\mathrm{d}s\mathrm{d}t$$
$$=-J_0\sin\omega_0 t+(R/L)\int_T^t J_0\sin\omega_0 s\,\mathrm{d}s+2E_0\sqrt{L}\int_T^t e^{(R/L)(s-T)}\sin\omega_0 s\Big/\left(\tilde{L}_0(s-T)\right)\mathrm{d}s$$
$$+Z_0J_0\int_T^t\sin\omega_0 s\Big/\left(\tilde{L}_0(s-T)\right)\mathrm{d}s-2\sqrt{L}\int_T^t\frac{e^{(R/L)(s-T)}}{\tilde{L}_0(s-T)}\frac{\bar{C}_0^{-1}}{2\sqrt{L}}\left(\int_T^s e^{-(R/L)(\theta-T)}J_0\sin\omega_0\theta\mathrm{d}\theta\right)\mathrm{d}s$$
$$-\left(\frac{t-T}{T_0}-\frac{1}{2}\right)\Bigg[2E_0\sqrt{L}\int_T^{T+T_0}e^{(R/L)(s-T)}\sin\omega_0 s\Big/\left(\tilde{L}_0(s-T)\right)\mathrm{d}s+Z_0J_0\int_T^{T+T_0}\sin\omega_0 s\Big/\left(\tilde{L}_0(s-T)\right)\mathrm{d}s$$
$$-2\sqrt{L}\int_T^{T+T_0}\frac{e^{(R/L)(s-T)}}{\tilde{L}_0(s-T)}\bar{C}_0^{-1}\left(\frac{J_0}{2\sqrt{L}}\int_T^s e^{-(R/L)(\theta-T)}\sin\omega_0\theta\mathrm{d}\theta\right)\mathrm{d}s\Bigg]$$
$$-\frac{1}{T_0}\int_T^{T+T_0}\Bigg[-J_0\sin\omega_0 t+(R/L)\int_T^t J_0\sin\omega_0 s\mathrm{d}s+2E_0\sqrt{L}\int_T^t e^{(R/L)(s-T)}\sin\omega_0 s\tilde{L}_0(s-T)\mathrm{d}s$$
$$+Z_0J_0\int_T^t\sin\omega_0 s\Big/\left(\tilde{L}_0(s-T)\right)\mathrm{d}s-2\sqrt{L}\int_T^t\frac{e^{(R/L)(s-T)}}{\tilde{L}_0(s-T)}\bar{C}_0^{-1}\left(\frac{J_0}{2\sqrt{L}}\int_T^s e^{-(R/L)(\theta-T)}\sin\omega_0\theta\,\mathrm{d}\theta\right)\mathrm{d}s\Bigg]\mathrm{d}t.$$

Since $\left|(R/L)\int_T^t J_0\sin\omega_0 s\,\mathrm{d}s\right|\approx J_0\cdot 10^{-9}$ and

$$2\sqrt{L}\left|\int_T^t\frac{e^{(R/L)(s-T)}}{\tilde{L}_0(s-T)}\bar{C}_0^{-1}\left(\frac{J_0}{2\sqrt{L}}\int_T^s e^{-(R/L)(\theta-T)}\sin\omega_0\theta\mathrm{d}\theta\right)\mathrm{d}s\right|\le W_0\frac{2\sqrt{L}}{\bar{L}}\frac{L}{R}\left(e^{(R/L)T_0}-1\right)\approx W_0 3.6\cdot 10^{-8}\approx 0$$

we can disregard this terms and obtain

$$W^{(1)}(t)=-J_0\sin\omega_0 t+2E_0\sqrt{L}\int_T^t\frac{e^{(R/L)(s-T)}\sin\omega_0 s}{\tilde{L}_0(s-T)}\mathrm{d}s+Z_0J_0\int_T^t\sin\omega_0 s\Big/\left(\tilde{L}_0(s-T)\right)\mathrm{d}s$$
$$-\left(\frac{t-T}{T_0}-\frac{1}{2}\right)\left(2E_0\sqrt{L}\int_T^{T+T_0}e^{(R/L)(s-T)}\sin\omega_0 s\Big/\left(\tilde{L}_0(s-T)\right)\mathrm{d}s+Z_0J_0\int_T^{T+T_0}\sin\omega_0 s\Big/\left(\tilde{L}_0(s-T)\right)\mathrm{d}s\right)$$
$$-\frac{1}{T_0}\int_T^{T+T_0}\left(2E_0\sqrt{L}\int_T^t e^{(R/L)(s-T)}\sin\omega_0 s\Big/\left(\tilde{L}_0(s-T)\right)\mathrm{d}s+Z_0J_0\int_T^t\sin\omega_0 s\Big/\left(\tilde{L}_0(s-T)\right)\mathrm{d}s\right)\mathrm{d}t,$$

and then

$$\left|W^{(1)}(t)-W^{(0)}(t)\right|$$
$$\le J_0+\left(2E_0L\sqrt{L}/R\bar{L}\right)\left(e^{(R/L)T_0}-1\right)+\left(Z_0J_0/\bar{L}\right)\left|\int_T^t e^{\mu(s-T)}\,\mathrm{d}s\right|+\left(E_0L\sqrt{L}/R\bar{L}\right)\left(e^{(R/L)T_0}-1\right)$$
$$+\left(Z_0J_0/2\bar{L}\right)\left|\int_T^{T+T_0}e^{\mu(s-T)}\,\mathrm{d}s\right|+\frac{1}{T_0}\int_T^{T+T_0}\left(\left(2E_0L\sqrt{L}/R\bar{L}\right)\left(e^{(R/L)T_0}-1\right)+\left(Z_0J_0/\bar{L}\right)\left|\int_T^t e^{\mu(s-T)}\,\mathrm{d}s\right|\right)\mathrm{d}t\le J_0$$
$$+\left(5L^{3/2}/\bar{L}R\right)\left(e^{(R/L)T_0}-1\right)E_0+5Z_0J_0\left(e^{\mu_0}-1\right)\Big/\left(2\bar{L}\mu\right)$$
$$\le e^{\mu_0}J_0=e^2J_0.$$

For the derivative we have

$$\dot{W}^{(1)}(t)= U(W^{(0)},J^{(0)})(t)-\frac{1}{T_0}\int_T^{T+T_0}U(W^{(0)},J^{(0)})(s)\mathrm{d}s$$

$$=-\omega_0 J_0\cos\omega_0 t+\frac{R}{L}J_0\sin\omega_0 t+\frac{2e^{(R/L)(t-T)}E_0\sqrt{L}}{\tilde{L}_0(t-T)}$$

$$+\frac{Z_0}{\tilde{L}_0}\quad J_0\sin\omega_0 t-\frac{2e^{(R/L)(t-T)}\sqrt{L}\ R\left(\Pi_0(W^{(0)},J^{(0)})(t)\right)}{\tilde{L}_0(t-T)}$$

$$-\frac{2e^{(R/L)(t-T)}\sqrt{L}}{\tilde{L}_0(t-T)}\bar{C}_0^{-1}\left(\int_T^t\Pi_0(W^{(0)},J^{(0)})(\theta)\mathrm{d}\theta\right)-\frac{1}{T_0}\left(2E\sqrt{L}\int_T^{T+T_0}e^{(R/L)(s-T)/\tilde{L}_0(s-T)}\mathrm{d}s\right)$$

$$+\frac{1}{T_0}\left(Z_0\int_T^{T+T_0}J_0\sin\omega_0\Big/\left(\tilde{L}_0(s-T)\right)\mathrm{d}s\ -2\sqrt{L}\int_T^{T+T_0}e^{(R/L)(s-T)}R_0\left(\Pi_0(W^{(0)},J^{(0)})(s)\right)\Big/\left(\tilde{L}_0(s-T)\right)\mathrm{d}s\right)$$

$$+\frac{1}{T_0}\left(2\sqrt{L}\int_T^{T+T_0}\frac{e^{(R/L)(s-T)}}{\tilde{L}_0(s-T)}\bar{C}_0^{-1}\left(\int_T^s\Pi_0(W^{(0)},J^{(0)})(\theta)\mathrm{d}\theta\right)\mathrm{d}s\right).$$

Then we obtain

$$\left|\dot{W}^{(1)}(t)-\dot{W}^{(0)}(t)\right|$$

$$\le\left(\omega_0+\frac{R}{L}+\frac{Z_0}{\bar{L}}+\frac{Z_0}{\bar{L}}(e^{\mu_0}-1)/\mu_0\right)J_0+\left(2\sqrt{L}/\bar{L}\right)\left(e^{RT_0/L}+\left(e^{RT_0/L}-1\right)/\left(RT_0/L\right)\right)\left(E_0+W_0\right)$$

$$+\left(2\sqrt{L}/\bar{L}\right)\left(0.028+0.125\right)\left(e^{RT_0/L}+(e^{RT_0/L}-1)/\left(RT_0/L\right)\right)+\left(2\sqrt{L}/\bar{L}\right)\left(e^{RT_0/L}-1\right)/\left(RT_0/L\right)W_0.$$

Since $\left(e^{RT_0/L}-1\right)/\left(RT_0/L\right)\approx 1,\ E_0=0.5\,V$ we have

$$\left|\dot{W}^{(1)}(t)-\dot{W}^{(0)}(t)\right|\le\left(2\pi\cdot 10^{12}+0,6.10^4+\frac{225(e^2-1)}{34}\right)J_0+\frac{3\sqrt{0.45\cdot 10^{-6}}}{17}\left(2+6W_0+0,612\right)\approx 3.768\cdot 10^7.$$

It follows $\rho\left(\dot{W}^{(1)},\dot{W}^{(0)}\right)\le 18.84\cdot 10^6$. Consequently

$$\rho((W^{(n+1)},J^{(n+1)}),(W^{(n)},J^{(n)}))\le\frac{\left(1.13\cdot 10^{-7}\right)^n}{1-1.13\cdot 10^{-7}}3.8\cdot 10^7\quad(n=0,1,\cdots)$$

or

$$\left|\dot{W}^{(n+1)}(t)-\dot{W}^{(n)}(t)\right|\le 7.4(1.13\cdot 10^{-7})^n 3.8\cdot 10^7\le(1.13\cdot 10^{-7})^n\cdot(2.82)\cdot 10^8.$$

In the same way we can obtain an estimate for the second component of the operator *B*:

$$J^{(1)}(t)=B_i(W^{(0)},J^{(0)})(t)=\int_T^t I(W^{(0)},J^{(0)})(s)\mathrm{d}s-\left(\frac{t-T}{T_0}-\frac{1}{2}\right)\int_T^{T+T_0}I(W^{(0)},J^{(0)})(s)\mathrm{d}s-\frac{1}{T_0}\int_T^{T+T_0}\int_T^t I(W^{(0)},J^{(0)})(s)\mathrm{d}s\mathrm{d}t$$

$$=-W_0\sin\omega_0 t+\frac{RW_0}{L}\int_T^t\sin\omega_0 s\,\mathrm{d}s+2E_0\sqrt{L}\int_T^t\frac{e^{R/L(s-T)}}{\tilde{L}_1(s-T)}\mathrm{d}s$$

$$+Z_0\int_T^t\frac{W_0\sin\omega_0 s}{\tilde{L}_1(s-T)}\mathrm{d}s-\left(\frac{t-T}{T_0}-\frac{1}{2}\right)\left(2E_0\sqrt{L}\int_T^{T+T_0}\frac{e^{(R/L)(s-T)}}{\tilde{L}_1(s-T)}\mathrm{d}s+Z_0\int_T^{T+T_0}\frac{W_0\sin\omega_0 s}{\tilde{L}_1(s-T)}\ \mathrm{d}s\right)$$

$$-\frac{1}{T_0}\int_T^{T+T_0}\left(2E_0\sqrt{L}\int_T^t\frac{e^{(R/L)(s-T)}}{\tilde{L}_1(s-T)}\mathrm{d}s+Z_0\int_T^t\frac{W_0\sin\omega_0 s}{\tilde{L}_1(s-T)}\ \mathrm{d}s\right)\mathrm{d}t.$$

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